Math 4707 Fall 2017 (Darij Grinberg): midterm 2
due date: Wednesday 8 Nov 2017 at the beginning of class, or before that by email or moodle
Please solve **at most 4** of the 7 exercises!

0.1. Idempotent maps

If S is a set, then a map $f : S \to S$ is said to be *idempotent* if and only if $f \circ f = f$. For instance, the map $[3] \to [3]$ sending $1, 2, 3$ to $1, 3, 3$ (respectively) is idempotent.

Exercise 1. Let $n \in \mathbb{N}$.

(a) Prove that a map $f : [n] \to [n]$ is idempotent if and only if $f(y) = y$ for every y in the image of f.

(b) Prove that the number of idempotent maps $[n] \to [n]$ is $\sum\limits_{k=0}^{n} \dbinom{n}{k} k^{n-k}$.

(c) Prove that the number of idempotent maps $[n] \to [n]$ has the form $an + 1$ for some $a \in \mathbb{N}$. (Of course, a will depend on n.)

[**Hint:** When is $\dbinom{n}{k} k^{n-k}$ divisible by n ?]

0.2. Fixed points

Exercise 2. Let S be a finite set. For any map $f : S \to S$, we let $\operatorname{Fix} f$ denote the set of all fixed points of f. (That is, $\operatorname{Fix} f = \{s \in S \mid f(s) = s\}$.)

(a) Prove that $|\operatorname{Fix}(f \circ g)| = |\operatorname{Fix}(g \circ f)|$ for any two maps $f : S \to S$ and $g : S \to S$.

(b) Is it true that every three maps f, g, h from S to S satisfy $|\operatorname{Fix}(f \circ g \circ h)| = |\operatorname{Fix}(g \circ f \circ h)|$?

[**Hint:** For **(a)**, find a bijection.]

0.3. A binomial coefficient in a denominator

Exercise 3. Let n and a be two integers with $n \geq a \geq 1$. Prove that

$$\sum_{k=a}^{n} \frac{(-1)^k}{k} \binom{n-a}{k-a} = \frac{(-1)^a}{a \dbinom{n}{a}}.$$

0.4. Derangements with at most 1 descent

Exercise 4. Let $n \in \mathbb{N}$. How many derangements σ of $[n]$ have at most 1 descent?

(See homework set #5 for the definitions of descents and of derangements.)

0.5. Connected permutations

Definition 0.1. Let n be a positive integer. A permutation σ of $[n]$ is said to be *connected* if and only if there exists no $k \in [n-1]$ such that $\sigma([k]) = [k]$.

For example, the permutation σ of $[5]$ sending $1,2,3,4,5$ to $2,4,1,5,3$ is connected, since it satisfies

$$\sigma([1]) = \{2\} \neq [1], \qquad \sigma([2]) = \{2,4\} \neq [2],$$
$$\sigma([3]) = \{2,4,1\} \neq [3], \qquad \sigma([4]) = \{2,4,1,5\} \neq [4].$$

But the permutation σ of $[4]$ sending $1,2,3,4$ to $2,1,4,3$ is not connected, because it satisfies $\sigma([2]) = [2]$. Likewise, a permutation σ of $[n]$ (for $n > 1$) satisfying $\sigma(1) = 1$ is never connected (since $\sigma([1]) = [1]$); the same holds for a permutation σ satisfying $\sigma(n) = n$ (since $\sigma([n-1]) = [n-1]$).

Exercise 5. For each positive integer n, let c_n denote the number of all connected permutations of $[n]$. (Thus, $c_1 = 1$, $c_2 = 1$ and $c_3 = 3$.)

Prove that

$$n! = \sum_{k=1}^{n} c_k (n-k)! \qquad \text{for each positive integer } n.$$

0.6. Permutations and intervals

An *integer interval* means a set of the form $\{a, a+1, \ldots, b\}$ for some integers a and b. (If $a > b$, then this set is understood to be empty.)

Exercise 6. Let $n \in \mathbb{N}$ and $r \in [n]$. A permutation σ of $[n]$ is said to be *r-friendly* if for each $k \in \{r, r+1, \ldots, n\}$, the set $\sigma([k])$ is an integer interval.

(For example, the permutation σ of $[6]$ sending $1,2,3,4,5,6$ to $2,4,3,5,1,6$ is 3-friendly (since $\sigma([3]) = \{2,3,4\}$, $\sigma([4]) = \{2,3,4,5\}$, $\sigma([5]) = \{1,2,3,4,5\}$ and $\sigma([6]) = \{1,2,3,4,5,6\}$ are integer intervals), and thus also r-friendly for each $r \geq 3$, but not 2-friendly (since $\sigma([2]) = \{2,4\}$ is not an integer interval).)

Prove that the number of r-friendly permutations of $[n]$ is $2^{n-r} r!$.

0.7. Inverting a power series

Exercise 7. Find and prove an explicit formula for the coefficient of x^n in the formal power series $\dfrac{1}{1 - x - x^2 + x^3}$.

[Hint: The standard strategy is to factor $1 - x - x^2 + x^3$, then do partial fraction decomposition. But it is perfectly legitimate to guess the formula based on solving

$$\left(1-x-x^2+x^3\right)\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+\cdots\right)=1$$

for the first few of the unknown coefficients $b_0, b_1, b_2, \ldots$, and then prove it by multiplying out. Either option works.]