Math 4707 Fall 2017 (Darij Grinberg): midterm 1 [corrected]
due date: Friday 13 Oct 2017 by moodle or email (or in office hours before)
Please solve **at most 5** of the 6 exercises.

Exercise 1. Let $n \in \mathbb{N}$.
(a) Find the number of all triples (A, B, C) of subsets of $[n]$ satisfying $A \cup B \cup C = [n]$ and $A \cap B \cap C = \varnothing$.
(b) Find the number of all triples (A, B, C) of subsets of $[n]$ satisfying $B \cap C = C \cap A = A \cap B$.

Recall that if $n \in \mathbb{N}$ and $k \in \mathbb{N}$, then $\operatorname{sur}(n, k)$ denotes the number of surjections $[n] \to [k]$, and $\left\{ n \atop k \right\}$ denotes the Stirling number of the 2nd kind (defined as $\operatorname{sur}(n, k) / k!$).

Exercise 2. Let n be a positive integer. Let $k \in \mathbb{N}$.
(a) Prove that

$$\operatorname{sur}(n, k) = k \sum_{i=0}^{k} (-1)^{k-i} \binom{k-1}{i-1} i^{n-1}.$$

(b) Prove that

$$\left\{ n \atop k \right\} = \sum_{i=0}^{k} (-1)^{k-i} \frac{i^n}{i! (k-i)!}.$$

Exercise 3. A set S of integers is said to be 2-*lacunar* if every $i \in S$ satisfies $i + 1 \notin S$ and $i + 2 \notin S$. (That is, any two distinct elements of S are at least a distance of 3 apart on the real axis.) For example, $\{1, 5, 8\}$ is 2-lacunar, but $\{1, 5, 7\}$ is not.
For any $n \in \mathbb{N}$, we let $h(n)$ denote the number of all 2-lacunar subsets of $[n]$.
(a) Prove that $h(n) = h(n-1) + h(n-3)$ for each $n \geq 3$.
(b) Prove that $h(n) = \sum_{\substack{k \in \mathbb{N}; \\ 2k \leq n+2}} \binom{n+2-2k}{k}$ for each $n \in \mathbb{N}$.

Exercise 4. A set S of integers is said to be *shadowed* if it has the following property: Whenever an **odd** integer i belongs to S, the next integer $i + 1$ must also belong to S. (For example, $\varnothing$, $\{2, 4\}$ and $\{1, 2, 5, 6, 8\}$ are shadowed, but $\{1, 5, 6\}$ is not, since 1 belongs to $\{1, 5, 6\}$ but 2 does not.)
Let $n \in \mathbb{N}$ be even. How many shadowed subsets of $[n]$ exist?

Exercise 5. Let n and k be positive integers. A k-*smord* will mean a k-tuple $(a_1, a_2, \ldots, a_k) \in [n]^k$ such that no two consecutive entries of the k-tuple are equal (i.e., we have $a_i \neq a_{i+1}$ for all $i \in [k-1]$). For example, $(3, 1, 3, 2)$ is a 4-smord (when $n \geq 3$), but $(1, 3, 3, 2)$ is not.
Compute the number of all k-smords.

Exercise 6. This continues Exercise 7 from homework set 2.

Let n be a positive integer. Let X be a set.

We define a map $c : X^n \to X^n$ by

$$c(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_n, x_1) \qquad \text{for all } (x_1, x_2, \ldots, x_n) \in X^n.$$

(In other words, the map c transforms any n-tuple $(x_1, x_2, \ldots, x_n) \in X^n$ by "rotating" it one step to the left, or, equivalently, moving its first entry to the last position.)

For two n-tuples $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{x} \sim \mathbf{y}$ if there exists some $k \in \mathbb{N}$ such that $\mathbf{y} = c^k(\mathbf{x})$. (For example, $(1,5,2,4) \sim (2,4,1,5)$, because $(2,4,1,5) = c^2(1,5,2,4)$.)

(a) Prove that $\sim$ is an equivalence relation, i.e., is reflexive, transitive and symmetric. (For example, symmetry boils down to showing that if there exists some $k \in \mathbb{N}$ satisfying $\mathbf{y} = c^k(\mathbf{x})$, then there exists some $\ell \in \mathbb{N}$ satisfying $\mathbf{x} = c^\ell(\mathbf{y})$.)

(b) An *n-necklace* (over X) shall mean a $\sim$-equivalence class. We denote the $\sim$-equivalence class of a tuple $\mathbf{x} \in X^n$ by $[\mathbf{x}]_\sim$.

Let $\mathbf{x} \in X^n$ be an n-tuple. Let m be the smallest nonzero period of the n-tuple $\mathbf{x} \in X^n$.

Prove that $[\mathbf{x}]_\sim = \{c^0(\mathbf{x}), c^1(\mathbf{x}), \ldots, c^{m-1}(\mathbf{x})\}$.

(c) Show that the m tuples $c^0(\mathbf{x}), c^1(\mathbf{x}), \ldots, c^{m-1}(\mathbf{x})$ are distinct. Conclude that $|[\mathbf{x}]_\sim| = m$.