UMTYMP Advanced Topics

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(just for today)

Plan for today

1. Cartesian products + relations ← Books!
2. equivalence relations

BREAK

3. equivalence classes + partitions
4. induction (if time)

For each topic, I'll give a brief introduction, then you'll work through examples in groups

§1 : Cartesian products and relations

Suppose we have two sets, X and Y.

<u>Ex</u> $X = \{0,1\}$ $Y = \{1,2,3,4\}$

We define the Cartesian product of X and Y to be the set of ordered pairs (x,y) where $x \in X$ and $y \in Y$. We denote this set by $X \times Y$.

So, $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$

<u>Ex</u> $X \times Y = \{(0,1),(0,2),(0,3),(0,4),$
$(1,1),(1,2),(1,3),(1,4)\}$

Question Have you seen this anywhere before?

Ex $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a,b) \mid a \in \mathbb{R}, b \in \mathbb{R}\}$

Things to pay attention to: $(1,2) \in X \times Y, \notin Y \times X$

- order matters - $X \times Y \neq Y \times X$ if X and Y are different sets

- X and Y could be any sets; they could be completely unrelated, or they could be the same set.

- order <u>really</u> matters - $X \times (Y \times Z) \neq (X \times Y) \times Z$

$(x_1, (y_1, z_1))$ $((x_1, y_1), z_1)$

bijection (later)

In mathematics, we often want to talk about how two objects are related to each other

Ex

- a number a equals a number b
- a number a is less than a number b
- an integer a is a divisor of an integer b
- a line L_1 is parallel to a line L_2
- a line L_1 is a graph of a linear function
- a function F is an antiderivative of a function f

Cartesian products give us a way to make this idea rigorous:

A relation between sets X and Y is a subset of $X \times Y$.

<u>Ex</u> • a number a equals a number b $\quad \mathbb{R}$

$$R = \{(a,a) \mid a \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$$

• a number a is less than a number b

$$R = \{(a,b) \mid a < b\} \subset \mathbb{R} \times \mathbb{R}$$

• an integer a is a divisor of an integer b

$$R = \{(a,b) \mid b = na \text{ for some integer } n\}$$

• a line L_1 is parallel to a line L_2

$$R = \{(L_1, L_2) \mid \text{same slope}\}$$ or dot product, cross pr.

Thinking about relations this way is cumbersome and not very enlightening –

so we usually don't. $x \sim y$

We typically use notation like xRy if $(x,y) \in R$, $\subset X \times Y$

Ex
- a number a equals a number b

  $a = b$

- a number a is less than a number b

  $a < b$

- an integer a is a divisor of an integer b

  $a \mid b$

- a line L_1 is parallel to a line L_2

  $L_1 \parallel L_2$

§2: Equivalence relations

One of the most common types of relations in mathematics is defining what it means for two objects to be "the same".

Ex
- a number a equals a number b
- a triangle A is congruent to a triangle B
- a function f equals a function g

(+ less obvious examples, in group work)

Determining when two objects are the same motivates the definition of an equivalence relation:

A relation $\sim$ on a set X is an equivalence relation if:

- it is reflexive,

  $x \sim x$ for all $x \in X$

- it is symmetric,

  If $x \sim y$, then $y \sim x$

- and it is transitive.

  If $x \sim y$ and $y \sim z$, then $x \sim z$

<u>Ex</u> • a number a equals a number b

§3: Equivalence classes and partitions

Once we have defined an equivalence relation, we want to think about equivalent objects as a single unit, called an equivalence class:

Suppose $\sim$ is an equivalence relation on a set X. For an element $x \in X$, the equivalence class of x is

$$E_x = \{y \in X \mid x \sim y\}$$

(the set of elements equivalent to x)

<u>Ex</u> Let $X = \mathbb{Z}$, $x \sim y$ if x and y have the same remainder when divided by 3.

$$E_0 = \{\ldots, 0, 3, 6, 9, \ldots\} = E_3 = E_6 = E_9$$

$$E_1 = \{\ldots, 1, 4, 7, 10, \ldots\} = E_4$$

$$E_2 = \{\ldots, 2, 5, 8, 11, \ldots\}$$

Ex Let $X=\mathbb{Z}$, $x \sim y$ if x and y have the same remainder when divided by 3.

$$E_0 = \{\dots, -3, 0, 3, 6, 9, \dots\} = E_3 = E_6 \cdots$$
$$E_1 = \{\dots, -2, 1, 4, 7, 10, \dots\}$$
$$E_2 = \{\dots, -1, 2, 5, 8, 11, \dots\}$$

So our relation "partitions" the set $\mathbb{Z}$ into three disjoint subsets.

Note

- different equivalence classes are disjoint.

  If $E_x \neq E_y$, then $E_x \cap E_y = \emptyset$

- Every element in X is in an equivalence class.

  $x \in E_x$

A partition of a set X is a collection $\mathcal{P}$ of nonempty subsets of X s.t.

- every element x is in some $A \in \mathcal{P}$
- For any $A, B \in \mathcal{P}$, either $A = B$ or $A \cap B = \emptyset$

(every element is in exactly one subset)

Ex $X = \{1,2,3,4\}$ Which are partitions?

- $\{\{1\},\{2\},\{3\},\{4\}\}$ ✓
- $\{\{1,2,3,4\}\}$ ✗
- $\{\{\},\{1,2\},\{3,4\}\}$ ✗
- $\{\{1\},\{2,3,4\}\}$ ✓
- $\{\{1\},\{1,2\},\{3\}\}$ ✗

Thm Consider a set X.

1) If $\sim$ is an equivalence relation on X, the equivalence classes form a partition of X.

2) If $\mathcal{P}$ is a partition of X, the relation

$$x \sim y \iff x, y \text{ are in the same set in } \mathcal{P}$$

is an equivalence relation.

Pf groupwork!

New groups!

you can't be in the same group as anyone you were just in a group with.

start by comparing grids

§4: Induction

Claim $n! > 2^n$ for $n \geq 4$

Pf If $n=4$, $n! = 4! = 24 > 16 = 2^4 = 2^n$, so true.

Assume $n! > 2^n$, want to show $(n+1)! > 2^{n+1}$

$$
\begin{aligned}
(n+1)! &= (n+1)\,n! \\
&> (n+1)\,2^n \quad \text{(by assumption)} \\
&\geq 2 \cdot 2^n \quad \text{(since } n \geq 4\text{)} \\
&= 2^{n+1}
\end{aligned}
$$

$(n+1)! > 2^{n+1}$

$\vdots$

$n! > 2^n$

Basic structure:

To show $P(n)$ is true for all $n \in \mathbb{N}$,

- show $P(1)$ is true
- assume $P(n)$ is true, show $P(n+1)$ is true.

Things to watch for:

- make sure your base case is clear

  $1=1$ ✓ (not good)

- don't start with what you're trying to prove!

Strong Induction

To show $P(n)$ is true for all $n \in \mathbb{N}$,

- show $P(1)$ is true
- assume $P(k)$ is true for all $k < n$, show $P(n)$ is true

(examples in group work)

HW

Equiv Classes + Partitions #4

Induction #2,3,4

a, part 2: Hint: If $E_x \cap E_y \neq \phi$, show $E_x = E_y$ as sets