Why the log and exp series are mutually inverse

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The purpose of this note is to detail an argument that I have only briefly mentioned in class: namely, the (algebraic) proof of the fact that the power series exp and log (or, more precisely, $\exp x - 1$ and $\log(1+x)$) are mutually inverse. Along the way, I will prove a few basic properties of the derivative of formal power series. A significant part of this note is copy-pasted from [GriRei18, solution to Exercise 1.7.20].

First, let us recall the setting in which we are working.

We let $\mathbf{k}$ be a commutative $\mathbb{Q}$-algebra. For example, $\mathbf{k}$ can be one of the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. (If you are curious: A $\mathbb{Q}$-*algebra* is a ring whose elements can be divided by $1, 2, 3, \ldots$. Thus, $\mathbb{Z}$ is not a $\mathbb{Q}$-algebra, since 2 cannot be divided by 3 **inside** $\mathbb{Z}$.)

(In class, I used the notation K for $\mathbf{k}$.)

We consider the ring $\mathbf{k}[[x]]$ of formal power series in one indeterminate x over $\mathbf{k}$. We shall abbreviate the notion "formal power series" by "*FPS*".

If $f \in \mathbf{k}[[x]]$ and $g \in \mathbf{k}[[x]]$ are two FPSs such that the constant term of g is 0, then we can define a new FPS $f[g] \in \mathbf{k}[[x]]$ by

$$f[g] = \sum_{n\geq 0} f_n g^n,$$

where $f_0, f_1, f_2, \ldots$ are the coefficients of f (so that $f = \sum_{n\geq 0} f_n x^n$). The sum $\sum_{n\geq 0} f_n g^n$ is well-defined, because the family $(f_n g^n)_{n\in\mathbb{N}}$ is summable (i.e., for each $i \in \mathbb{N}$, only finitely many entries of this family have a nonzero x^i-coefficient); this is thanks to our assumption that the constant term of g is 0. The FPS $f[g]$ is called the *composition* of f with g (or the result of *substituting* g into f). An alternative notation for $f[g]$ is $f \circ g$. (Some authors also write $f(g)$ instead of $f[g]$, but this is dangerous notation, since $f(g)$ may just as well mean the product of f with g; thus, we shall stick to the notation $f[g]$.)

We define two FPSs $\exp \in \mathbf{k}[[x]]$ and $\overline{\log} \in \mathbf{k}[[x]]$ by

$$\exp = \sum_{n\geq 0} \frac{1}{n!} x^n \qquad \text{and} \qquad \overline{\log} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n.$$

Here, we are using that $\mathbf{k}$ is a $\mathbb{Q}$-algebra. (If we only knew that $\mathbf{k}$ is a ring, then we wouldn't be able to divide by $n!$ and by n in these formulas.)

The FPS $\overline{\log}$ defined above is commonly called $\log(1+x)$ (because it is precisely the Taylor series of the $\log(1+z)$ function from complex analysis). However, I will avoid the "$\log(1+x)$" notation, because it looks like it is a combination of something called "log" with something called "$1+x$", but I have not introduced anything called "log". (In fact, there is no FPS called "log".)

The FPS $\overline{\log}$ has constant term 0 (because the sum in its definition starts at $n = 1$). The FPS exp has constant term $\frac{1}{0!} = 1$; thus, the FPS $\exp - 1$ has constant term

$1-1=0$. Let us denote the FPS $\exp -1$ by $\overline{\exp}$. Then, both FPSs $\overline{\exp}$ and $\overline{\log}$ have constant term 0; hence, they can be substituted into one another. We now claim the following:

Theorem 0.1. We have $\overline{\exp}\left[\overline{\log}\right] = x$ and $\overline{\log}\left[\overline{\exp}\right] = x$.

Theorem 0.1 is an "algebraic analogue" of the well-known fact from calculus that the exponential function and the natural logarithm function are each other's inverse. It is often used in enumerative combinatorics (for computing generating functions). We shall give a purely algebraic proof (somewhat similar to the one given in [Loehr11, Example 7.67]). Other proofs (some combinatorial, some analytic) can be found in the literature.

The proof of Theorem 0.1 uses the concept of the *derivative* of an FPS. This concept is very simple (a lot simpler than the concept of derivative in analysis, where it requires dealing with the intricacies of convergence and differentiability): The *derivative* of an FPS $f \in \mathbf{k}[[x]]$ is defined to be the FPS $\sum\limits_{n\geq 1} n f_n x^{n-1}$, where $f_0, f_1, f_2, \ldots$ are the coefficients of f (so that $f = \sum\limits_{n\geq 0} f_n x^n$). This derivative is denoted by f' or by $\dfrac{d}{dx} f$.

The following properties of derivatives are easy to check:

Proposition 0.2. **(a)** We have $\dfrac{d}{dx}(f+g) = \dfrac{d}{dx} f + \dfrac{d}{dx} g$ for any $f \in \mathbf{k}[[x]]$ and $g \in \mathbf{k}[[x]]$.

(b) We have $\dfrac{d}{dx}(\lambda f) = \lambda \dfrac{d}{dx} f$ for any $f \in \mathbf{k}[[x]]$ and $\lambda \in \mathbf{k}$.

(c) We have $\dfrac{d}{dx}(fg) = \left(\dfrac{d}{dx} f\right) g + f\left(\dfrac{d}{dx} g\right)$ for any $f \in \mathbf{k}[[x]]$ and $g \in \mathbf{k}[[x]]$. (Keep in mind that $f\left(\dfrac{d}{dx} g\right)$ means the **product** of f with $\dfrac{d}{dx} g$, not the composition of f with $\dfrac{d}{dx} g$. The latter would be denoted by $f\left[\dfrac{d}{dx} g\right]$.)

(d) We have $\dfrac{d}{dx}(w^n) = n\left(\dfrac{d}{dx} w\right) w^{n-1}$ for any $w \in \mathbf{k}[[x]]$ and any positive integer n.

(e) If $v \in \mathbf{k}[[x]]$ is an FPS that has a multiplicative inverse v^{-1}, then $\dfrac{d}{dx}\left(v^{-1}\right) = -v^{-2}\left(\dfrac{d}{dx} v\right)$.

Proposition 0.2 **(c)** is known as the *Leibniz rule*.

Proof of Proposition 0.2. We leave the easy proofs of parts **(a)** and **(b)** to the reader.

(c) Let $f \in \mathbf{k}[[x]]$ and $g \in \mathbf{k}[[x]]$.

Let $f_0, f_1, f_2, \ldots$ be the coefficients of f; thus,

$$f = \sum_{n\geq 0} f_n x^n. \tag{1}$$

Hence, the definition of $\frac{d}{dx}f$ yields

$$\frac{d}{dx}f = \sum_{n\geq 1} n f_n x^{n-1} = \sum_{n\geq 0} (n+1) f_{n+1} x^n \tag{2}$$

(here, we have substituted $n+1$ for n in the sum).

Let $g_0, g_1, g_2, \ldots$ be the coefficients of g; thus,

$$g = \sum_{n\geq 0} g_n x^n. \tag{3}$$

Hence, the definition of $\frac{d}{dx}g$ yields

$$\frac{d}{dx}g = \sum_{n\geq 1} n g_n x^{n-1} = \sum_{n\geq 0} (n+1) g_{n+1} x^n \tag{4}$$

(here, we have substituted $n+1$ for n in the sum).

Multiplying the equalities (2) and (3), we obtain

$$\begin{aligned}\left(\frac{d}{dx}f\right) g &= \left(\sum_{n\geq 0} (n+1) f_{n+1} x^n\right)\left(\sum_{n\geq 0} g_n x^n\right)\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} (k+1) f_{k+1} g_{n-k}\right) x^n \qquad (5)\end{aligned}$$

(by the definition of the product of two FPSs).

Multiplying the equalities (1) and (4), we obtain

$$\begin{aligned}f\left(\frac{d}{dx}g\right) &= \left(\sum_{n\geq 0} f_n x^n\right)\left(\sum_{n\geq 0} (n+1) g_{n+1} x^n\right)\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} f_k (n-k+1) g_{n-k+1}\right) x^n \qquad (6)\end{aligned}$$

(by the definition of the product of two FPSs).

Adding the equalities (5) and (6) together, we find

$$\begin{aligned}&\left(\frac{d}{dx}f\right) g + f\left(\frac{d}{dx}g\right)\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} (k+1) f_{k+1} g_{n-k}\right) x^n + \sum_{n\geq 0}\left(\sum_{k=0}^{n} f_k (n-k+1) g_{n-k+1}\right) x^n\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} (k+1) f_{k+1} g_{n-k} + \sum_{k=0}^{n} f_k (n-k+1) g_{n-k+1}\right) x^n \qquad (7)\end{aligned}$$

(by the definition of the sum of two FPSs).

Multiplying the equalities (1) and (3), we obtain

$$fg = \left(\sum_{n\geq 0} f_n x^n\right)\left(\sum_{n\geq 0} g_n x^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} f_k g_{n-k}\right) x^k$$

(by the definition of the product of two FPSs). Thus, $\sum_{k=0}^{0} f_k g_{0-k}, \sum_{k=0}^{1} f_k g_{1-k}, \sum_{k=0}^{2} f_k g_{2-k}, \ldots$ are the coefficients of the FPS fg. Hence, the definition of a derivative yields

$$\frac{d}{dx}(fg) = \sum_{n\geq 1} n\left(\sum_{k=0}^{n} f_k g_{n-k}\right) x^{n-1}. \tag{8}$$

But each $n \in \mathbb{N}$ satisfies

$$\begin{aligned}
& n\left(\sum_{k=0}^{n} f_k g_{n-k}\right) \\
&= \sum_{k=0}^{n} \underbrace{n}_{=k+(n-k)} f_k g_{n-k} = \sum_{k=0}^{n} \underbrace{(k+(n-k)) f_k g_{n-k}}_{=kf_k g_{n-k}+f_k(n-k)g_{n-k}} \\
&= \sum_{k=0}^{n} \left(kf_k g_{n-k} + f_k (n-k) g_{n-k}\right) \\
&= \underbrace{\sum_{k=0}^{n} kf_k g_{n-k}}_{=0f_0 g_{n-0}+\sum_{k=1}^{n} kf_k g_{n-k}} + \underbrace{\sum_{k=0}^{n} f_k (n-k) g_{n-k}}_{=\sum_{k=0}^{n-1} f_k(n-k)g_{n-k}+f_n(n-n)g_{n-n}} \\
&= \underbrace{0f_0 g_{n-0}}_{=0} + \sum_{k=1}^{n} kf_k g_{n-k} + \sum_{k=0}^{n-1} f_k (n-k) g_{n-k} + \underbrace{f_n (n-n) g_{n-n}}_{\substack{=0 \\ (\text{since } n-n=0)}} \\
&= \sum_{k=1}^{n} kf_k g_{n-k} + \sum_{k=0}^{n-1} f_k (n-k) g_{n-k} \\
&= \sum_{k=0}^{n-1} (k+1) f_{k+1} g_{n-(k+1)} + \sum_{k=0}^{n-1} f_k (n-k) g_{n-k} \\
&\qquad (\text{here, we have substituted } k+1 \text{ for } k \text{ in the first sum}).
\end{aligned}$$

Thus, (8) becomes

$$\begin{aligned}
&\frac{d}{dx}(fg) \\
&= \sum_{n\geq 1} \underbrace{n\left(\sum_{k=0}^{n} f_k g_{n-k}\right)}_{=\sum_{k=0}^{n-1}(k+1)f_{k+1}g_{n-(k+1)}+\sum_{k=0}^{n-1}f_k(n-k)g_{n-k}} x^{n-1} \\
&= \sum_{n\geq 1} \left(\sum_{k=0}^{n-1}(k+1)f_{k+1}g_{n-(k+1)} + \sum_{k=0}^{n-1} f_k(n-k)g_{n-k}\right) x^{n-1} \\
&= \sum_{n\geq 0} \left(\underbrace{\sum_{k=0}^{(n+1)-1}}_{=\sum_{k=0}^{n}} (k+1) f_{k+1} \underbrace{g_{(n+1)-(k+1)}}_{=g_{n-k}} + \underbrace{\sum_{k=0}^{(n+1)-1}}_{=\sum_{k=0}^{n}} f_k \underbrace{((n+1)-k)}_{=n-k+1} \underbrace{g_{(n+1)-k}}_{=g_{n-k+1}}\right) x^n \\
&\qquad \text{(here, we have substituted } n+1 \text{ for } n \text{ in the sum)} \\
&= \sum_{n\geq 0} \left(\sum_{k=0}^{n}(k+1)f_{k+1}g_{n-k} + \sum_{k=0}^{n} f_k(n-k+1)g_{n-k+1}\right) x^n \\
&= \left(\frac{d}{dx}f\right) g + f\left(\frac{d}{dx}g\right)
\end{aligned}$$

(by (7)). This proves Proposition 0.2 **(c)**.

(d) Proposition 0.2 **(d)** follows easily by induction over n using Proposition 0.2 **(c)**.

(e) Let $v \in \mathbf{k}[[x]]$ be an FPS that has a multiplicative inverse v^{-1}. The Leibniz rule (applied to v and v^{-1}) yields

$$\frac{d}{dx}\left(v \cdot v^{-1}\right) = \left(\frac{d}{dx}v\right) v^{-1} + v\frac{d}{dx}\left(v^{-1}\right).$$

Comparing this with $\frac{d}{dx}\underbrace{\left(v \cdot v^{-1}\right)}_{=1} = \frac{d}{dx}1 = 0$, we obtain $\left(\frac{d}{dx}v\right) v^{-1} + v\frac{d}{dx}\left(v^{-1}\right) = 0$. Solving this equality for $\frac{d}{dx}\left(v^{-1}\right)$, we find

$$\frac{d}{dx}\left(v^{-1}\right) = -\frac{1}{v}\left(\frac{d}{dx}v\right) v^{-1} = -v^{-2}\left(\frac{d}{dx}v\right).$$

This proves Proposition 0.2 **(e)**. □

Next, we notice that

$$\overline{\exp} = \underbrace{\exp}_{=\sum_{n\geq 0}\frac{1}{n!}x^n} - 1 = \sum_{n\geq 0}\frac{1}{n!}x^n - 1 = \underbrace{\frac{1}{0!}}_{=\frac{1}{1}=1}\underbrace{x^0}_{=1} + \sum_{n\geq 1}\frac{1}{n!}x^n - 1$$

(here, we have split off the addend for $n = 0$ from the sum)

$$= 1 + \sum_{n\geq 1}\frac{1}{n!}x^n - 1 = \sum_{n\geq 1}\frac{1}{n!}x^n. \qquad (9)$$

Hence, the FPS $\overline{\exp}$ has constant term 0. Hence, the FPS $\overline{\log}\left[\overline{\exp}\right]$ is well-defined.

Also,

$$\overline{\log} = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}x^n. \qquad (10)$$

Hence, the FPS $\overline{\log}$ has constant term 0. Hence, the FPS $\overline{\exp}\left[\overline{\log}\right]$ is well-defined.

For each $n \geq 1$, we have

$$\left(\text{the constant term of } \overline{\log}^n\right) = 0 \qquad (11)$$

[1].

Substituting $\overline{\log}$ for x on both sides of the equality (9), we obtain

$$\overline{\exp}\left[\overline{\log}\right] = \sum_{n\geq 1}\frac{1}{n!}\overline{\log}^n.$$

Hence,

$$\begin{aligned}
&\left(\text{the constant term of } \overline{\exp}\left[\overline{\log}\right]\right) \\
&= \left(\text{the constant term of } \sum_{n\geq 1}\frac{1}{n!}\overline{\log}^n\right) \\
&= \sum_{n\geq 1}\frac{1}{n!}\underbrace{\left(\text{the constant term of } \overline{\log}^n\right)}_{\substack{=0 \\ \text{(by (11))}}} = \sum_{n\geq 1}\frac{1}{n!}0 = 0.
\end{aligned}$$

In other words, the FPS $\overline{\exp}\left[\overline{\log}\right]$ has constant term 0. A similar argument (with the roles of $\overline{\exp}$ and $\overline{\log}$ switched) shows that the FPS $\overline{\log}\left[\overline{\exp}\right]$ has constant term 0.

Next, we prove some simple lemmas:

[1]*Proof of (11):* Let $n \geq 1$. The FPS $\overline{\log}$ is divisible by x (since it has constant term 0). Hence, the FPS $\overline{\log}^n$ is divisible by x^n. Thus, the FPS $\overline{\log}^n$ is also divisible by x (since x^n is divisible by x (since $n \geq 1$)), and therefore has constant term 0. In other words, we have $\left(\text{the constant term of } \overline{\log}^n\right) = 0$. This proves (11).

Lemma 0.3. Let $u \in \mathbf{k}[[x]]$ and $v \in \mathbf{k}[[x]]$ be two FPSs having the same constant term. Assume that $\frac{d}{dx}u = \frac{d}{dx}v$. Then, $u = v$.

Proof of Lemma 0.3. Let $u_0, u_1, u_2, \ldots$ be the coefficients of the FPS u (so that $u = \sum_{n\geq 0} u_n x^n$). Thus, $\frac{d}{dx}u = \sum_{n\geq 1} n u_n x^{n-1}$ (by the definition of the derivative).

Let $v_0, v_1, v_2, \ldots$ be the coefficients of the FPS v (so that $v = \sum_{n\geq 0} v_n x^n$). Thus, $\frac{d}{dx}v = \sum_{n\geq 1} n v_n x^{n-1}$ (by the definition of the derivative).

Now,

$$\sum_{n\geq 1} n u_n x^{n-1} = \frac{d}{dx}u = \frac{d}{dx}v = \sum_{n\geq 1} n v_n x^{n-1}.$$

Comparing coefficients in front of x^{n-1} on both sides of this equality, we obtain

$$n u_n = n v_n \qquad \text{for each integer } n \geq 1. \tag{12}$$

On the other hand, the FPS u has constant term u_0 (since $u = \sum_{n\geq 0} u_n x^n$), and the FPS v has constant term v_0 (similarly). Thus, the constant terms of u and v are u_0 and v_0, respectively. Therefore, $u_0 = v_0$ (since the FPSs u and v have the same constant term).

Now, each $n \in \mathbb{N}$ satisfies $u_n = v_n$ [2]. Hence, $\sum_{n\geq 0} \underbrace{u_n}_{=v_n} x^n = \sum_{n\geq 0} v_n x^n$. Thus, $u = \sum_{n\geq 0} u_n x^n = \sum_{n\geq 0} v_n x^n = v$. This proves Lemma 0.3. □

Lemma 0.4. Let $w \in \mathbf{k}[[x]]$ be an FPS having constant term 0. Then,

$$\frac{d}{dx}\left(\overline{\exp}[w]\right) = \left(\frac{d}{dx}w\right) \cdot \exp[w] \tag{13}$$

and

$$\frac{d}{dx}\left(\overline{\log}[w]\right) = \left(\frac{d}{dx}w\right) \cdot \frac{1}{1+w}. \tag{14}$$

Proof of Lemma 0.4. Substituting w for x on both sides of the equality (9), we obtain

$$\overline{\exp}[w] = \sum_{n\geq 1} \frac{1}{n!} w^n.$$

[2] *Proof.* Let $n \in \mathbb{N}$. We must prove that $u_n = v_n$.

If $n = 0$, then this follows immediately from $u_0 = v_0$. Hence, we WLOG assume that we don't have $n = 0$. Thus, $n \geq 1$ (since $n \in \mathbb{N}$). Therefore, (12) yields $n u_n = n v_n$. We can multiply both sides of this equality by $\frac{1}{n}$ (since $\mathbf{k}$ is a $\mathbb{Q}$-algebra), and thus obtain $u_n = v_n$, qed.

Applying the operator $\frac{d}{dx}$ to this equality, we find

$$\frac{d}{dx}\overline{\exp}[w] = \frac{d}{dx}\sum_{n\geq 1}\frac{1}{n!}w^n = \sum_{n\geq 1}\frac{1}{n!}\cdot \underbrace{\frac{d}{dx}(w^n)}_{\substack{=n\left(\frac{d}{dx}w\right)w^{n-1} \\ \text{(by Proposition 0.2 \textbf{(d)})}}} = \sum_{n\geq 1}\underbrace{\frac{1}{n!}\cdot n}_{=\frac{1}{(n-1)!}}\left(\frac{d}{dx}w\right)w^{n-1}$$

$$= \sum_{n\geq 1}\frac{1}{(n-1)!}\left(\frac{d}{dx}w\right)w^{n-1} = \sum_{n\geq 0}\frac{1}{n!}\left(\frac{d}{dx}w\right)w^n$$

$$\text{(here, we have substituted } n \text{ for } n-1 \text{ in the sum)}.$$

Comparing this with

$$\left(\frac{d}{dx}w\right)\cdot \underbrace{\exp[w]}_{\substack{=\sum_{n\geq 0}\frac{1}{n!}w^n \\ \text{(since } \exp=\sum_{n\geq 0}\frac{1}{n!}x^n\text{)}}} = \left(\frac{d}{dx}w\right)\cdot\sum_{n\geq 0}\frac{1}{n!}w^n = \sum_{n\geq 0}\frac{1}{n!}\left(\frac{d}{dx}w\right)w^n,$$

we obtain $\frac{d}{dx}\left(\overline{\exp}[w]\right) = \left(\frac{d}{dx}w\right)\cdot\exp[w]$. This proves (13).

Substituting w for x on both sides of the equality (10), we obtain

$$\overline{\log}[w] = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}w^n.$$

Applying the operator $\frac{d}{dx}$ to this equality, we find

$$\frac{d}{dx}\overline{\log}[w] = \frac{d}{dx}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}w^n = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\cdot \underbrace{\frac{d}{dx}(w^n)}_{\substack{=n\left(\frac{d}{dx}w\right)w^{n-1} \\ \text{(by Proposition 0.2 \textbf{(d)})}}}$$

$$= \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\cdot n\left(\frac{d}{dx}w\right)w^{n-1}$$

$$= \sum_{n\geq 1}(-1)^{n-1}\left(\frac{d}{dx}w\right)w^{n-1} = \sum_{n\geq 0}(-1)^n\left(\frac{d}{dx}w\right)w^n$$

$$\text{(here, we have substituted } n \text{ for } n-1 \text{ in the sum)}.$$

Comparing this with

$$\left(\frac{d}{dx}w\right)\cdot\underbrace{\frac{1}{1+w}}_{=\sum_{n\geq0}(-1)^n w^n}=\left(\frac{d}{dx}w\right)\cdot\sum_{n\geq0}(-1)^n w^n=\sum_{n\geq0}(-1)^n\left(\frac{d}{dx}w\right)w^n,$$

we obtain $\frac{d}{dx}\left(\overline{\log}[w]\right)=\left(\frac{d}{dx}w\right)\cdot\frac{1}{1+w}$. This proves (14). Thus, Lemma 0.4 is proven. □

Lemma 0.5. Let $u\in\mathbf{k}[[x]]$ and $v\in\mathbf{k}[[x]]$ be two FPSs having constant term 1. Assume that $\left(\frac{d}{dx}u\right)\cdot v=\left(\frac{d}{dx}v\right)\cdot u$. Then, $u=v$.

Proof of Lemma 0.5. The FPS v has constant term 1, and thus has a multiplicative inverse v^{-1}. The Leibniz rule (applied to u and v^{-1}) yields

$$\frac{d}{dx}\left(uv^{-1}\right)=\left(\frac{d}{dx}u\right)v^{-1}+u\underbrace{\frac{d}{dx}\left(v^{-1}\right)}_{\substack{=-v^{-2}\left(\frac{d}{dx}v\right)\\ \text{(by Proposition 0.2 (e))}}}=\left(\frac{d}{dx}u\right)v^{-1}+u\left(-v^{-2}\left(\frac{d}{dx}v\right)\right)$$

$$=v^{-2}\underbrace{\left(\left(\frac{d}{dx}u\right)\cdot v-\left(\frac{d}{dx}v\right)\cdot u\right)}_{\substack{=0\\ \text{(since }\left(\frac{d}{dx}u\right)\cdot v=\left(\frac{d}{dx}v\right)\cdot u\text{)}}}=v^{-2}0=0=\frac{d}{dx}1.$$

Moreover, the FPSs uv^{-1} and 1 have the same constant term[3]. Hence, Lemma 0.3 (applied to uv^{-1} and 1 instead of u and v) shows that $uv^{-1}=1$. Thus, $u=v$. This proves Lemma 0.5. □

Proof of Theorem 0.1. The equality (14) (applied to $w=x$) yields $\frac{d}{dx}\left(\overline{\log}[x]\right)=\underbrace{\left(\frac{d}{dx}x\right)}_{=1}\cdot\frac{1}{1+x}=\frac{1}{1+x}$. In other words, $\frac{d}{dx}\overline{\log}=\frac{1}{1+x}$ (since $\overline{\log}=\overline{\log}[x]$).

Now, (13) (applied to $w=\overline{\log}$) shows that

$$\frac{d}{dx}\left(\overline{\exp}\left[\overline{\log}\right]\right)=\underbrace{\left(\frac{d}{dx}\overline{\log}\right)}_{=\frac{1}{1+x}}\cdot\exp\left[\overline{\log}\right]=\frac{1}{1+x}\cdot\exp\left[\overline{\log}\right].$$

[3] *Proof.* The FPS v has constant term 1. Hence, its inverse v^{-1} has constant term $1^{-1}=1$. Now, both FPSs u and v^{-1} have constant term 1. Hence, their product uv^{-1} has constant term $1\cdot1=1$. Since the FPS 1 also has constant term 1, this shows that the FPSs uv^{-1} and 1 have the same constant term (namely, 1).

But $\overline{\exp} = \exp - 1$ and thus $\exp = \overline{\exp} + 1$. Substituting $\overline{\log}$ for x in this equality, we find $\exp\left[\overline{\log}\right] = \overline{\exp}\left[\overline{\log}\right] + 1$. Hence,

$$\frac{d}{dx}\left(\exp\left[\overline{\log}\right]\right) = \frac{d}{dx}\left(\overline{\exp}\left[\overline{\log}\right] + 1\right) = \frac{d}{dx}\overline{\exp}\left[\overline{\log}\right] + \underbrace{\frac{d}{dx}1}_{=0}$$
$$= \frac{d}{dx}\overline{\exp}\left[\overline{\log}\right] = \frac{1}{1+x} \cdot \exp\left[\overline{\log}\right].$$

Multiplying this equality by $1+x$, we find

$$\left(\frac{d}{dx}\left(\exp\left[\overline{\log}\right]\right)\right) \cdot (1+x) = \exp\left[\overline{\log}\right].$$

Comparing this with $\underbrace{\left(\frac{d}{dx}(1+x)\right)}_{=1} \cdot \exp\left[\overline{\log}\right] = \exp\left[\overline{\log}\right]$, we find

$$\left(\frac{d}{dx}\left(\exp\left[\overline{\log}\right]\right)\right) \cdot (1+x) = \left(\frac{d}{dx}(1+x)\right) \cdot \exp\left[\overline{\log}\right].$$

Since both FPSs $\exp\left[\overline{\log}\right]$ and $1+x$ have constant term 1 [4], we can thus apply Lemma 0.5 to $u = \exp\left[\overline{\log}\right]$ and $v = 1+x$. We thus conclude that $\exp\left[\overline{\log}\right] = 1 + x$. Comparing this with $\exp\left[\overline{\log}\right] = \overline{\exp}\left[\overline{\log}\right] + 1$, we obtain $\overline{\exp}\left[\overline{\log}\right] + 1 = 1 + x$. Subtracting 1 from this equality, we find $\overline{\exp}\left[\overline{\log}\right] = x$.

The equality (13) (applied to $w = x$) yields $\frac{d}{dx}\left(\overline{\exp}[x]\right) = \underbrace{\left(\frac{d}{dx}x\right)}_{=1} \cdot \underbrace{\exp[x]}_{=\exp} = \exp$.

In other words, $\frac{d}{dx}\overline{\exp} = \exp$ (since $\overline{\exp} = \overline{\exp}[x]$).

On the other hand, (14) (applied to $w = \overline{\exp}$) shows that

$$\frac{d}{dx}\left(\overline{\log}\left[\overline{\exp}\right]\right) = \underbrace{\left(\frac{d}{dx}\overline{\exp}\right)}_{=\exp=\overline{\exp}+1=1+\overline{\exp}} \cdot \frac{1}{1+\overline{\exp}} = \left(1+\overline{\exp}\right) \cdot \frac{1}{1+\overline{\exp}} = 1 = \frac{d}{dx}x.$$

[4] *Proof.* It is clear that the FPS $1+x$ has constant term 1. Thus, it remain to prove that the FPS $\exp\left[\overline{\log}\right]$ has constant term 1.

Recall that the FPS $\overline{\exp}\left[\overline{\log}\right]$ has constant term 0. Hence, the FPS $\overline{\exp}\left[\overline{\log}\right] + 1$ has constant term $0 + 1 = 1$. In other words, the FPS $\exp\left[\overline{\log}\right]$ has constant term 1 (since $\exp\left[\overline{\log}\right] = \overline{\exp}\left[\overline{\log}\right] + 1$). Qed.

Since the two FPSs $\overline{\log}\left[\overline{\exp}\right]$ and x have the same constant term[5], we can thus apply Lemma 0.3 to $u = \overline{\log}\left[\overline{\exp}\right]$ and $v = x$. We thus conclude that $\overline{\log}\left[\overline{\exp}\right] = x$. The proof of Theorem 0.1 is thus complete. □

References

[Loehr11] Nicholas A. Loehr, *Bijective Combinatorics*, Chapman & Hall/CRC 2011.

[GriRei18] Darij Grinberg, Victor Reiner, *Hopf algebras in Combinatorics*, version of 11 May 2018, `arXiv:1409.8356v5`.
See also `http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf` for a version that gets updated.

[5]This is because the FPS $\overline{\log}\left[\overline{\exp}\right]$ has constant term 0, and the FPS x also has constant term 0.