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Fall 2017, Math 4990, Homework Set 9

Exercise 1

1.1 Exercise 1

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{-2}{k} = (-1)^n \left\lfloor \frac{n+2}{2} \right\rfloor.$$

1.2 Solution

We will do this with induction over n.

Base case: When $n = 0$, we have

$$\sum_{k=0}^{0} \binom{-2}{k} = \binom{-2}{0} = 1 = 1 \cdot 1 = (-1)^0 \left\lfloor \frac{0+2}{2} \right\rfloor.$$

Inductive step: We assume (as the induction hypothesis) that

$$\sum_{k=0}^{n} \binom{-2}{k} = (-1)^n \left\lfloor \frac{n+2}{2} \right\rfloor,$$

and wish to show that

$$\sum_{k=0}^{n+1} \binom{-2}{k} = (-1)^{n+1} \left\lfloor \frac{(n+1)+2}{2} \right\rfloor.$$

The upper negation identity for binomial coefficients yields

$$\binom{-2}{n+1} = (-1)^{n+1} \binom{n+1-(-2)-1}{n+1} = (-1)^{n+1} \binom{n+2}{n+1} = (-1)^{n+1} \binom{n+2}{1} = (-1)^{n+1}(n+2) \tag{1}$$

(where we used the symmetry of Pascal's triangle to get $\binom{n+2}{n+1} = \binom{n+2}{1}$).

Now,

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{-2}{k} &= \sum_{k=0}^{n} \binom{-2}{k} + \binom{-2}{n+1} \\
&= (-1)^n \left\lfloor \frac{n+2}{2} \right\rfloor + \binom{-2}{n+1} \qquad \text{(by the induction hypothesis)} \\
&= (-1)^n \left\lfloor \frac{n+2}{2} \right\rfloor + (-1)^{n+1}(n+2) \qquad (2)
\end{aligned}$$

(by (1)).

We have two cases to consider: when n is odd, and when n is even. Consider the case when n is even. Thus, the equality (2) becomes

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{-2}{k} &= (-1)^n \left\lfloor \frac{n+2}{2} \right\rfloor + (-1)^{n+1}(n+2) = 1 \cdot \left(\frac{n}{2} + 1\right) + (-1)(n+2) \qquad \text{(since } n \text{ is even)} \\
&= -\frac{n+2}{2}. \qquad (3)
\end{aligned}$$

Since $n+1$ is odd, $\left\lfloor \frac{(n+1)+2}{2} \right\rfloor = \frac{n+2}{2}$ and $(-1)^{n+1} = -1$. Thus, $(-1)^{n+1} \left\lfloor \frac{(n+1)+2}{2} \right\rfloor = -\frac{n+2}{2}$. Comparing this with (3), we find

$$\sum_{k=0}^{n+1} \binom{-2}{k} = (-1)^{n+1} \left\lfloor \frac{(n+1)+2}{2} \right\rfloor,$$

as desired.

On the other hand, consider the case when n is odd. Thus, the equality (2) becomes

$$\sum_{k=0}^{n+1} \binom{-2}{k} = (-1)^n \left\lfloor \frac{n+2}{2} \right\rfloor + (-1)^{n+1}(n+2) = (-1)\frac{n+1}{2} + 1(n+2) \qquad \text{(since } n \text{ is odd)}$$
$$= \frac{n+3}{2}. \tag{4}$$

Since $n+1$ is even, $(n+1)+2$ is also even and, hence, divisible by 2. So, $\left\lfloor \frac{(n+1)+2}{2} \right\rfloor = \frac{(n+1)+2}{2} = \frac{n+3}{2}$, and thus also $(-1)^{n+1} \left\lfloor \frac{(n+1)+2}{2} \right\rfloor = \frac{n+3}{2}$ (since $(-1)^{n+1} = 1$). Comparing this with (4), we find

$$\sum_{k=0}^{n+1} \binom{-2}{k} = (-1)^{n+1} \left\lfloor \frac{(n+1)+2}{2} \right\rfloor,$$

as desired.

Hence, regardless of the parity of n, we have

$$\sum_{k=0}^{n+1} \binom{-2}{k} = (-1)^{n+1} \left\lfloor \frac{(n+1)+2}{2} \right\rfloor.$$

This completes the induction step, and so solves the exercise.

Exercise 3

2.1 Exercise 3

Let $G = (V, E, \varphi)$ be a connected multigraph. Let $v \in V$ be any vertex.

(a) Pick any $w \in V$ such that $d(v, w)$ is maximum (among all $w \in V$). Prove that w is a non-cut vertex of G.

(b) Let $n = |V|$. Prove that $\sum_{u \in V} d(v, u) \leq \binom{n}{2}$.

2.2 Solution

(a) Note: we'll call a vertex v "cut" if v is not non-cut (that is, if $G \setminus v$ is not connected and has at least one vertex).

We have $w \in V$ and thus $V \neq \varnothing$. Hence, $|V| > 0$.

In the case where V contains only one vertex v, the vertex w is indeed non-cut because the multigraph $G \setminus w$ has no vertices. So WLOG, we'll also assume that V has more than one vertex.

Now we will prove the following proposition:

Proposition 1

Let a be a cut vertex in G. Then there exists a vertex b in $G \setminus a$ such that there is no walk $v \to b$ in $G \setminus a$.

Proof of Proposition 1. Assume the contrary. Then, for each vertex $p \in V \setminus a$, there is a walk[1] $v \to p$. This also tells us that for all p, there is a walk $p \to v$ (such a walk can be created by listing the entries in the walk $v \to p$

[1] In this proof of Proposition 1, "walk" means "walk in $G \setminus a$".

in reverse order). Now fix two vertices $u, w \in V \setminus a$. We can find a walk from u to w by taking the walk $u \to v$ and appending to it all entries but the first of the walk $v \to w$ (we omit the first entry, v, because the walk $u \to v$ already ends with v). Now forget we fixed u and w. We've shown that there is a walk $u \to w$ for all $u, w \in V \setminus a$. Additionally, since we have assumed that $|V| > 1$, the multigraph $G \setminus a$ has at least one vertex. But this tells us that $G \setminus a$ is connected, so a is non-cut. Since we assumed that a was a cut vertex, we have a contradiction. □

We will now assume that **(a)** is false – that is, that there exists a cut vertex $w \in V$ such that $d(v,w)$ is maximum – and show that we end up with a contradiction. By Proposition 1, there is a vertex b of $G \setminus w$ for which there is no walk $v \to b$ in $G \setminus w$. Consider this b. Notice that $b \neq w$ (since b is a vertex in $G \setminus w$).

But since G is connected, there is a path $v \to b$ in G having length $d(v,b)$. Let p be such a path. Then, p cannot be a path in $G \setminus w$ (since there is no walk $v \to b$ in $G \setminus w$); thus, it must contain the vertex w (since the only vertices and edges removed from G to form $G \setminus w$ were w and edges containing w). Therefore, p contains a path q from v to w. The length of q must of course be $\geq d(v,w)$ (since $d(v,w)$ is the smallest length of a path from v to w). But the path p cannot end at w (since p ends at b, but $b \neq w$). Thus, q is not the whole path p. Hence, $(\text{the length of } p) > (\text{the length of } q)$.

Thus,

$$d(v,b) = (\text{the length of } p) > (\text{the length of } q) \geq d(v,w).$$

This contradicts our assumption that $d(v,w)$ was maximum. Hence, part **(a)** is solved.

(b) When $n = 0$, we have $\sum_{u\in V} d(v,u) = (\text{empty sum}) = 0 = \binom{0}{2}$. So without loss of generality, we will assume $n > 0$.

We will now use induction over n.

Base case: When $n = 1$, there is only one vertex in V, namely v. The shortest path from v to itself has no edges (the path is (v)), so $d(v,v) = 0$. Thus, $\sum_{u\in V} d(v,u) = d(v,v) = 0 = \binom{1}{2}$.

Inductive step: Assume that $\sum_{u\in V} d(v,u) \leq \binom{n}{2}$ for each n-vertex connected multigraph G and each vertex v of G. We wish to show that $\sum_{u\in V'} d(a,u) \leq \binom{n+1}{2}$ for each $n+1$-vertex connected multigraph $G' = (V', E', \varphi')$ and each vertex a of G'.

Fix some such G' and a. Find a vertex b in V' for which $d(a,b)$ is maximum. By part **(a)** of this exercise, the vertex b will be non-cut. Since $|V'| = n+1 > n \geq 1$, the graph $G' \setminus b$ has at least one vertex and hence is connected (since b is non-cut). If $d(a,b)$ would be 0, then we would have $d(a,w) = 0$ for **each** vertex $w \in V'$ (due to our choice of b); but this would yield that $V' = \{a\}$, which would contradict $|V'| > 1$. Hence, $d(a,b)$ cannot be 0. Thus, $d(a,b) \neq 0$, so that $a \neq b$. Hence, $a \in V' \setminus \{b\}$.

If p and q are two vertices of G', then we will use the notation $d'(p,q)$ to denote the length of the shortest path $p \to q$ in G'. If p and q are two vertices of $G \setminus b$, then we will use the notation $d(p,q)$ to denote the length of the shortest path $p \to q$ in $G' \setminus b$. (This is well-defined since $G' \setminus b$ is connected.) Note that $d(p,q)$ may be distinct from $d'(p,q)$. But since all edges of $G' \setminus b$ are also edges of G', for all vertices p, q of $G' \setminus b$, any path $p \to q$ of length $d(p,q)$ in $G' \setminus b$ is also a path in G'. Thus, for all vertices p and q of $G' \setminus b$, we have $d'(p,q) \leq d(p,q)$.

The multigraph $G' \setminus b$ is connected and has n vertices. Hence, by our induction hypothesis,

$$\sum_{u\in V'\setminus\{b\}} d(v,u) \leq \binom{n}{2} \tag{5}$$

for all vertices v in $V' \setminus \{b\}$.

Recall that for all vertices p and q of $G' \setminus b$, we have $d'(p,q) \leq d(p,q)$. Hence, for all vertices u of $G' \setminus b$, we have $d'(a,u) \leq d(a,u)$. This tells us

$$\sum_{u\in V'\setminus\{b\}} d'(a,u) \leq \sum_{u\in V'\setminus\{b\}} d(a,u) \leq \binom{n}{2} \tag{6}$$

(by (5)). Now, we can write

$$\sum_{u\in V'} d'(a,u) = \sum_{u\in V'\setminus\{b\}} d'(a,u) + d'(a,b) \leq \binom{n}{2} + d'(a,b)$$

(by (6)).

What upper bound can we put on $d'(a,b)$? Since all vertices in a path need to be distinct, each of the $n+1$ vertices of V' appears at most once in the path $a \to b$ in G'. A maximum-possible-length path in any $n+1$-vertex graph, where each vertex appears once, would have $(n+1)-1 = n$ edges, since exactly one edge appears in between each pair of consecutive vertices in the path. So $d'(a,b) \leq n$. This now gives us

$$\begin{aligned}\sum_{u\in V'} d'(a,u) &\leq \binom{n}{2} + d'(a,b)\\ &\leq \binom{n}{2} + n\\ &= \binom{n}{2} + \binom{n}{1}\\ &= \binom{n+1}{2} \text{ by Pascal's identity.}\end{aligned}$$

Now forget we fixed G' and a. We've shown that $\sum\limits_{u\in V'} d'(a,u) \leq \binom{n+1}{2}$ for each $n+1$-vertex connected multigraph $G' = (V', E', \varphi')$ and each vertex a of G'. This completes the induction step. Thus, part **(b)** is solved.