

Fall 2017, Math 4990, Homework Set 8

November 28, 2017

Exercise 2

1.1 Exercise 2

Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $p = \min\{m, n\}$.

(a) Prove that

$$\sum_{k=-p}^p (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k} = \binom{m+n}{m}.$$

(b) Prove that

$$T(m, n) = \sum_{k=-p}^p (-1)^k \binom{2m}{m+k} \binom{2n}{n-k}.$$

Here, $T(m, n)$ is defined as in Exercise 1 (that is, by $T(m, n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}$).

1.2 Solution

The symmetry of Pascal's triangle shows that $\binom{m+n}{m} = \binom{m+n}{n}$. Thus, the statement of the exercise is symmetric in m and n . We will therefore assume, without loss of generality, that $m \leq n$. Thus, $p = m$.

(a) From $p = m$, we obtain

$$\sum_{k=-p}^p (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k} = \sum_{k=-m}^m (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k}.$$

Substituting $k - m$ for k in the sum gives us

$$\begin{aligned} \sum_{k=-p}^p (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k} &= \sum_{k=-m}^m (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k} \\ &= \sum_{k=0}^{2m} (-1)^{k-m} \binom{m+n}{m+(k-m)} \binom{m+n}{n+(k-m)} \\ &= (-1)^{-m} \sum_{k=0}^{2m} (-1)^k \binom{m+n}{k} \binom{m+n}{n+k-m} \\ &= (-1)^{-m} \sum_{k=0}^{2m} (-1)^k \binom{m+n}{k} \binom{m+n}{m+n-(n+k-m)} \\ &= (-1)^{-m} \sum_{k=0}^{2m} (-1)^k \binom{m+n}{k} \binom{m+n}{2m-k}. \end{aligned}$$

Recall the following from class (see [Grinbe16, Exercise 3.22] for a proof):

Proposition.

Let $v \in \mathbb{Z}$ and $u \in \mathbb{N}$. Then,

$$\sum_{a=0}^u (-1)^a \binom{v}{a} \binom{v}{u-a} = \begin{cases} (-1)^{u/2} \binom{v}{u/2} & \text{if } u \text{ is even} \\ 0 & \text{if } u \text{ is odd} \end{cases}. \quad (1)$$

Since $m \in \mathbb{N}$, the integer $2m$ is even. Using (1) with $u = 2m$, $a = k$, and $v = m + n$ thus gives us

$$\sum_{k=0}^{2m} (-1)^k \binom{m+n}{k} \binom{m+n}{2m-k} = (-1)^{2m/2} \binom{m+n}{2m/2} = (-1)^m \binom{m+n}{m}.$$

So our above computation becomes

$$\begin{aligned} \sum_{k=-p}^p (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k} &= (-1)^{-m} \sum_{k=0}^{2m} (-1)^k \binom{m+n}{k} \binom{m+n}{2m-k} \\ &= (-1)^{-m} (-1)^m \binom{m+n}{m} \\ &= \binom{m+n}{m}. \end{aligned}$$

(b) Since $m, n \in \mathbb{N}$, we have $0 \leq m \leq n$, and $m \leq m + n$. So, we can write

$$\binom{m+n}{m} = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m+n)!}{m!n!}.$$

Now, we observe that

$$\begin{aligned} T(m, n) &= \frac{(2m)!(2n)!}{m!n!(m+n)!} = \frac{(2m)!(2n)!(m+n)!}{(m+n)!(m+n)!m!n!} = \frac{(2m)!(2n)!}{((m+n)!)^2} \binom{m+n}{m} \\ &= \frac{(2m)!(2n)!}{((m+n)!)^2} \sum_{k=-p}^p (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k} \end{aligned}$$

because of part (a).

Since we assumed $m \leq n$, we know $p = m$ and so each k in the above sum satisfies $-m \leq k \leq m \leq n$, so we have $0 \leq m+k \leq 2m \leq m+n$ and $0 \leq n-m \leq n+k \leq m+n$. This allows us to write

$$\binom{m+n}{m+k} = \frac{(m+n)!}{(m+k)!(m+n-(m+k))!} = \frac{(m+n)!}{(m+k)!(n-k)!}$$

and

$$\binom{m+n}{n+k} = \frac{(m+n)!}{(n+k)!(m+n-(n+k))!} = \frac{(m+n)!}{(n+k)!(m-k)!}.$$

Returning to $T(m, n)$, we now have

$$\begin{aligned}
 T(m, n) &= \frac{(2m)!(2n)!}{((m+n)!)^2} \sum_{k=-p}^p (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k} \\
 &= \frac{(2m)!(2n)!}{((m+n)!)^2} \sum_{k=-p}^p (-1)^k \frac{(m+n)!}{(m+k)!(n-k)!} \cdot \frac{(m+n)!}{(n+k)!(m-k)!} \\
 &= \sum_{k=-p}^p (-1)^k \frac{(2m)!(2n)!}{(m+k)!(n-k)!(n+k)!(m-k)!} \\
 &= \sum_{k=-p}^p (-1)^k \frac{(2m)!}{(m+k)!(m-k)!} \frac{(2n)!}{(n-k)!(n+k)!}.
 \end{aligned}$$

Now, because $-m \leq k \leq m$, we have $0 \leq m+k \leq 2m$, and we can write

$$\binom{2m}{m+k} = \frac{(2m)!}{(m+k)!(2m-(m+k))!} = \frac{(2m)!}{(m+k)!(m-k)!}.$$

Likewise, since $0 \leq m \leq n$ and $-m \leq k \leq m$, we know $2n \geq n+m \geq n-k \geq n-m \geq 0$, and we have

$$\binom{2n}{n-k} = \frac{(2n)!}{(n-k)!(2n-(n-k))!} = \frac{(2n)!}{(n-k)!(n+k)!}.$$

So,

$$\begin{aligned}
 T(m, n) &= \sum_{k=-p}^p (-1)^k \frac{(2m)!}{(m+k)!(m-k)!} \frac{(2n)!}{(n-k)!(n+k)!} \\
 &= \sum_{k=-p}^p (-1)^k \binom{2m}{m+k} \binom{2n}{n-k}.
 \end{aligned}$$

Exercise 3

2.1 Exercise 3

Let $p \in \mathbb{N}$ and $q \in \mathbb{N}$. Let $n = pq$. Consider the permutation $\sigma \in S_n$ that maps $(i-1)q + j$ to $(j-1)p + i$ for every $i \in [p]$ and $j \in [q]$.

(This permutation σ can be visualized as follows: Fill in a $p \times q$ -matrix A with the entries $1, 2, \dots, n$ by going row by row from top to bottom:

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & q \\ q+1 & q+2 & q+3 & \cdots & 2q \\ 2q+1 & 2q+2 & 2q+3 & \cdots & 3q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)q+1 & (p-1)q+2 & (p-1)q+3 & \cdots & pq \end{pmatrix}.$$

Fill in a $p \times q$ -matrix B with the entries $1, 2, \dots, n$ by going column by column from left to right:

$$B = \begin{pmatrix} 1 & p+1 & 2p+1 & \cdots & (q-1)p+1 \\ 2 & p+2 & 2p+2 & \cdots & (q-1)p+2 \\ 3 & p+3 & 2p+3 & \cdots & (q-1)p+3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p & 2p & 3p & \cdots & qp \end{pmatrix}.$$

The permutation σ then sends each entry of A to the corresponding entry of B .)

Find the length $\ell(\sigma)$ of the permutation σ .

2.2 Solution

Fix an (i, j) . Let us find the number of elements $x \in [pq]$ that are greater than $(i-1)q + j$ but have images $\sigma(x) < \sigma((i-1)q + j) = (j-1)p + i$.

Using the matrix visualization of σ , we note that the entries of A that are greater than the entry at position (i, j) are all entries (l, m) in row i where $m > j$, as well all the entries of the matrix at position (l, m) where the row number $l > i$. (When $(i, j) = (2, 2)$, these entries, and their corresponding entries of B , are the bolded ones in the following:

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & q \\ q+1 & q+2 & \mathbf{q+3} & \cdots & \mathbf{2q} \\ \mathbf{2q+1} & \mathbf{2q+2} & \mathbf{2q+3} & \cdots & \mathbf{3q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)q+1 & (p-1)q+2 & (p-1)q+3 & \cdots & pq \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & p+1 & 2p+1 & \cdots & (q-1)p+1 \\ 2 & p+2 & \mathbf{2p+2} & \cdots & \mathbf{(q-1)p+2} \\ \mathbf{3} & \mathbf{p+3} & \mathbf{2p+3} & \cdots & \mathbf{(q-1)p+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{p} & \mathbf{2p} & \mathbf{3p} & \cdots & \mathbf{qp} \end{pmatrix}.)$$

Meanwhile, the elements of $[pq]$ that are less than the image $\sigma((i-1)q + j)$ (this is the image of the entry of A at position (i, j)) are the entries of B at positions (l, m) in column j where $l < i$, and all entries of B in positions (l, m) where $m < j$.

(These entries of B for $(i, j) = (2, 2)$ are marked below in bold along with their corresponding entries of A :

$$B = \begin{pmatrix} \mathbf{1} & \mathbf{p+1} & 2p+1 & \cdots & (q-1)p+1 \\ \mathbf{2} & p+2 & 2p+2 & \cdots & (q-1)p+2 \\ \mathbf{3} & p+3 & 2p+3 & \cdots & (q-1)p+3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{p} & 2p & 3p & \cdots & qp \end{pmatrix}$$

$$A = \begin{pmatrix} \mathbf{1} & 2 & 3 & \cdots & q \\ \mathbf{q+1} & q+2 & q+3 & \cdots & 2q \\ \mathbf{2q+1} & \mathbf{2q+2} & \mathbf{2q+3} & \cdots & \mathbf{3q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)q+1 & (p-1)q+2 & (p-1)q+3 & \cdots & pq \end{pmatrix}.)$$

So we wish to find all pairs (l, m) satisfying the following criteria:

1. $l \geq i$;
2. if $l = i$, then $m > j$;
3. $m \leq j$;
4. if $m = j$, then $l < i$.

Since we must have $m \leq j$, we cannot have $l = i$, as this would require that $m > j$. Similarly, we cannot have $m = j$, as this would require that $l < i$, whereas we need $l \geq i$. So the valid pairs (l, m) are pairs where $l > i$ and $m < j$. Hence, there are $p-i$ valid l 's and $j-1$ m 's, and so we have a total of $(p-i)(j-1)$ inversions of the form $((i-1)q + j, x)$ where $\sigma(x) < \sigma((i-1)q + j) = (j-1)p + i$ for some $x > (i-1)q + j$ in $[pq]$. Visually,

this number counts the x 's in the $(p-i) \times (j-1)$ rectangle of entries located below and to the left of $(i-1)q+j$ in A . (When $(i, j) = (2, 2)$, this “rectangle” is

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & q \\ q+1 & q+2 & q+3 & \cdots & 2q \\ 2q+1 & 2q+2 & 2q+3 & \cdots & 3q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)q+1 & (p-1)q+2 & (p-1)q+3 & \cdots & pq \end{pmatrix}.$$

Summing $(p-i)(j-1)$ over all (i, j) gives us

$$\begin{aligned} \sum_{i \in [p]} \sum_{j \in [q]} (p-i)(j-1) &= \left(\sum_{i \in [p]} (p-i) \right) \left(\sum_{j \in [q]} (j-1) \right) = \left(\sum_{i \in [p]} (i-1) \right) \left(\sum_{j \in [q]} (j-1) \right) \\ &\quad \text{(here, we have substituted } p+1-i \text{ for } i \text{ in the first sum)} \\ &= \frac{p(p-1)}{2} \cdot \frac{q(q-1)}{2} \end{aligned}$$

(since $\sum_{i \in [p]} (i-1) = \left(\sum_{i \in [p]} i \right) - p = \frac{p(p+1)}{2} - p = \frac{p(p-1)}{2}$ and similarly $\sum_{j \in [q]} (j-1) = \frac{q(q-1)}{2}$). So this is the total number of inversions of σ , that is, the length of σ .

Exercise 4

3.1 Exercise 4

Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Prove that

$$\begin{aligned} &(\text{the number of all } (i, j) \in [n] \times [n] \text{ such that } i \geq j > \sigma(i)) \\ &= (\text{the number of all } (i, j) \in [n] \times [n] \text{ such that } \sigma(i) \geq j > i). \end{aligned}$$

3.2 Solution

Let's first look at the left hand side. Fix an $i \in [n]$ for which $i > \sigma(i)$. The number of $j \in [n]$ such that $i \geq j > \sigma(i)$ is the number of elements of $[n]$ in the interval $(\sigma(i), i] = \{\sigma(i)+1, \sigma(i)+2, \dots, i\}$. This is just $i - \sigma(i)$, since $[n]$ is a set of consecutive integers that contains $\sigma(i)$ and i . Now forget we fixed i . Our left hand side can be expressed as

$$(\text{the number of all } (i, j) \in [n] \times [n] \text{ such that } i \geq j > \sigma(i)) = \sum_{\substack{i \in [n]; \\ i > \sigma(i)}} (i - \sigma(i)). \quad (2)$$

We can do something similar for the right hand side. Fix an $i \in [n]$ for which $\sigma(i) > i$. The number of $j \in [n]$ that satisfy $\sigma(i) \geq j > i$ is $\sigma(i) - i$, the number of elements of $[n]$ in the interval $(i, \sigma(i)] = \{i+1, i+2, \dots, \sigma(i)\}$. Forget we fixed i . By summing $\sigma(i) - i$ for all $i \in [n]$ such that $\sigma(i) > i$, we see that the right hand side is

$$(\text{the number of all } (i, j) \in [n] \times [n] \text{ such that } \sigma(i) \geq j > i) = \sum_{\substack{i \in [n]; \\ i < \sigma(i)}} (\sigma(i) - i). \quad (3)$$

Now let's consider the following sum:

$$\sum_{i \in [n]} (\sigma(i) - i).$$

On one hand, we have

$$\sum_{i \in [n]} (\sigma(i) - i) = \sum_{i \in [n]} \sigma(i) - \sum_{i \in [n]} i.$$

Since σ is a permutation of $[n]$, the sum $\sum_{i \in [n]} \sigma(i)$ is just the sum of all the elements of $[n]$, which is the same as $\sum_{i \in [n]} i$. So our expression becomes

$$\begin{aligned} \sum_{i \in [n]} (\sigma(i) - i) &= \sum_{i \in [n]} \sigma(i) - \sum_{i \in [n]} i \\ &= \sum_{i \in [n]} i - \sum_{i \in [n]} i \\ &= 0. \end{aligned}$$

On the other hand, we can rewrite the sum as

$$\begin{aligned} \sum_{i \in [n]} (\sigma(i) - i) &= \sum_{\substack{i \in [n]; \\ i < \sigma(i)}} (\sigma(i) - i) + \sum_{\substack{i \in [n]; \\ i > \sigma(i)}} (\sigma(i) - i) + \underbrace{\sum_{\substack{i \in [n]; \\ i = \sigma(i)}} (\sigma(i) - i)}_{=0, \text{ since here } i = \sigma(i)} \\ &= \sum_{\substack{i \in [n]; \\ i < \sigma(i)}} (\sigma(i) - i) + \sum_{\substack{i \in [n]; \\ i > \sigma(i)}} (\sigma(i) - i) \\ &= \sum_{\substack{i \in [n]; \\ i < \sigma(i)}} (\sigma(i) - i) - \sum_{\substack{i \in [n]; \\ i > \sigma(i)}} (i - \sigma(i)). \end{aligned}$$

We now have

$$0 = \sum_{i \in [n]} (\sigma(i) - i) = \sum_{\substack{i \in [n]; \\ i < \sigma(i)}} (\sigma(i) - i) - \sum_{\substack{i \in [n]; \\ i > \sigma(i)}} (i - \sigma(i)),$$

so that

$$\sum_{\substack{i \in [n]; \\ i > \sigma(i)}} (i - \sigma(i)) = \sum_{\substack{i \in [n]; \\ i < \sigma(i)}} (\sigma(i) - i).$$

In view of (2) and (3), this rewrites as

$$\begin{aligned} &(\text{the number of all } (i, j) \in [n] \times [n] \text{ such that } i \geq j > \sigma(i)) \\ &= (\text{the number of all } (i, j) \in [n] \times [n] \text{ such that } \sigma(i) \geq j > i), \end{aligned}$$

as desired.

Exercise 7

4.1 Exercise 7

Let $n \in \mathbb{N}$ and $p \in \mathbb{N}$. Let a be the number of all partitions (i_1, i_2, \dots, i_k) of n satisfying $k \geq p$ and $i_1 = i_2 = \dots = i_p$. Let b be the number of all nonempty partitions (i_1, i_2, \dots, i_k) of n such that all of i_1, i_2, \dots, i_k are $\geq p$. Prove that $a = b$.

4.2 Solution

We will let A denote the set of partitions (i_1, i_2, \dots, i_k) of n satisfying $k \geq p$ and $i_1 = i_2 = \dots = i_p$. Thus, $a = |A|$.

We will let B denote the set of partitions (i_1, i_2, \dots, i_k) of n such that all of i_1, i_2, \dots, i_k are $\geq p$. Thus, $b = |B|$.

Each partition in A or B can be represented as a Young diagram. The partitions in A are in a bijection with Young diagrams with n squares that have $\geq p$ rows, the first p of which are of equal length. The partitions in B are in a bijection with Young diagrams with n squares in which all rows contain $\geq p$ squares. For both A and B , the mappings are as follows:

1. The mapping from the set of partitions to the respective set of Young diagrams sends each partition to the Young diagram in which the number of squares in the i -th row is the i -th part of the partition.

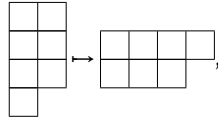
2. The inverse of this mapping creates a tuple in which the i -th entry is the number of squares in the i -th row of the Young diagram.

Since the composition of bijections is itself a bijection, we can show that $a = |A| = |B| = b$ by finding a bijection between the Young diagram representations of partitions in A and the Young diagram representations of partitions in B . If we call this desired mapping between Young diagrams f , the bijection from A to its Young diagrams g , and the bijection from the Young diagram representations of B to B itself h , then a bijection from A to B would be $h \circ f \circ g$.

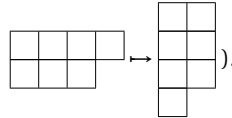
Consider the following maps (the *conjugate* of a Young diagram is formed by interchanging the rows and columns of the original):

1. {Young diagrams representing elements of A } \rightarrow {Young diagrams representing elements of B }, maps a diagram to its conjugate diagram.
2. {Young diagrams representing elements of B } \rightarrow {Young diagrams representing elements of A }, maps a diagram to its conjugate diagram.

(For example, if $n = 7$ and $p = 2$, the first map sends



while the second map sends



It is clear that, if well-defined, these two maps are inverses, but we still need to show that they map the elements of their domains to their intended codomains.

The domain of the first map consists of Young diagrams with n squares that have $\geq p$ rows, the first p of which are of equal length. Fix a diagram in this domain. We'll call the length of its first row m . Since the first p rows are of equal length, m , each of the m columns contains a square from each of the first p rows. This tells us that the length of each column is $\geq p$. Now we know that the conjugate of this diagram has m rows, each of which is of length $\geq p$. Forget we fixed our diagram. We've shown that the image of each element of the domain is indeed a Young diagram representation of a partition in B .

The domain of the second map consists of Young diagrams with n squares in which all rows contain $\geq p$ squares. Fix such a diagram. Since each row contains $\geq p$ squares, we have $\geq p$ columns. Now, we would like to show that the first p columns all have equal length. Indeed, assume the contrary. Let i (with $2 \leq i \leq p$) be the first column whose length differs from the length of the previous column. Since the squares in a Young diagram are left-aligned, the length of the i -th column is smaller than that of its preceding column (otherwise, we would not have a Young diagram, since this would mean that there exists some row in which there is a gap followed by a square). This tells us that there exists a row that is missing its i -th square and, hence, has its last square at index $i - 1 < p$. This contradicts our assumption that each row has $\geq p$ squares. So, the first p columns must all have equal length. Since we have $\geq p$ columns, the first p of which have equal length, the conjugate diagram is a diagram with $\geq p$ rows, the first p of which are of the same length. Forget that we fixed our diagram. We've shown that all elements of our domain are mapped to Young diagram representations of partitions in A .

Since our maps are well-defined and mutual inverses, they are bijections, and we can now conclude that $a = |A| = |B| = b$.

References

- [Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
<http://www.cip.ifi.lmu.de/~grinberg/primess2015/sols.pdf>

The numbering of theorems and formulas in this link might shift when the project gets updated; for a “frozen” version whose numbering is guaranteed to match that in the citations above, see <https://github.com/darijgr/detnotes/releases/tag/2019-01-10> .