Math 4990 Fall 2017 (Darij Grinberg): homework set 8 [corrected 17 Dec 2017]
due date: Tuesday 28 Nov 2017 at the beginning of class, or before that by email or moodle
Please solve **at most 4** of the 7 exercises!

0.1. Strange integers

Exercise 1. For any $m \in \mathbb{N}$ and $n \in \mathbb{N}$, define a rational number $T(m,n)$ by

$$T(m,n) = \frac{(2m)!\,(2n)!}{m!n!\,(m+n)!}.$$

(a) Prove that $4T(m,n) = T(m+1,n) + T(m,n+1)$ for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

(b) Prove that $T(m,n) \in \mathbb{N}$ for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

(c) Prove that $T(m,n)$ is an **even** integer for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$ unless $(m,n) = (0,0)$.

(d) If $m \in \mathbb{N}$ and $n \in \mathbb{N}$ are such that $m+n$ is odd and $m+n > 1$, then prove that $4 \mid T(m,n)$.

[**Hint:** Don't be afraid to use induction. Part **(b)** suggests that the numbers $T(m,n)$ count something, but no one has so far discovered what; combinatorial proofs aren't always the easiest to find. For **(c)**, start by showing that $\dbinom{2g}{g}$ is even whenever g is a positive integer. For **(d)**, start by showing that $\dbinom{2g-1}{g-1}$ is even whenever $g > 1$ is odd.]

Exercise 2. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $p = \min\{m,n\}$.

(a) Prove that

$$\sum_{k=-p}^{p} (-1)^k \binom{m+n}{m+k} \binom{m+n}{n+k} = \binom{m+n}{m}.$$

(b) Prove that

$$T(m,n) = \sum_{k=-p}^{p} (-1)^k \binom{2m}{m+k} \binom{2n}{n-k},$$

where $T(m,n)$ is defined as in Exercise 1.

[**Hint:** Part **(a)** should follow from something done in class. Then, compare part **(b)** with part **(a)**.]

0.2. The length of a permutation

Definition 0.1. Let $n \in \mathbb{N}$.

(a) We let S_n denote the set of all permutations of $[n]$.

Let $\sigma \in S_n$ be a permutation of $[n]$.

(b) An *inversion* of σ means a pair (i, j) of elements of $[n]$ satisfying $i < j$ and $\sigma(i) > \sigma(j)$.

(c) The *length* of σ is defined to be the number of inversions of σ. This length is denoted by $\ell(\sigma)$.

(d) The *sign* of σ is defined to be the integer $(-1)^{\ell(\sigma)}$. It is denoted by $(-1)^{\sigma}$.

Exercise 3. Let $p \in \mathbb{N}$ and $q \in \mathbb{N}$. Let $n = pq$. Consider the permutation $\sigma \in S_n$ that maps $(i-1)q + j$ to $(j-1)p + i$ for every $i \in [p]$ and $j \in [q]$.

(This permutation σ can be visualized as follows: Fill in a $p \times q$-matrix A with the entries $1, 2, \ldots, n$ by going row by row from top to bottom:

$$A = \begin{pmatrix} 1 & 2 & 3 & \cdots & q \\ q+1 & q+2 & q+3 & \cdots & 2q \\ 2q+1 & 2q+2 & 2q+3 & \cdots & 3q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (p-1)q+1 & (p-1)q+2 & (p-1)q+3 & \cdots & pq \end{pmatrix}.$$

Fill in a $p \times q$-matrix B with the entries $1, 2, \ldots, n$ by going column by column from left to right:

$$B = \begin{pmatrix} 1 & p+1 & 2p+1 & \cdots & (q-1)p+1 \\ 2 & p+2 & 2p+2 & \cdots & (q-1)p+2 \\ 3 & p+3 & 2p+3 & \cdots & (q-1)p+3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p & 2p & 3p & \cdots & qp \end{pmatrix}.$$

The permutation σ then sends each entry of A to the corresponding entry of B.)

Find the length $\ell(\sigma)$ of the permutation σ.

0.3. Two equal counts

Exercise 4. Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Prove that

$$\begin{aligned} &(\text{the number of all } (i,j) \in [n] \times [n] \text{ such that } i \geq j > \sigma(i)) \\ &= (\text{the number of all } (i,j) \in [n] \times [n] \text{ such that } \sigma(i) \geq j > i). \end{aligned}$$

0.4. Lehmer codes

Recall the following definition from the preceding homework set:

Definition 0.2. Let $n \in \mathbb{N}$. Let $\sigma \in S_n$ be a permutation. For any $i \in [n]$, we let $\ell_i(\sigma)$ denote the number of $j \in \{i+1, i+2, \ldots, n\}$ such that $\sigma(i) > \sigma(j)$.

Exercise 5. Let $n \in \mathbb{N}$. Let G be the set of all n-tuples $(j_1, j_2, \ldots, j_n)$ of integers satisfying $0 \leq j_k \leq n-k$ for each $k \in [n]$. (In other words, $G = \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, n-2\} \times \cdots \times \{0, 1, \ldots, n-n\}$.)

(a) For any $\sigma \in S_n$ and $i \in [n]$, prove that $\sigma(i)$ is the $(\ell_i(\sigma) + 1)$-th smallest element of the set $[n] \setminus \{\sigma(1), \sigma(2), \ldots, \sigma(i-1)\}$.

(b) For any $\sigma \in S_n$, prove that

$$(\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma)) \in G.$$

(c) Prove that the map

$$S_n \to G,$$
$$\sigma \mapsto (\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma))$$

is bijective.

(d) Show that $\ell(\sigma) = \ell_1(\sigma) + \ell_2(\sigma) + \cdots + \ell_n(\sigma)$ for each $\sigma \in S_n$.

(e) Show that

$$\sum_{\sigma \in S_n} x^{\ell(\sigma)} = (1+x)\left(1+x+x^2\right)\cdots\left(1+x+x^2+\cdots+x^{n-1}\right)$$

(an equality between polynomials in x). (If $n \leq 1$, then the right hand side of this equality is an empty product, and thus equals 1.)

Note that the n-tuple $(\ell_1(\sigma), \ell_2(\sigma), \ldots, \ell_n(\sigma))$ is known as the *Lehmer code* of the permutation σ.

0.5. Permutations as composed transpositions

Recall a basic notation regarding permutations, which we shall now extend:

Definition 0.3. Let $n \in \mathbb{N}$. Let i and j be two distinct elements of $[n]$. We let $t_{i,j}$ be the permutation in S_n which switches i with j while leaving all other elements of $[n]$ unchanged. Such a permutation is called a *transposition*.

Let us furthermore set $t_{i,i} = \text{id}$ for each $i \in [n]$. Thus, $t_{i,j}$ is defined even when i and j are not distinct.

Exercise 6. Let $n \in \mathbb{N}$. Let $\sigma \in S_n$.

(a) Prove that there is a unique n-tuple $(i_1, i_2, \ldots, i_n) \in [1] \times [2] \times \cdots \times [n]$ such that

$$\sigma = t_{1,i_1} \circ t_{2,i_2} \circ \cdots \circ t_{n,i_n}.$$

(b) Consider this n-tuple $(i_1, i_2, \ldots, i_n)$. Define the relation $\sim$ and the $\sim$-equivalence classes $E_1, E_2, \ldots, E_m$ as in Exercise 7 on homework set #7 (for $X = [n]$). (Thus, m is the number of cycles in the cycle decomposition of σ.)

Prove that m is the number of all $k \in [n]$ satisfying $i_k = k$.

0.6. Another partition identity

Recall the following:

Definition 0.4. Let $n \in \mathbb{Z}$. A *partition* of n means a finite list $(i_1, i_2, \ldots, i_k)$ of positive integers satisfying

$$i_1 \geq i_2 \geq \cdots \geq i_k \qquad \text{and} \qquad i_1 + i_2 + \cdots + i_k = n.$$

Exercise 7. Let $n \in \mathbb{N}$ and $p \in \mathbb{N}$. Let a be the number of all partitions $(i_1, i_2, \ldots, i_k)$ of n satisfying $k \geq p$ and $i_1 = i_2 = \cdots = i_p$. Let b be the number of all nonempty partitions $(i_1, i_2, \ldots, i_k)$ of n such that all of $i_1, i_2, \ldots, i_k$ are $\geq p$. Prove that $a = b$.

Example 0.5. Let $n = 9$ and $p = 3$. Then, the partitions counted by a in Exercise 7 are

$$(3,3,3), \qquad (2,2,2,2,1), \qquad (2,2,2,1,1,1), \qquad (1,1,1,1,1,1,1,1,1).$$

Meanwhile, the partitions counted by b in Exercise 7 are

$$(9), \qquad (6,3), \qquad (5,4), \qquad (3,3,3).$$

Thus, $a = 4$ and $b = 4$ in this case.

Further reading on partitions includes:

- Herbert S. Wilf, *Lectures on Integer Partitions*, 2009.
  `https://www.math.upenn.edu/~wilf/PIMS/PIMSLectures.pdf`
- George E. Andrews, Kimmo Eriksson, *Integer Partitions*, Cambridge University Press 2004.
- Igor Pak, *Partition bijections, a survey*, Ramanujan Journal, vol. 12 (2006), pp. 5–75.
  `http://www.math.ucla.edu/~pak/papers/psurvey.pdf`

The Wikipedia articles on partitions, the pentagonal number theorem and Ramanujan's congruences are also useful. That said, none of these is necessary for the above exercise.