Math 4990 Fall 2017 (Darij Grinberg): homework set 6 with solutions

Contents

0.1. Reminders on binomial coefficients

Let us first recall some facts about binomial coefficients:

Proposition 0.1. For every $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$ and $n \in \mathbb{N}$, we have

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}.$$

Proposition 0.2. Let a and b be two integers such that $a \geq b \geq 0$. Then,

$$\binom{a}{b} = \frac{a!}{b!\,(a-b)!}.$$

Proposition 0.3. We have

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$$

for any $n \in \mathbb{Q}$ and $k \in \mathbb{N}$.

Proposition 0.4. We have $k\binom{n}{k} = n\binom{n-1}{k-1}$ for any $n \in \mathbb{Q}$ and any positive integer k.

Proposition 0.5. We have

$$\binom{m}{n} = 0$$

for every $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m < n$.

Proposition 0.6. If $n \in \mathbb{Q}$ and if a and b are two integers such that $a \geq b \geq 0$, then

$$\binom{n}{a}\binom{a}{b} = \binom{n}{b}\binom{n-b}{a-b}.$$

Proposition 0.7. We have $\binom{m}{m} = 1$ for every $m \in \mathbb{N}$.

Proposition 0.8. We have

$$\binom{m}{n} = \binom{m}{m-n} \qquad (1)$$

for any $m \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfying $m \geq n$.

Proposition 0.1 is the *Vandermonde convolution identity*, and is proven in multiple places[1]. Proposition 0.2 was proven in the solutions to homework set 1. Proposition 0.3 is Exercise 2 **(a)** in homework set 1. Proposition 0.4 is Exercise 2 **(b)** in homework set 1. Proposition 0.5 is fundamental and easy to prove. Proposition 0.6 is Exercise 2 **(c)** in homework set 1. Proposition 0.7 and Proposition 0.8 are easy to check.

0.2. Idempotent maps

If S is a set, then a map $f : S \to S$ is said to be *idempotent* if and only if $f \circ f = f$. For instance, the map $[3] \to [3]$ sending $1, 2, 3$ to $1, 3, 3$ (respectively) is idempotent.

Exercise 1. Let $n \in \mathbb{N}$.

(a) Prove that a map $f : [n] \to [n]$ is idempotent if and only if every $y \in f([n])$ satisfies $f(y) = y$.

(b) Prove that the number of idempotent maps $[n] \to [n]$ is $\sum\limits_{k=0}^{n} \binom{n}{k} k^{n-k}$.

(c) Prove that the number of idempotent maps $[n] \to [n]$ has the form $an + 1$ for some $a \in \mathbb{N}$. (Of course, a will depend on n.)

[**Hint:** When is $\binom{n}{k} k^{n-k}$ divisible by n ?]

Solution to Exercise 1 (sketched). **(a)** Let $f : [n] \to [n]$ be a map. We must prove that f is idempotent if and only if every $y \in f([n])$ satisfies $f(y) = y$.

$\Longrightarrow$: Assume that f is idempotent. We must prove that every $y \in f([n])$ satisfies $f(y) = y$.

We have assumed that f is idempotent. In other words, $f \circ f = f$. Now, let $y \in f([n])$. Thus, there exists some $x \in [n]$ satisfying $y = f(x)$. Consider this

[1] For an elementary proof, see, e.g., [Grinbe16, first proof of Theorem 3.29].

x. We have $f\left(\underbrace{y}_{=f(x)}\right) = f(f(x)) = \underbrace{(f \circ f)}_{=f}(x) = f(x) = y$. Now, forget that we fixed y. We thus have shown that every $y \in f([n])$ satisfies $f(y) = y$. This proves the "$\Longrightarrow$" direction of Exercise 1 **(a)**.

$\Longleftarrow$: Assume that every $y \in f([n])$ satisfies $f(y) = y$. We must prove that f is idempotent.

Let $x \in [n]$. Thus, $f(x) \in f([n])$. But we assumed that every $y \in f([n])$ satisfies $f(y) = y$. Applying this to $y = f(x)$, we obtain $f(f(x)) = f(x)$ (since $f(x) \in f([n])$). Hence, $(f \circ f)(x) = f(f(x)) = f(x)$. Now, forget that we fixed x. We thus have proven that $(f \circ f)(x) = f(x)$ for each $x \in [n]$. In other words, $f \circ f = f$. In other words, f is idempotent (by the definition of "idempotent"). This proves the "$\Longleftarrow$" direction of Exercise 1 **(a)**.

Hence, Exercise 1 **(a)** is solved (since we have proven both of its directions).

(b) The following algorithm constructs every idempotent map $f : [n] \to [n]$:

- First, we choose an integer $k \in \{0, 1, \ldots, n\}$. This integer k shall be the size $|f([n])|$ of the image of f. (Of course, this size has to be in $\{0, 1, \ldots, n\}$, because $f([n])$ must be a subset of $[n]$.)
- Next, we choose a k-element subset S of $[n]$. This subset S shall be the image $f([n])$ of f. There are $\dbinom{n}{k}$ choices for S (since the number of k-element subsets of $[n]$ is $\dbinom{n}{k}$).
- At this point, the values of f on all elements of S are already uniquely determined: Indeed, Exercise 1 **(a)** shows that every $y \in f([n])$ has to satisfy $f(y) = y$ for f to be idempotent; in other words, every $y \in S$ has to satisfy $f(y) = y$ (since we want $f([n])$ to be S).
- Finally, we choose the values of f on all remaining elements of $[n]$ (that is, on all elements of $[n] \setminus S$). These values must belong to S (because we want $f([n])$ to be S), but are otherwise unconstrained[2]. Thus, there are $|S|^{|[n] \setminus S|}$ choices at this step. In other words, there are k^{n-k} choices at this step (since $|S| = k$ and $|[n] \setminus S| = n - \underbrace{|S|}_{=k} = n - k$).

It is easy to check that this algorithm really constructs idempotent maps $f : [n] \to [n]$, and constructs each of them exactly once. Thus, the number of idempotent

[2] At this step, we do not need to ensure that every element of S is taken as a value of f, because this has already been ensured (indeed, every $y \in S$ satisfies $f(y) = y$, so that y is already a value of f).

maps $f : [n] \to [n]$ is

$$\sum_{k\in\{0,1,\ldots,n\}} \binom{n}{k} k^{n-k}$$

(since we get to choose $k \in \{0, 1, \ldots, n\}$ in the first step of the algorithm, then we have $\binom{n}{k}$ choices in the second step, then a unique choice in the third step, and finally k^{n-k} choices in the fourth step). Hence, the number of idempotent maps $f : [n] \to [n]$ is

$$\sum_{k\in\{0,1,\ldots,n\}} \binom{n}{k} k^{n-k} = \sum_{k=0}^{n} \binom{n}{k} k^{n-k}.$$

This solves Exercise 1 **(b)**.

(c) If $n = 0$, then the number of idempotent maps $[n] \to [n]$ is 1, which clearly has the form $an + 1$ for some $a \in \mathbb{N}$ (namely, for $a = 0$). Hence, for the rest of this solution, we WLOG assume that we don't have $n = 0$. Thus, $n > 0$, so that $n \geq 1$.

For each $k \in \{1, 2, \ldots, n-1\}$, the number $\binom{n-1}{k-1} k^{n-k-1}$ is a nonnegative integer[3]. Hence, $\sum_{k=1}^{n-1} \binom{n-1}{k-1} k^{n-k-1}$ is a sum of nonnegative integers, thus itself a nonnegative integer. In other words,

$$\sum_{k=1}^{n-1} \binom{n-1}{k-1} k^{n-k-1} \in \mathbb{N}.$$

[3] *Proof.* Let $k \in \{1, 2, \ldots, n-1\}$. Thus, $1 \leq k \leq n-1$. From $1 \leq k$, we obtain $k - 1 \in \mathbb{N}$. Also, $n - 1 \in \mathbb{N}$ (since $n \geq 1$). Hence, $\binom{n-1}{k-1}$ is the number of $(k-1)$-element subsets of the set $[n-1]$ (by the combinatorial interpretation of binomial coefficients). Therefore, $\binom{n-1}{k-1}$ is a nonnegative integer. Also, $n - \underbrace{k}_{\leq n-1} - 1 \geq n - (n-1) - 1 = 0$, so that $n - k - 1 \in \mathbb{N}$. Thus, $k^{n-k-1} \in \mathbb{N}$ (since $k \in \{1, 2, \ldots, n-1\} \subseteq \mathbb{N}$). In other words, k^{n-k-1} is a nonnegative integer. Hence, the number $\binom{n-1}{k-1} k^{n-k-1}$ is a product of two nonnegative integers, hence itself a nonnegative integer. Qed.

Exercise 1 **(b)** shows that the number of idempotent maps $[n] \to [n]$ is

$$\sum_{k=0}^{n} \binom{n}{k} k^{n-k} = \sum_{k=0}^{n-1} \binom{n}{k} \underbrace{k^{n-k}}_{\substack{=kk^{n-k-1} \\ \text{(since } n-k>0 \\ \text{(since } k\leq n-1<n))}} + \underbrace{\binom{n}{n}}_{\substack{=1 \\ \text{(by Proposition 0.7,} \\ \text{applied to } m=n)}} \underbrace{n^{n-n}}_{=n^0=1}$$

$$= \underbrace{\sum_{k=0}^{n-1} \binom{n}{k} kk^{n-k-1}}_{\substack{=\binom{n}{0} 0\cdot 0^{n-0-1} + \sum_{k=1}^{n-1} \binom{n}{k} kk^{n-k-1} \\ \text{(since } n\geq 1)}} +1$$

$$= \underbrace{\binom{n}{0} 0 \cdot 0^{n-0-1}}_{=0} + \sum_{k=1}^{n-1} \underbrace{\binom{n}{k} k}_{\substack{=k\binom{n}{k} = n\binom{n-1}{k-1} \\ \text{(by Proposition 0.4)}}} k^{n-k-1} + 1$$

$$= \underbrace{\sum_{k=1}^{n-1} n \binom{n-1}{k-1} k^{n-k-1}}_{=\left(\sum_{k=1}^{n-1} \binom{n-1}{k-1} k^{n-k-1}\right) n} +1 = \left(\sum_{k=1}^{n-1} \binom{n-1}{k-1} k^{n-k-1} \right) n + 1.$$

Hence, this number has the form $an + 1$ for some $a \in \mathbb{N}$ (namely, for $a = \sum_{k=1}^{n-1} \binom{n-1}{k-1} k^{n-k-1}$). This solves Exercise 1 **(c)**. □

0.3. Fixed points

Exercise 2. Let S be a finite set. For any map $f : S \to S$, we let $\operatorname{Fix} f$ denote the set of all fixed points of f. (That is, $\operatorname{Fix} f = \{s \in S \mid f(s) = s\}$.)

(a) Prove that $|\operatorname{Fix}(f \circ g)| = |\operatorname{Fix}(g \circ f)|$ for any two maps $f : S \to S$ and $g : S \to S$.

(b) Is it true that every three maps f, g, h from S to S satisfy $|\operatorname{Fix}(f \circ g \circ h)| = |\operatorname{Fix}(g \circ f \circ h)|$?

[**Hint:** For **(a)**, find a bijection.]

Solution to Exercise 2. **(a)** Let $f : S \to S$ and $g : S \to S$ be two maps. Then, $g(x) \in \operatorname{Fix}(g \circ f)$ for each $x \in \operatorname{Fix}(f \circ g)$ [4]. The same argument (applied to g and f instead of f and g) shows that $f(x) \in \operatorname{Fix}(f \circ g)$ for each $x \in \operatorname{Fix}(g \circ f)$.

[4] *Proof.* Let $x \in \operatorname{Fix}(f \circ g)$. We must show that $g(x) \in \operatorname{Fix}(g \circ f)$.

Let γ be the map

$$\operatorname{Fix}(f\circ g)\to\operatorname{Fix}(g\circ f), \qquad x\mapsto g(x).$$

(This is well-defined, because $g(x)\in\operatorname{Fix}(g\circ f)$ for each $x\in\operatorname{Fix}(f\circ g)$.)

Let φ be the map

$$\operatorname{Fix}(g\circ f)\to\operatorname{Fix}(f\circ g), \qquad x\mapsto f(x).$$

(This is well-defined, because $f(x)\in\operatorname{Fix}(f\circ g)$ for each $x\in\operatorname{Fix}(g\circ f)$.)

We have $\varphi\circ\gamma=\operatorname{id}$ [5]. The same argument (applied to g, f, γ and φ instead of f, g, φ and γ) shows that $\gamma\circ\varphi=\operatorname{id}$.

The maps φ and γ are mutually inverse (since $\varphi\circ\gamma=\operatorname{id}$ and $\gamma\circ\varphi=\operatorname{id}$), and thus bijections. Hence, we have found a bijection $\operatorname{Fix}(f\circ g)\to\operatorname{Fix}(g\circ f)$ (namely, γ). Hence, $|\operatorname{Fix}(f\circ g)|=|\operatorname{Fix}(g\circ f)|$. This solves Exercise 2 **(a)**.

(b) It is false. Here is one of many possible counterexamples:

Let S be the 3-element set $[3]=\{1,2,3\}$. Let s_1 be the permutation of $[3]$ that switches 1 and 2 while leaving 3 unchanged. Let s_2 be the permutation of $[3]$ that switches 2 and 3 while leaving 1 unchanged. Then, $s_1\circ s_1=\operatorname{id}$ and $s_2\circ s_2=\operatorname{id}$.

Now, set $f=s_1$, $g=s_2$ and $h=s_1\circ s_2$. Then, the map $f\circ g\circ h=s_1\circ s_2\circ s_1\circ s_2$ has no fixed points at all (check this!), so that $|\operatorname{Fix}(f\circ g\circ h)|=0$. But

$$g\circ f\circ h=s_2\circ\underbrace{s_1\circ s_1}_{=\operatorname{id}}\circ s_2=s_2\circ s_2=\operatorname{id},$$

and thus $|\operatorname{Fix}(g\circ f\circ h)|=|\operatorname{Fix}(\operatorname{id})|=|S|=3$. Hence, $|\operatorname{Fix}(f\circ g\circ h)|\neq|\operatorname{Fix}(g\circ f\circ h)|$. This solves Exercise 2 **(b)**. □

We know that $x\in\operatorname{Fix}(f\circ g)$. In other words, x is a fixed point of $f\circ g$ (since $\operatorname{Fix}(f\circ g)$ is defined as the set of all fixed points of $f\circ g$). In other words, $(f\circ g)(x)=x$. Thus, $f(g(x))=(f\circ g)(x)=x$. Hence,

$$(g\circ f)(g(x))=g\left(\underbrace{f(g(x))}_{=x}\right)=g(x).$$

In other words, $g(x)$ is a fixed point of $g\circ f$. In other words, $g(x)\in\operatorname{Fix}(g\circ f)$ (since $\operatorname{Fix}(g\circ f)$ is defined as the set of all fixed points of $g\circ f$). This completes our proof.

[5] *Proof.* Let $x\in\operatorname{Fix}(f\circ g)$. Thus, $\gamma(x)=g(x)$ (by the definition of γ). But $\gamma(x)\in\operatorname{Fix}(g\circ f)$, so that $\varphi(\gamma(x))=f(\gamma(x))$ (by the definition of φ).

We have $x\in\operatorname{Fix}(f\circ g)$. In other words, x is a fixed point of $f\circ g$ (since $\operatorname{Fix}(f\circ g)$ is defined as the set of all fixed points of $f\circ g$). In other words, $(f\circ g)(x)=x$. Now,

$$(\varphi\circ\gamma)(x)=\varphi(\gamma(x))=f\left(\underbrace{\gamma(x)}_{=g(x)}\right)=f(g(x))=(f\circ g)(x)=x=\operatorname{id}(x).$$

Now, forget that we fixed x. We thus have proven that $(\varphi\circ\gamma)(x)=\operatorname{id}(x)$ for each $x\in\operatorname{Fix}(f\circ g)$. In other words, $\varphi\circ\gamma=\operatorname{id}$.

Remark 0.9. Exercise 2 **(a)** can be slightly generalized:

Proposition 0.10. Let S and T be two finite sets. Then, $|\operatorname{Fix}(f \circ g)| = |\operatorname{Fix}(g \circ f)|$ for any two maps $f : S \to T$ and $g : T \to S$.

Setting $T = S$ in Proposition 0.10, we recover Exercise 2 **(a)**. The proof of Proposition 0.10 is analogous to the above solution of Exercise 2 **(a)**.

Remark 0.11. Exercise 2 (and Proposition 0.10) might remind you of something you have seen in linear algebra. Namely, if A is an $n \times m$-matrix and B is an $m \times n$-matrix (so that both products AB and BA are well-defined), then

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA), \tag{2}$$

but it is not true that any three $n \times n$-matrices A, B, C satisfy $\operatorname{Tr}(ABC) = \operatorname{Tr}(BAC)$.

This is more than just a similarity. You can actually prove Proposition 0.10 using (2). Here is an outline of this proof: To any map $w : [p] \to [q]$ (where p and q are two nonnegative integers), we assign a $q \times p$-matrix M_w whose (i, j)-th entry is $[i = w(j)]$ (where we are using the Iverson bracket notation). In other words, M_w is the matrix whose j-th column has a 1 in its $w(j)$-th row and 0's everywhere else. If $p = q$ (so that w is a map from the set $[p]$ to itself), then this matrix M_w is a square matrix with trace

$$\operatorname{Tr}(M_w) = |\operatorname{Fix} w| \tag{3}$$

(check this!). Now, let $f : S \to T$ and $g : T \to S$ be two maps. WLOG assume that $S = [n]$ and $T = [m]$ for two nonnegative integers n and m (otherwise, "relabel" the elements of S and T). Then, it is easy to see that $M_{f \circ g} = M_f M_g$ and $M_{g \circ f} = M_g M_f$. Meanwhile, (3) yields $\operatorname{Tr}(M_{f \circ g}) = |\operatorname{Fix}(f \circ g)|$ and $\operatorname{Tr}(M_{g \circ f}) = |\operatorname{Fix}(g \circ f)|$. Hence,

$$\begin{aligned} |\operatorname{Fix}(f \circ g)| &= \operatorname{Tr}\left(\underbrace{M_{f \circ g}}_{= M_f M_g}\right) = \operatorname{Tr}(M_f M_g) = \operatorname{Tr}\left(\underbrace{M_g M_f}_{= M_{g \circ f}}\right) \qquad \text{(by (2))} \\ &= \operatorname{Tr}(M_{g \circ f}) = |\operatorname{Fix}(g \circ f)|. \end{aligned}$$

So we have again proved Proposition 0.10.

0.4. A binomial coefficient in a denominator

Exercise 3. Let n and a be two integers with $n \geq a \geq 1$. Prove that

$$\sum_{k=a}^{n} \frac{(-1)^k}{k} \binom{n-a}{k-a} = \frac{(-1)^a}{a\binom{n}{a}}.$$

Exercise 3 is an identity that tends to creep up in various seemingly(?) unrelated situations in mathematics. I have first encountered it in [Schmit04, proof of Theorem 9.5] (where it appears with an incorrect power of -1 on the right hand side). It also has recently appeared on math.stackexchange ([dilemi17], with a, $n-a$ and $k-a$ renamed as p, n and k), where it has been proven in three different ways: once using the beta function, once using residues, and once (by myself in the comments) using finite differences. Let me here give a different, elementary proof.[6]

We begin with the following identities:

Proposition 0.12. Let $i \in \mathbb{Z}$, $n \in \mathbb{Z}$ and $j \in \mathbb{N}$. Then:

(a) We have

$$\sum_{k=0}^{j} (-1)^k \binom{n}{j-k} \binom{k+i-1}{k} = \binom{n-i}{j}.$$

(b) If i is positive, then

$$\sum_{k=0}^{j} \frac{(-1)^k}{k+i} \binom{n}{j-k} \binom{k+i}{i} = \frac{1}{i} \binom{n-i}{j}.$$

Proof of Proposition 0.12. **(a)** Proposition 0.1 (applied to $-i$, n and j instead of x, y

[6] This proof also appears in [Grinbe16, solution to Exercise 3.16].

and n) yields

$$\binom{(-i)+n}{j} = \sum_{k=0}^{j} \underbrace{\binom{-i}{k}}_{\substack{=(-1)^k \binom{k-(-i)-1}{k} \\ \text{(by Proposition 0.3,} \\ \text{applied to } -i \text{ instead of } n)}} \binom{n}{j-k}$$
$$= \sum_{k=0}^{j} (-1)^k \underbrace{\binom{k-(-i)-1}{k}}_{\substack{=\binom{k+i-1}{k} \\ \text{(since } k-(-i)-1=k+i-1)}} \binom{n}{j-k}$$
$$= \sum_{k=0}^{j} (-1)^k \binom{k+i-1}{k} \binom{n}{j-k} = \sum_{k=0}^{j} (-1)^k \binom{n}{j-k} \binom{k+i-1}{k}.$$

Thus,

$$\sum_{k=0}^{j} (-1)^k \binom{n}{j-k} \binom{k+i-1}{k} = \binom{(-i)+n}{j} = \binom{n-i}{j}.$$

This proves Proposition 0.12 **(a)**.

(b) Assume that i is positive. Let $k \in \mathbb{N}$. Then, $i-1 \in \mathbb{N}$ (since i is a positive integer). Thus, $i-1 \geq 0$. Also, $\underbrace{k}_{\geq 0} + i - 1 \geq i-1$. Hence, Proposition 0.8 (applied to $k+i-1$ and $i-1$ instead of m and n) yields

$$\binom{k+i-1}{i-1} = \binom{k+i-1}{(k+i-1)-(i-1)} = \binom{k+i-1}{k}$$

(since $(k+i-1)-(i-1) = k$).

Furthermore, Proposition 0.4 (applied to $k+i$ and i instead of n and k) yields

$$i\binom{k+i}{i} = (k+i) \underbrace{\binom{k+i-1}{i-1}}_{=\binom{k+i-1}{k}} = (k+i)\binom{k+i-1}{k}.$$

Hence,

$$\binom{k+i}{i} = \frac{1}{i}(k+i)\binom{k+i-1}{k}.$$

Thus,

$$\begin{aligned}
&\frac{(-1)^k}{k+i}\binom{n}{j-k}\underbrace{\binom{k+i}{i}}_{=\frac{1}{i}(k+i)\binom{k+i-1}{k}} \\
&= \frac{(-1)^k}{k+i}\binom{n}{j-k}\cdot\frac{1}{i}(k+i)\binom{k+i-1}{k} \\
&= \frac{1}{i}(-1)^k\binom{n}{j-k}\binom{k+i-1}{k}. \qquad (4)
\end{aligned}$$

Now, forget that we fixed k. We thus have proven (4) for each $k \in \mathbb{N}$. Now,

$$\begin{aligned}
&\sum_{k=0}^{j}\underbrace{\frac{(-1)^k}{k+i}\binom{n}{j-k}\binom{k+i}{i}}_{\substack{=\frac{1}{i}(-1)^k\binom{n}{j-k}\binom{k+i-1}{k} \\ \text{(by (4))}}} \\
&= \sum_{k=0}^{j}\frac{1}{i}(-1)^k\binom{n}{j-k}\binom{k+i-1}{k} \\
&= \frac{1}{i}\underbrace{\sum_{k=0}^{j}(-1)^k\binom{n}{j-k}\binom{k+i-1}{k}}_{\substack{=\binom{n-i}{j} \\ \text{(by Proposition 0.12 (a))}}} = \frac{1}{i}\binom{n-i}{j}.
\end{aligned}$$

This proves Proposition 0.12 **(b)**. □

Let us now solve the actual exercise:

Solution to Exercise 3. From $n \geq a$, we obtain $n - a \in \mathbb{N}$. Also, $n \geq a \geq 1 \geq 0$, and therefore Proposition 0.2 (applied to n and a instead of a and b) yields $\binom{n}{a} = \frac{n!}{a!\,(n-a)!} \neq 0$ (since $n! \neq 0$).

Any $k \in \{a, a+1, \ldots, n\}$ satisfies

$$\binom{n}{a}\binom{n-a}{k-a} = \binom{n}{n-k}\binom{k}{a} \qquad (5)$$

[7]. We have

$$\binom{n}{a}\sum_{k=a}^{n}\frac{(-1)^k}{k}\binom{n-a}{k-a}$$
$$=\sum_{k=a}^{n}\frac{(-1)^k}{k}\underbrace{\binom{n}{a}\binom{n-a}{k-a}}_{\substack{=\binom{n}{n-k}\binom{k}{a}\\ \text{(by (5))}}}$$
$$=\sum_{k=a}^{n}\frac{(-1)^k}{k}\binom{n}{n-k}\binom{k}{a}=\sum_{k=0}^{n-a}\underbrace{\frac{(-1)^{k+a}}{k+a}}_{\substack{=\frac{(-1)^k(-1)^a}{k+a}\\ \text{(since }(-1)^{k+a}=(-1)^k(-1)^a)}}\underbrace{\binom{n}{n-(k+a)}}_{\substack{=\binom{n}{(n-a)-k}\\ \text{(since }n-(k+a)=(n-a)-k)}}\binom{k+a}{a}$$

(here, we have substituted $k+a$ for k in the sum)

$$=\sum_{k=0}^{n-a}\frac{(-1)^k(-1)^a}{k+a}\binom{n}{(n-a)-k}\binom{k+a}{a}=(-1)^a\underbrace{\sum_{k=0}^{n-a}\frac{(-1)^k}{k+a}\binom{n}{(n-a)-k}\binom{k+a}{a}}_{\substack{=\frac{1}{a}\binom{n-a}{n-a}\\ \text{(by Proposition 0.12 \textbf{(b)},}\\ \text{applied to }j=n-a\text{ and }i=a)}}$$
$$=(-1)^a\frac{1}{a}\underbrace{\binom{n-a}{n-a}}_{\substack{=1\\ \text{(by Proposition 0.7,}\\ \text{applied to }m=n-a)}}=(-1)^a\frac{1}{a}.$$

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[7]*Proof of (5):* Let $k\in\{a,a+1,\ldots,n\}$. Thus, $a\leq k\leq n$, so that $k\geq a\geq 1\geq 0$, so that $k\in\mathbb{N}$. Hence, Proposition 0.8 (applied to n and k instead of m and n) yields $\binom{n}{k}=\binom{n}{n-k}$.

But Proposition 0.6 (applied to k and a instead of a and b) shows that

$$\binom{n}{k}\binom{k}{a}=\binom{n}{a}\binom{n-a}{k-a}.$$

Hence,

$$\binom{n}{a}\binom{n-a}{k-a}=\underbrace{\binom{n}{k}}_{=\binom{n}{n-k}}\binom{k}{a}=\binom{n}{n-k}\binom{k}{a}.$$

This proves (5).

We can divide both sides of this equality by $\dbinom{n}{a}$ (since $\dbinom{n}{a} \neq 0$). Thus, we find

$$\sum_{k=a}^{n} \frac{(-1)^k}{k} \binom{n-a}{k-a} = (-1)^a \frac{1}{a} / \binom{n}{a} = \frac{(-1)^a}{a \dbinom{n}{a}}.$$

This solves Exercise 3. □

0.5. Derangements with at most 1 descent

Exercise 4. Let $n \in \mathbb{N}$. How many derangements σ of $[n]$ have at most 1 descent?

(See homework set #5 for the definitions of descents and of derangements.)

Recall that for any $n \in \mathbb{N}$, we let S_n denote the set of all permutations of $[n]$.

The main work of the solution to Exercise 4 is done by the following fact:

Proposition 0.13. Let $n \in \mathbb{N}$. Let $i \in [n-1]$. Then,

$$\begin{aligned} &(\text{the number of derangements of } [n] \text{ whose only descent is } i) \\ &= \binom{n-2}{i-1}. \end{aligned}$$

Proposition 0.13 is a result by Gessel and Reutenauer [GesReu93, Theorem 9.5], which they obtained using the theory of quasisymmetric functions. We shall instead prove it by elementary combinatorics.

To simplify its proof, let us first verify a lemma:

Lemma 0.14. Let $n \in \mathbb{N}$. Let $i \in [n-1]$. Let $\sigma \in S_n$ be a permutation satisfying $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$ and $\sigma(1) \neq 1$ and $\sigma(n) \neq n$. Then, σ is a derangement of $[n]$ whose only descent is i.

Proof of Lemma 0.14. Let k be a fixed point of σ. We shall derive a contradiction (from which we will, of course, conclude that σ has no fixed points).

We have $\sigma(k) = k$ (since k is a fixed point of σ).

Assume first that $k \leq i$. We have $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$, thus $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ (since $k \leq i$). Thus, the k integers $\sigma(1), \sigma(2), \ldots, \sigma(k)$ are distinct.

Also, $\sigma(1) \neq 1$, so that $\sigma(1) > 1$, so that $\sigma(1) \geq 2$. Hence,

$$2 \leq \sigma(1) < \sigma(2) < \cdots < \sigma(k) = k.$$

Hence, the k integers $\sigma(1), \sigma(2), \ldots, \sigma(k)$ all lie between 2 and k; that is, they lie in the set $\{2, 3, \ldots, k\}$. Since this set $\{2, 3, \ldots, k\}$ has only $k-1$ elements, this shows (by the pigeonhole principle) that at least two of the k integers $\sigma(1), \sigma(2), \ldots, \sigma(k)$

must be equal. But this contradicts the fact that the k integers $\sigma(1), \sigma(2), \ldots, \sigma(k)$ are distinct.

This contradiction shows that our assumption (that $k \leq i$) was false. Hence, we don't have $k \leq i$. In other words, we have $k > i$. Hence, $k \geq i+1$.

We have $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$, thus $\sigma(k) < \sigma(k+1) < \cdots < \sigma(n)$ (since $k \geq i+1$). Thus, the $n-k+1$ integers $\sigma(k), \sigma(k+1), \ldots, \sigma(n)$ are distinct.

But $\sigma(n) \neq n$, so that $\sigma(n) < n$, so that $\sigma(n) \leq n-1$. From $\sigma(k) = k$, we obtain

$$k = \sigma(k) < \sigma(k+1) < \cdots < \sigma(n) \leq n-1.$$

Hence, the $n-k+1$ integers $\sigma(k), \sigma(k+1), \ldots, \sigma(n)$ all lie between k and $n-1$; that is, they lie in the set $\{k, k+1, \ldots, n-1\}$. Since this set $\{k, k+1, \ldots, n-1\}$ has only $n-k$ elements, this shows (by the pigeonhole principle) that at least two of the $n-k+1$ integers $\sigma(k), \sigma(k+1), \ldots, \sigma(n)$ must be equal. But this contradicts the fact that the $n-k+1$ integers $\sigma(k), \sigma(k+1), \ldots, \sigma(n)$ are distinct.

Now, forget that we fixed k. We thus have derived a contradiction for any fixed point k of σ. Hence, there exists no fixed point k of σ. In other words, the permutation σ has no fixed points. In other words, σ is a derangement.

Next, we shall show that the only descent of σ is i.

The map σ is a permutation, thus injective. Hence, $\sigma(i) \neq \sigma(i+1)$.

Assume (for the sake of contradiction) that $\sigma(i) \leq \sigma(i+1)$. Hence, $\sigma(i) < \sigma(i+1)$ (since $\sigma(i) \neq \sigma(i+1)$). Now,

$$\sigma(1) < \sigma(2) < \cdots < \sigma(i) < \sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n).$$

In other words, $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$. Hence, the numbers $\sigma(1), \sigma(2), \ldots, \sigma(n)$ are precisely the elements of $[n]$ written down in increasing order (since σ is a permutation). In other words, $\sigma(k) = k$ for each $k \in [n]$. Applying this to $k = 1$, we find $\sigma(1) = 1$. This contradicts $\sigma(1) \neq 1$.

This contradiction shows that our assumption (that $\sigma(i) \leq \sigma(i+1)$) was false. Thus, we have $\sigma(i) > \sigma(i+1)$. In other words, i is a descent of σ. Moreover, σ cannot have any other descents than i (since $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$); thus, i is the only descent of σ. Hence, σ is a derangement of $[n]$ whose only descent is i. This proves Lemma 0.14. □

Proof of Proposition 0.13. From $i \in [n-1]$, we obtain $1 \leq i \leq n-1$, so that $n-1 \geq 1$. Hence, $n \geq 2$.

Let us first analyze how a derangement of $[n]$ whose only descent is i looks like.

Let σ be a derangement of $[n]$ whose only descent is i. Thus,

$$\sigma(1) < \sigma(2) < \cdots < \sigma(i) \tag{6}$$

and

$$\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n) \tag{7}$$

but $\sigma(i) > \sigma(i+1)$.

Since σ is a derangement, we know that σ has no fixed points. Hence, in particular, 1 cannot be a fixed point of σ. Thus, $\sigma(1) \neq 1$. Also, (6) shows that the numbers $\sigma(2), \sigma(3), \ldots, \sigma(i)$ are greater than $\sigma(1)$, and therefore greater than 1 (since $\sigma(1) \geq 1$); thus, they are distinct from 1. Hence, all the i numbers $\sigma(1), \sigma(2), \ldots, \sigma(i)$ are distinct from 1.

Furthermore, n cannot be a fixed point of σ (since σ has no fixed points). Hence, $\sigma(n) \neq n$. Also, (7) shows that the numbers $\sigma(i+1), \sigma(i+2), \ldots, \sigma(n-1)$ are smaller than $\sigma(n)$, and thus smaller than n (since $\sigma(n) \leq n$); thus, they are distinct from n. Hence, all the $n-i$ numbers $\sigma(i+1), \sigma(i+2), \ldots, \sigma(n)$ are distinct from n. Therefore, $\sigma^{-1}(n) \notin \{i+1, i+2, \ldots, n\}$. Thus,

$$\sigma^{-1}(n) \in [n] \setminus \{i+1, i+2, \ldots, n\} = \{1, 2, \ldots, i\}.$$

In other words, $n = \sigma(j)$ for some $j \in \{1, 2, \ldots, i\}$. In other words, one of the numbers $\sigma(1), \sigma(2), \ldots, \sigma(i)$ is n.

So we know that all the i numbers $\sigma(1), \sigma(2), \ldots, \sigma(i)$ are distinct from 1, but one of these numbers is n. Thus, $\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$ is a subset of $\{2, 3, \ldots, n\}$ that contains n. This subset is clearly an i-element subset (since $\sigma(1), \sigma(2), \ldots, \sigma(i)$ are distinct).

Now, forget that we fixed σ. We thus have proven that if σ is a derangement of $[n]$ whose only descent is i, then $\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$ is an i-element subset of $\{2, 3, \ldots, n\}$ that contains n. Furthermore, if we know this subset, then we can uniquely reconstruct the whole permutation σ: Indeed, its first i values $\sigma(1), \sigma(2), \ldots, \sigma(i)$ are simply the elements of this subset written in increasing order (because of (6)), whereas the remaining $n-i$ values $\sigma(i+1), \sigma(i+2), \ldots, \sigma(n)$ are the remaining $n-i$ elements of $[n]$ written in increasing order (because of (7)).

Thus, the following algorithm constructs every possible derangement σ of $[n]$ whose only descent is i:

- First, choose an i-element subset S of $\{2, 3, \ldots, n\}$ that contains n to become the set $\{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$. There are $\dbinom{n-2}{i-1}$ choices here[8].
- Then, the values of $\sigma(1), \sigma(2), \ldots, \sigma(i)$ are uniquely determined (indeed, they have to be the i elements of S in increasing order).
- Furthermore, the values of $\sigma(i+1), \sigma(i+2), \ldots, \sigma(n)$ are also uniquely determined (indeed, they have to be the $n-i$ elements of $[n] \setminus S$ in increasing order).

[8]Here, we are using the fact that the number of i-element subsets S of $\{2, 3, \ldots, n\}$ that contain n is $\dbinom{n-2}{i-1}$. This is easy to prove (in fact, choosing such a subset means choosing its $i-1$ elements other than n; and these $i-1$ elements are chosen from the $(n-2)$-element set $\{2, 3, \ldots, n-1\}$).

Furthermore, every permutation σ constructed by this algorithm is actually a derangement σ of $[n]$ whose only descent is i [9]. Hence, the number of derangements σ of $[n]$ whose only descent is i equals the number of ways to perform the above algorithm. But the latter number is clearly $\dbinom{n-2}{i-1}$ (since there are $\dbinom{n-2}{i-1}$ choices in the first step, and the next two steps are uniquely determined). Thus, the number of derangements σ of $[n]$ whose only descent is i equals $\dbinom{n-2}{i-1}$. This proves Proposition 0.13. □

Let us next recall a fundamental fact (which has already been proven in the solutions to homework set 1):

Proposition 0.15. Let $m \in \mathbb{N}$. Then,

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m.$$

Solution to Exercise 4 (sketched). The answer is

$$\begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n = 1; \\ 2^{n-2}, & \text{if } n > 1 \end{cases}.$$

Proof. We WLOG assume that $n > 1$ (since the cases when $n = 0$ and when $n = 1$ can be easily dealt with). Thus, $n \geq 2$, so that $n - 2 \in \mathbb{N}$.

The only permutation having no descents is id, and this is not a derangement (since id has $n > 0$ fixed points). Thus, there exists no derangement having no descents. In other words,

$$(\text{the number of all derangements having no descents}) = 0.$$

[9] *Proof.* Let σ be a permutation constructed by this algorithm. From the definition of the algorithm, it follows immediately that $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$ and $\{\sigma(1), \sigma(2), \ldots, \sigma(i)\} = S$ (where S is the i-element subset of $\{2, 3, \ldots, n\}$ that was chosen during the algorithm).

We have $i \geq 1$, so that $\sigma(1) \in \{\sigma(1), \sigma(2), \ldots, \sigma(i)\} = S \subseteq \{2, 3, \ldots, n\}$. Thus, $\sigma(1) \geq 2 > 1$, so that $\sigma(1) \neq 1$.

The definition of S yields $n \in S = \{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$, so that $\sigma^{-1}(n) \in \{1, 2, \ldots, i\}$ and thus $\sigma^{-1}(n) \leq i \leq n - 1 < n$. Thus, $\sigma^{-1}(n) \neq n$, so that $\sigma(n) \neq n$.

Thus, Lemma 0.14 shows that σ is a derangement of $[n]$ whose only descent is i. This is what we wanted to prove.

Now,

$$\begin{aligned}
&(\text{the number of all derangements having at most 1 descent}) \\
&= \underbrace{(\text{the number of all derangements having no descents})}_{=0} \\
&\qquad + (\text{the number of all derangements having exactly 1 descent}) \\
&= (\text{the number of all derangements having exactly 1 descent}) \\
&= \underbrace{\sum_{i\in[n-1]}}_{=\sum_{i=1}^{n-1}} \underbrace{(\text{the number of derangements of } [n] \text{ whose only descent is } i)}_{\substack{=\binom{n-2}{i-1} \\ \text{(by Proposition 0.13)}}} \\
&= \sum_{i=1}^{n-1} \binom{n-2}{i-1} = \sum_{k=0}^{n-2} \binom{n-2}{k} \qquad \left(\begin{array}{c}\text{here, we have substituted } k \\ \text{for } i-1 \text{ in the sum}\end{array}\right) \\
&= 2^{n-2} \qquad (\text{by Proposition 0.15, applied to } m = n-2) \\
&= \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n = 1; \\ 2^{n-2}, & \text{if } n > 1 \end{cases} \qquad (\text{since } n > 1).
\end{aligned}$$

This completes the proof. □

0.6. Connected permutations

Definition 0.16. Let n be a positive integer. A permutation σ of $[n]$ is said to be *connected* if and only if there exists no $k \in [n-1]$ such that $\sigma([k]) = [k]$.

For example, the permutation σ of $[5]$ sending $1,2,3,4,5$ to $2,4,1,5,3$ is connected, since it satisfies

$$\sigma([1]) = \{2\} \neq [1], \qquad \sigma([2]) = \{2,4\} \neq [2],$$
$$\sigma([3]) = \{2,4,1\} \neq [3], \qquad \sigma([4]) = \{2,4,1,5\} \neq [4].$$

But the permutation σ of $[4]$ sending $1,2,3,4$ to $2,1,4,3$ is not connected, because it satisfies $\sigma([2]) = [2]$. Likewise, a permutation σ of $[n]$ (for $n > 1$) satisfying $\sigma(1) = 1$ is never connected (since $\sigma([1]) = [1]$); the same holds for a permutation σ satisfying $\sigma(n) = n$ (since $\sigma([n-1]) = [n-1]$).

Exercise 5. For each positive integer n, let c_n denote the number of all connected permutations of $[n]$. (Thus, $c_1 = 1$, $c_2 = 1$ and $c_3 = 3$.)

Prove that

$$n! = \sum_{k=1}^{n} c_k (n-k)! \qquad \text{for each positive integer } n.$$

Exercise 5 appears (with an outline of a solution) in [Camero16, §3.4, Example: Connected permutations]. See also Sequence A003319 in the OEIS database for the sequence $(c_1, c_2, c_3, \ldots)$.

Solution to Exercise 5 (sketched). Let n be a positive integer.

If $\sigma \in S_n$, then a *return* of σ shall denote a positive integer $k \in [n]$ satisfying $\sigma([k]) = [k]$. Every $\sigma \in S_n$ has at least one return, namely the number n (since $\sigma([n]) = [n]$); thus, the smallest return of σ is well-defined.

It is easy to see that a permutation $\sigma \in S_n$ is connected if and only if its smallest return is n. We shall not use this, but we shall use the following closely connected fact:

> *Observation 1:* Let $j \in [n]$. The number of permutations $\sigma \in S_n$ whose smallest return is j equals $c_j (n-j)!$.

[*Proof of Observation 1:* Let $\sigma \in S_n$ be a permutation whose smallest return is j. Then, j is a return of σ; in other words, $\sigma([j]) = [j]$ (by the definition of a return). Hence, $\sigma(h) \in \sigma([j]) = [j]$ for each $h \in [j]$. Thus, we can define a map

$$\tau : [j] \to [j], \qquad h \mapsto \sigma(h).$$

(This map τ is essentially the restriction of σ to $[j]$, but the codomain is also being restricted to $[j]$.) The map τ is injective (since σ is injective), and thus is a permutation of $[j]$ (since it is an injective map between two finite sets of equal size). Moreover, the permutation τ of $[j]$ is connected[10].

Now, forget that we fixed σ. Thus, for each permutation $\sigma \in S_n$ whose smallest return is j, we have constructed a connected permutation τ of $[j]$. Let us denote this τ by τ_σ (to stress its dependence on σ). We now have the following algorithm to construct every permutation $\sigma \in S_n$ whose smallest return is j:

- First, pick any connected permutation τ of $[j]$; this permutation shall be the τ_σ corresponding to our σ. There are c_j choices at this step, since the number of connected permutations of $[j]$ is c_j (by the definition of c_j).
- The values $\sigma(1), \sigma(2), \ldots, \sigma(j)$ are now uniquely determined (because they are the values of the already chosen permutation $\tau_\sigma = \tau$), and are simply the numbers $1, 2, \ldots, j$ in some order. Next, choose the remaining values $\sigma(j+1), \sigma(j+2), \ldots, \sigma(n)$. These $n-j$ values must be the numbers $j+1, j+2, \ldots, n$ in some order; the only choice at this step is which order they are in. Thus, there are $(n-j)!$ choices at this step.

[10] *Proof.* If there was some $k \in [j-1]$ such that $\tau([k]) = [k]$, then this k would be a return of σ (because the definition of τ shows that $\tau([k]) = \sigma([k])$, so that $\sigma([k]) = \tau([k]) = [k]$), which would contradict the fact that j is the **smallest** return of σ (indeed, k is smaller than j). Hence, there exists no $k \in [j-1]$ such that $\tau([k]) = [k]$. In other words, the permutation τ of $[j]$ is connected (by the definition of "connected").

Our argument above shows that each permutation $\sigma \in S_n$ whose smallest return is j can be constructed by this algorithm. Conversely, it is easy to see (more or less by reversing the above argument) that every σ constructed by this algorithm is a permutation $\sigma \in S_n$ whose smallest return is j. Moreover, every set of choices during the algorithm yields a different such permutation. Thus, the number of permutations $\sigma \in S_n$ whose smallest return is j equals the number of ways of making choices during the algorithm. But the latter number is $c_j (n-j)!$ (since we have c_j choices at the first step, and $(n-j)!$ choices at the second step). Thus, the number of permutations $\sigma \in S_n$ whose smallest return is j equals $c_j (n-j)!$. This proves Observation 1.]

Now,

$$\begin{aligned} n! &= (\text{the number of permutations } \sigma \in S_n) \\ &= \underbrace{\sum_{j \in [n]}}_{= \sum_{j=1}^{n}} \underbrace{(\text{the number of permutations } \sigma \in S_n \text{ whose smallest return is } j)}_{\substack{= c_j (n-j)! \\ \text{(by Observation 1)}}} \\ &= \sum_{j=1}^{n} c_j (n-j)! = \sum_{k=1}^{n} c_k (n-k)! \end{aligned}$$

(here, we have renamed the summation index j as k). This solves Exercise 5. □

Remark 0.17. Notice the similarity between our above solution of Exercise 5 and the proof of the recursion

$$m_n = \sum_{k=0}^{n-1} m_k m_{n-k-1}$$

(for $n > 0$) for the Catalan numbers $m_0, m_1, m_2, \ldots$ (see, e.g., [Galvin17, §24, problem (2)]).

0.7. Permutations and intervals

An *integer interval* means a set of the form $\{a, a+1, \ldots, b\}$ for some integers a and b. (If $a > b$, then this set is understood to be empty.)

Exercise 6. Let $n \in \mathbb{N}$ and $r \in [n]$. A permutation σ of $[n]$ is said to be *r-friendly* if for each $k \in \{r, r+1, \ldots, n\}$, the set $\sigma([k])$ is an integer interval.

(For example, the permutation σ of $[6]$ sending $1,2,3,4,5,6$ to $2,4,3,5,1,6$ is 3-friendly (since $\sigma([3]) = \{2,3,4\}$, $\sigma([4]) = \{2,3,4,5\}$, $\sigma([5]) = \{1,2,3,4,5\}$ and $\sigma([6]) = \{1,2,3,4,5,6\}$ are integer intervals), and thus also r-friendly for each $r \geq 3$, but not 2-friendly (since $\sigma([2]) = \{2,4\}$ is not an integer interval).)

Prove that the number of r-friendly permutations of $[n]$ is $2^{n-r} r!$.

See `https://artofproblemsolving.com/community/c6h1542350_rfriendly_permutations_sending_some_intervals_to_intervals` for a discussion of this exercise.

Before we solve this exercise, let us state two simple lemmas:

Lemma 0.18. Let S and T be two integer intervals such that $T \subseteq S$ and $|S| = |T| + 1$. Then, either $T = S \setminus \{\max S\}$ or $T = S \setminus \{\min S\}$ (or both).

Proof of Lemma 0.18 (sketched). Write the integer interval S in the form $S = \{a, a+1, \ldots, b\}$ for some integers a and b.

From $T \subseteq S$, we obtain $|S \setminus T| = |S| - |T| = 1$ (since $|S| = |T| + 1$). In other words, $S \setminus T$ is a 1-element set. In other words, $S \setminus T = \{k\}$ for some object k. Consider this k.

From $T \subseteq S$, we obtain $T = S \setminus \underbrace{(S \setminus T)}_{=\{k\}} = S \setminus \{k\}$. Thus, $S \setminus \{k\}$ is an integer interval (since T is an integer interval).

Also, $k \in \{k\} = S \setminus T \subseteq S = \{a, a+1, \ldots, b\}$. Hence, k is an integer satisfying $a \leq k \leq b$.

From $S = \{a, a+1, \ldots, b\}$ (and $a \leq b$), we obtain $\min S = a$ and $\max S = b$.

Recall that $k \in \{a, a+1, \ldots, b\}$. Thus, we are in one of the following three cases:

Case 1: We have $k = a$.

Case 2: We have $a < k < b$.

Case 3: We have $k = b$.

Let us first consider Case 1. In this case, we have $k = a$. Thus, $T = S \setminus \left\{ \underbrace{k}_{=a=\min S} \right\} = S \setminus \{\min S\}$. Hence, either $T = S \setminus \{\max S\}$ or $T = S \setminus \{\min S\}$ (or both). Therefore, Lemma 0.18 is proven in Case 1.

Let us now consider Case 2. In this case, we have $a < k < b$. Now,

$$T = \underbrace{S}_{=\{a,a+1,\ldots,b\}} \setminus \{k\} = \{a, a+1, \ldots, b\} \setminus \{k\}$$
$$= \{a, a+1, \ldots, k-1, k+1, k+2, \ldots, b\}$$

(since $a < k < b$). Hence, the set T contains the two elements a and b but not the element k that lies between them (in the sense of being larger than a but smaller than b). Therefore, the set T is not an integer interval. This contradicts the fact that T is an integer interval. Hence, Case 2 is impossible.

Let us finally consider Case 3. In this case, we have $k = b$. Thus, $T = S \setminus \left\{ \underbrace{k}_{=b=\max S} \right\} = S \setminus \{\max S\}$. Hence, either $T = S \setminus \{\max S\}$ or $T = S \setminus \{\min S\}$ (or both). Therefore, Lemma 0.18 is proven in Case 3.

We have thus proven Lemma 0.18 in the two Cases 1 and 3. Since these two cases are the only possibilities (because we have shown that Case 2 is impossible), we thus conclude that Lemma 0.18 always holds. □

Lemma 0.19. Let $n \in \mathbb{N}$ and $r \in [n]$. Let σ be an r-friendly permutation of $[n]$. Let $k \in \{r+1, r+2, \ldots, n\}$. Then, the following holds:

(a) Both $\sigma([k-1])$ and $\sigma([k])$ are integer intervals.

(b) The integer interval $\sigma([k-1])$ is obtained from the integer interval $\sigma([k])$ by removing either its largest or its smallest element.

(c) This element removed from $\sigma([k])$ is $\sigma(k)$.

Proof of Lemma 0.19. **(a)** We have $k \in \{r+1, r+2, \ldots, n\}$, so that $k-1 \in \{r, r+1, \ldots, n-1\} \subseteq \{r, r+1, \ldots, n\}$.

But σ is r-friendly. Thus, $\sigma([k-1])$ is an integer interval (since $k-1 \in \{r, r+1, \ldots, n\}$). For the same reason, $\sigma([k])$ is an integer interval (since $k \in \{r+1, r+2, \ldots, n\} \subseteq \{r, r+1, \ldots, n\}$). Thus, Lemma 0.19 **(a)** is proven.

(b) From Lemma 0.19 **(a)**, we know that both $\sigma([k-1])$ and $\sigma([k])$ are integer intervals. Moreover, the map σ is injective; therefore, $|\sigma([k])| = |[k]| = k$ and similarly $|\sigma([k-1])| = k-1$. Hence, $|\sigma([k-1])| + 1 = k$. Comparing this with $|\sigma([k])| = k$, we obtain $|\sigma([k])| = |\sigma([k-1])| + 1$.

But $[k-1] \subseteq [k]$, so that $\sigma([k-1]) \subseteq \sigma([k])$. Hence, Lemma 0.18 (applied to $S = \sigma([k])$ and $T = \sigma([k-1])$) shows that either $\sigma([k-1]) = \sigma([k]) \setminus \{\max(\sigma([k]))\}$ or $\sigma([k-1]) = \sigma([k]) \setminus \{\min(\sigma([k]))\}$ (or both). In other words, the integer interval $\sigma([k-1])$ is obtained from the integer interval $\sigma([k])$ by removing either its largest or its smallest element. This proves Lemma 0.19 **(b)**.

(c) The element removed from $\sigma([k])$ is the unique element of $\sigma([k]) \setminus \sigma([k-1])$. Thus, we must prove that the unique element of $\sigma([k]) \setminus \sigma([k-1])$ is $\sigma(k)$.

The map σ is a bijection. Hence, $\sigma(X \setminus Y) = \sigma(X) \setminus \sigma(Y)$ for any two subsets X and Y of $[n]$. Applying this to $X = \sigma([k])$ and $Y = \sigma([k-1])$, we obtain $\sigma([k] \setminus [k-1]) = \sigma([k]) \setminus \sigma([k-1])$. Hence,

$$\sigma([k]) \setminus \sigma([k-1]) = \sigma\left(\underbrace{[k] \setminus [k-1]}_{=\{k\}}\right) = \sigma(\{k\}) = \{\sigma(k)\}.$$

Hence, the unique element of $\sigma([k]) \setminus \sigma([k-1])$ is $\sigma(k)$. This proves Lemma 0.19 **(c)**. □

Solution to Exercise 6 (sketched). I claim that the following algorithm constructs every r-friendly permutation σ of $[n]$:

- We construct a sequence $(I_n, I_{n-1}, \ldots, I_r)$ of subsets of $[n]$ with the property that $I_n \supseteq I_{n-1} \supseteq \cdots \supseteq I_r$ and that each I_i is an integer interval of size i. This construction proceeds recursively (i.e., we start with I_n, then construct I_{n-1}, then I_{n-2}, and so on until I_r); it begins by setting $I_n = [n]$ (which is clearly a subset of $[n]$ and an integer interval of size n). Then, whenever a subset I_p of $[n]$ is constructed (with $p \in \{r+1, r+2, \ldots, n\}$), we define I_{p-1} by removing either the largest or the smallest element from I_p.

This construction clearly guarantees that I_{p-1} will be a subset of $[n]$ and an integer interval of size $p-1$, provided that I_p is a subset of $[n]$ and an integer interval of size p. Also, it clearly guarantees that $I_n \supseteq I_{n-1} \supseteq \cdots \supseteq I_r$.

- Now, we define a map $\sigma : [n] \to [n]$ as follows: For each $p \in \{r+1, r+2, \ldots, n\}$, we let $\sigma(p)$ be the element of I_p that was removed in the construction of I_{p-1} (that is, the unique element of I_p that is not in I_{p-1}). Then, we define $\sigma(1), \sigma(2), \ldots, \sigma(r)$ to be the r elements of I_r in any order.

  Thus, a map $\sigma : [n] \to [n]$ is defined.

This algorithm has the following properties:

Observation 1: The algorithm constructs an r-friendly permutation σ of $[n]$ (whatever choices were made during the algorithm).

Observation 2: Every r-friendly permutation of $[n]$ can be obtained through the algorithm. (That is, if τ is any r-friendly permutation of $[n]$, then we can make the choices in the algorithm in such a way that the resulting permutation σ will be τ.)

Observation 3: The algorithm can be performed in $2^{n-r}r!$ ways (i.e., there is a total of $2^{n-r}r!$ options for the choices made during the algorithm).

Observation 4: Any two of these $2^{n-r}r!$ ways give rise to different permutations σ.

Clearly, once these four Observations are proven, we will immediately see that the number of r-friendly permutations of $[n]$ is $2^{n-r}r!$. Thus, Exercise 6 will be solved. Hence, it remains to prove the four Observations.

[*Proof of Observation 1:* We must prove that the map $\sigma : [n] \to [n]$ constructed by the algorithm is an r-friendly permutation of $[n]$.

Indeed, this map σ has the property that $\{\sigma(1), \sigma(2), \ldots, \sigma(r)\} = I_r$ (because $\sigma(1), \sigma(2), \ldots, \sigma(r)$ were defined to be the r elements of I_r in any order). Hence,

$$\sigma([r]) = \sigma(\{1, 2, \ldots, r\}) = \{\sigma(1), \sigma(2), \ldots, \sigma(r)\} = I_r.$$

Now, it is easy to check that

$$\sigma([p]) = I_p \tag{8}$$

for each $p \in \{r, r+1, \ldots, n\}$ [11]. Applying this to $p = n$, we obtain $\sigma([n]) = I_n = [n]$. Hence, the map σ is surjective. Thus, σ is bijective (since any surjective

[11] *Proof of (8):* We shall prove (8) by induction over p:

The *induction base* (i.e., the case $p = r$) follows from $\sigma([r]) = I_r$.

Now, let us handle the *induction step*. Thus, we must prove $\sigma([p]) = I_p$ under the assumption that $\sigma([p-1]) = I_{p-1}$. Recall that $\sigma(p)$ is the unique element of I_p that is not in I_{p-1} (this is how $\sigma(p)$ was defined). Thus, $I_p \setminus I_{p-1} = \{\sigma(p)\}$, so that $I_p = I_{p-1} \cup \{\sigma(p)\}$ (since $I_p \supseteq I_{p-1}$).

map between two finite sets of equal sizes is bijective). In other words, σ is a permutation of $[n]$. It remains to prove that σ is r-friendly.

For each $p \in \{r, r+1, \ldots, n\}$, the set $\sigma([p])$ is the set I_p (by (8)), and thus is an integer interval (since I_p is an integer interval). Renaming p as k in this statement, we obtain the following: For each $k \in \{r, r+1, \ldots, n\}$, the set $\sigma([k])$ is an integer interval. In other words, the permutation σ is r-friendly (by the definition of "r-friendly"). Thus, Observation 1 is proven.]

[*Proof of Observation 2:* Let τ be an r-friendly permutation of $[n]$. We must show that τ can be obtained through the algorithm. In other words, we have to explain which options we need to choose in order for the resulting permutation σ to be τ.

This is actually easy. There are two kinds of choices in the algorithm: The first kind of choice is the one made in the construction of I_{p-1} from I_p, in which we have to choose whether to remove the largest or the smallest element from I_p. The second kind of choice is the choice of order in which the elements of I_r are set to be $\sigma(1), \sigma(2), \ldots, \sigma(r)$.

So which options do we choose? In the first kind of choice, we choose to remove the element $\tau(r)$;

TODO

Let σ be an r-friendly permutation of $[n]$. Let $k \in [n-r]$. Thus, $k \leq n-r$, so that $n-k \geq r$. Hence, the two numbers $n-k$ and $n-k+1$ are both $\geq r$. Therefore, the sets $\sigma([n-k])$ and $\sigma([n-k+1])$ are two integer intervals (since σ is r-friendly).

TODO

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[*Proof of Observation 3:* TODO]

[*Proof of Observation 4:* TODO] □

0.8. Inverting a power series

Exercise 7. Find and prove an explicit formula for the coefficient of x^n in the formal power series $\dfrac{1}{1-x-x^2+x^3}$.

[**Hint:** The standard strategy is to factor $1-x-x^2+x^3$, then do partial fraction decomposition. But it is perfectly legitimate to guess the formula based on solving

$$\left(1-x-x^2+x^3\right)\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+\cdots\right)=1$$

Now,

$$\sigma\left(\underbrace{[p]}_{=[p-1]\cup\{p\}}\right)=\sigma([p-1]\cup\{p\})=\underbrace{\sigma([p-1])}_{=I_{p-1}}\cup\underbrace{\sigma(\{p\})}_{=\{\sigma(p)\}}=I_{p-1}\cup\{\sigma(p)\}$$
$$=I_p.$$

This completes the induction step. Thus, (8) is proven.

for the first few of the unknown coefficients $b_0, b_1, b_2, \ldots$, and then prove it by multiplying out. Either option works.]

Solution to Exercise 7 (sketched). The answer is

$$\frac{1}{1-x-x^2+x^3} = \sum_{n=0}^{\infty} \left\lfloor \frac{n+2}{2} \right\rfloor x^n$$
$$= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \cdots .$$

One simple way to prove this is to check that

$$\begin{aligned}
&\left(1-x-x^2+x^3\right)\left(\sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^n\right)\\
&=\sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^n - x\sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^n - x^2\sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^n + x^3\sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^n\\
&=\sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^n - \sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^{n+1} - \sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^{n+2} + \sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^{n+3}\\
&=\sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^n - \sum_{n=1}^{\infty}\left\lfloor\frac{n+1}{2}\right\rfloor x^n - \sum_{n=2}^{\infty}\left\lfloor\frac{n}{2}\right\rfloor x^n + \sum_{n=3}^{\infty}\left\lfloor\frac{n-1}{2}\right\rfloor x^n\\
&\qquad\left(\begin{array}{c}\text{here, we have substituted } n-1 \text{ for } n \text{ in the second sum,}\\ n-2 \text{ for } n \text{ in the third, and } n-3 \text{ for } n \text{ in the fourth}\end{array}\right)\\
&=\sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^n - \left(\sum_{n=0}^{\infty}\left\lfloor\frac{n+1}{2}\right\rfloor x^n - \left\lfloor\frac{0+1}{2}\right\rfloor x^0\right)\\
&\qquad - \left(\sum_{n=0}^{\infty}\left\lfloor\frac{n}{2}\right\rfloor x^n - \left\lfloor\frac{0}{2}\right\rfloor x^0 - \left\lfloor\frac{1}{2}\right\rfloor x^1\right)\\
&\qquad + \left(\sum_{n=0}^{\infty}\left\lfloor\frac{n-1}{2}\right\rfloor x^n - \left\lfloor\frac{0-1}{2}\right\rfloor x^0 - \left\lfloor\frac{1-1}{2}\right\rfloor x^1 - \left\lfloor\frac{2-1}{2}\right\rfloor x^2\right)\\
&\qquad\left(\begin{array}{c}\text{here, we have extended the ranges of the last three}\\ \text{sums in order for all of them to start at } n=0\text{; then, we}\\ \text{have subtracted back the extraneous addends}\end{array}\right)\\
&=\underbrace{\sum_{n=0}^{\infty}\left\lfloor\frac{n+2}{2}\right\rfloor x^n - \sum_{n=0}^{\infty}\left\lfloor\frac{n+1}{2}\right\rfloor x^n - \sum_{n=0}^{\infty}\left\lfloor\frac{n}{2}\right\rfloor x^n + \sum_{n=0}^{\infty}\left\lfloor\frac{n-1}{2}\right\rfloor x^n}_{=\sum_{n=0}^{\infty}\left(\left\lfloor\frac{n+2}{2}\right\rfloor - \left\lfloor\frac{n+1}{2}\right\rfloor - \left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n-1}{2}\right\rfloor\right)x^n}\\
&\qquad + \underbrace{\left\lfloor\frac{0+1}{2}\right\rfloor x^0 + \left\lfloor\frac{0}{2}\right\rfloor x^0 + \left\lfloor\frac{1}{2}\right\rfloor x^1 - \left\lfloor\frac{0-1}{2}\right\rfloor x^0 - \left\lfloor\frac{1-1}{2}\right\rfloor x^1 - \left\lfloor\frac{2-1}{2}\right\rfloor x^2}_{=1}\\
&=\sum_{n=0}^{\infty}\left(\left\lfloor\frac{n+2}{2}\right\rfloor - \left\lfloor\frac{n+1}{2}\right\rfloor - \left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n-1}{2}\right\rfloor\right)x^n + 1. \qquad (9)
\end{aligned}$$

But each $q \in \mathbb{Q}$ satisfies

$$\lfloor q+1\rfloor = \lfloor q\rfloor + 1 \qquad (10)$$

(this is easy to check). Hence, each $n \in \mathbb{N}$ satisfies

$$\begin{aligned}
&\underbrace{\left\lfloor \frac{n+2}{2} \right\rfloor}_{\substack{=\left\lfloor \frac{n}{2}+1\right\rfloor = \left\lfloor \frac{n}{2}\right\rfloor +1 \\ \text{(by (10))}}} - \underbrace{\left\lfloor \frac{n+1}{2} \right\rfloor}_{\substack{=\left\lfloor \frac{n-1}{2}+1\right\rfloor = \left\lfloor \frac{n-1}{2}\right\rfloor +1 \\ \text{(by (10))}}} - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor \\
&= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) - \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right) - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor \\
&= 0.
\end{aligned} \tag{11}$$

Thus, (9) becomes

$$\begin{aligned}
&\left(1 - x - x^2 + x^3\right) \left(\sum_{n=0}^{\infty} \left\lfloor \frac{n+2}{2} \right\rfloor x^n \right) \\
&= \sum_{n=0}^{\infty} \underbrace{\left(\left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor \right)}_{\substack{=0 \\ \text{(by (11))}}} x^n + 1 \\
&= \underbrace{\sum_{n=0}^{\infty} 0 x^n}_{=0} + 1 = 1.
\end{aligned}$$

Hence, $\dfrac{1}{1 - x - x^2 + x^3} = \sum\limits_{n=0}^{\infty} \left\lfloor \dfrac{n+2}{2} \right\rfloor x^n$.

[*Remark:* The above solution is a neat a-posteriori proof, but it does not explain how the answer $\sum\limits_{n=0}^{\infty} \left\lfloor \dfrac{n+2}{2} \right\rfloor x^n$ could have been found. Here is a quick sketch of this: The polynomial $1 - x - x^2 + x^3$ factors as $1 - x - x^2 + x^3 = (1+x)(1-x)^2$.

Thus, partial fraction decomposition yields

$$\begin{aligned}
\frac{1}{1-x-x^2+x^3} &= \frac{1}{4(1+x)} + \frac{1}{4(1-x)} + \frac{1}{2(1-x)^2} \\
&= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n + \frac{1}{4}\sum_{n=0}^{\infty} x^n + \frac{1}{2}\sum_{n=0}^{\infty}(n+1)x^n \\
&\qquad \left(\begin{array}{c}\text{here, we used the formulas } \dfrac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n, \\ \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ and } \dfrac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n\end{array}\right) \\
&= \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{4}(-1)^n + \frac{1}{4} + \frac{1}{2}(n+1)\right)}_{\substack{=\left\lfloor \frac{n+2}{2}\right\rfloor \\ \text{(this can be proven by a simple case distinction,} \\ \text{depending on } n \text{ being even or odd)}}} x^n \\
&= \sum_{n=0}^{\infty}\left\lfloor \frac{n+2}{2}\right\rfloor x^n.
\end{aligned}$$

This is also a valid proof.] □

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