Math 4707 Fall 2017 (Darij Grinberg): homework set 4 [corrected 27 Oct 2017]
due date: Wednesday 1 Nov 2017 at the beginning of class, or before that by email or moodle
Please solve **at most 4** of the 7 exercises!

0.1. The binomial transform, again

If $\mathbf{a} = (a_0, a_1, \ldots, a_N)$ is a list[1] of rational numbers, then the *binomial transform* of this list $\mathbf{a}$ is defined to be the list $(b_0, b_1, \ldots, b_N)$ of rational numbers, where

$$b_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i \qquad \text{for each } n \in \{0, 1, \ldots, N\}.$$

We shall denote the binomial transform of the list $\mathbf{a}$ by $B(\mathbf{a})$. We have already studied binomial transforms implicitly on the previous homework set: Namely, Exercise 5 on homework set #3 says that if $\mathbf{b}$ is the binomial transform of a list $\mathbf{a}$, then $\mathbf{a}$ is (in turn) the binomial transform of $\mathbf{b}$. In other words: If $\mathbf{b} = B(\mathbf{a})$, then $\mathbf{a} = B(\mathbf{b})$. In other words, if we regard B as a map that transforms lists into lists, then $B^2 = B \circ B = \mathrm{id}$.

Exercise 1. Let $N \in \mathbb{N}$.

(a) Find the binomial transform of the list $(1, 1, \ldots, 1)$ (with $N+1$ entries).

(b) For any given $a \in \mathbb{N}$, find the binomial transform of the list $\left(\binom{0}{a}, \binom{1}{a}, \ldots, \binom{N}{a}\right)$.

(c) For any given $q \in \mathbb{Z}$, find the binomial transform of the list $\left(q^0, q^1, \ldots, q^N\right)$.

(d) Find the binomial transform of the list $(1, 0, 1, 0, 1, 0, \ldots)$ (this ends with 1 if N is even, and with 0 if N is odd).

Exercise 2. Let $N \in \mathbb{N}$. If $\mathbf{a} = (a_0, a_1, \ldots, a_N)$ is a list of rational numbers, then $W(\mathbf{a})$ denotes the list $\left((-1)^N a_N, (-1)^N a_{N-1}, \ldots, (-1)^N a_0\right)$ of rational numbers. (Thus, the list $W(\mathbf{a})$ is obtained by reversing the list $\mathbf{a}$ and then multiplying each of its entries by $(-1)^N$.) Hence, W and B are two maps, each transforming lists into lists.

Prove that $B \circ W \circ B = W \circ B \circ W$ and $(B \circ W)^3 = \mathrm{id}$.

The equality $(B \circ W)^3 = \mathrm{id}$, spelt out in words, says that if we start with a list, apply the map W, apply the binomial transform, then apply the map W to the result, then again apply the binomial transform, then again apply the map W to the result, then apply the binomial transform once again, then we end up with the original list.

[1] The words "finite list", "tuple" and "finite sequence" mean the same thing. I only consider finite lists on this homework set.

0.2. Another recurrence

Exercise 3. Consider the sequence $(a_0, a_1, a_2, \ldots)$ of integers given by

$$a_0 = 2, \qquad a_1 = 20, \qquad a_n = 20a_{n-1} - 99a_{n-2} \qquad \text{for } n \geq 2.$$

Find an explicit formula for a_n.

[**Hint:** Use of generating functions is allowed. To solve Exercise 3 in the same way as I proved Binet's formula in class, partial fraction decomposition is needed. The Wikipedia examples can be useful.]

0.3. Counting permutations by descents

If σ is a permutation of $[n]$ for some $n \in \mathbb{N}$, then a *descent* of σ means an element $i \in [n-1]$ satisfying $\sigma(i) > \sigma(i+1)$. For example, the permutation σ of $[5]$ with $(\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)) = (3, 1, 4, 5, 2)$ has descents 1 (since $3 > 1$) and 4 (since $5 > 2$).

Exercise 4. Let n be a positive integer. How many permutations of $[n]$ have at most 1 descent?

(For example, the permutation σ of $[5]$ with $(\sigma(1), \sigma(2), \sigma(3), \sigma(4), \sigma(5)) = (1, 4, 2, 3, 5)$ has exactly 1 descent: namely, 2 is its only descent.)

0.4. Counting derangements squaring to the identity

Exercise 5. Let $n \in \mathbb{N}$. How many derangements σ of $[n]$ satisfy $\sigma^2 = \mathrm{id}$?

(For example, the derangement σ of $[6]$ sending $1, 2, 3, 4, 5, 6$ to $3, 6, 1, 5, 4, 2$ satisfies $\sigma^2 = \mathrm{id}$.)

[**Hint:** The answer will depend on whether n is even or odd.]

0.5. Iteration of maps on finite sets

The next two exercises study what happens if you apply a map from a finite set to itself several times.

Exercise 6. Let $n \in \mathbb{N}$. Let S be an n-element set. Let $f : S \to S$ be any map.

(a) Prove that $f^0(S) \supseteq f^1(S) \supseteq f^2(S) \supseteq \cdots$.

(b) Prove that $f^n(S) = f^k(S)$ for each integer $k \geq n$.

(c) Define a map $g : f^n(S) \to f^n(S)$ by

$$g(x) = f(x) \qquad \text{for each } x \in f^n(S).$$

(Thus, g is the restriction of f onto the image $f^n(S)$, regarded as a map from $f^n(S)$ to $f^n(S)$.)

Prove that g is well-defined (i.e., that $f(x)$ actually belongs to $f^n(S)$ for each $x \in f^n(S)$) and is a permutation of $f^n(S)$.

[Hint: For part **(b)**, first prove that there exists some $m \in \{0, 1, \ldots, n\}$ such that $f^m(S) = f^{m+1}(S)$. Then argue that $f^n(S) = f^{n+1}(S)$.]

Example 0.1. Let $n = 7$. Let $S = [7]$. Let $f : S \to S$ be the map with

$$(f(1), f(2), f(3), f(4), f(5), f(6), f(7)) = (4, 4, 5, 5, 2, 3, 3).$$

Then,

$$\begin{aligned} f^0(S) &= S = \{1, 2, 3, 4, 5, 6, 7\}; \\ f^1(S) &= f(S) = \{2, 3, 4, 5\}; \\ f^2(S) &= \{2, 4, 5\}; \\ f^k(S) &= \{2, 4, 5\} \qquad \text{for each } k \geq 2. \end{aligned}$$

Thus, in particular, $f^n(S) = \{2, 4, 5\}$. The map g is the permutation of this set $f^n(S) = \{2, 4, 5\}$ sending $2, 4, 5$ to $4, 5, 2$, respectively. It is perhaps worthwhile to draw the "cycle digraph" of f (which is not literally a cycle digraph, because f is not a permutation, but is constructed in the same way):

$$\begin{tikzcd} & 6 \arrow[dr] & & 7 \arrow[dl] \\ 1 \arrow[d] & & 3 \arrow[d] & \\ 4 \arrow[rr] & & 5 \arrow[dl] & \\ & 2 \arrow[ul] & & \end{tikzcd}.$$

Exercise 7. Let $n \in \mathbb{N}$. Let S be an n-element set. Let $f : S \to S$ be any map.

(a) If f is a permutation of S, then prove that there exists some $p \in [n!]$ such that $f^p = \mathrm{id}$.

(b) Prove in general (i.e., not only when f is a permutation) that there exist two integers u and v with $0 \leq u < v \leq n!$ and $f^u = f^v$.

[Hint: First prove part **(b)** in the case when f is a permutation (hint: what does the pigeonhole principle say about the permutations $f^0, f^1, \ldots, f^{n!}$?). Then, use this to show part **(a)**. Finally, prove part **(b)** in the general case, by applying part **(a)** to the map g from Exercise 6.]