Math 4707 Fall 2017 (Darij Grinberg): homework set 2
due date: Wednesday 4 Oct 2017 at the beginning of class
Please solve **at most 4** of the 7 exercises!

Exercise 1. Let $n \in \mathbb{N}$.

(a) Prove that

$$(2n-1) \cdot (2n-3) \cdot \cdots \cdot 1 = \frac{(2n)!}{2^n n!}.$$

(The left hand side is understood to be the product of all odd integers from 1 to $2n-1$.)

(b) Prove that

$$\binom{-1/2}{n} = \left(\frac{-1}{4}\right)^n \binom{2n}{n}.$$

(c) Prove that

$$\binom{-1/3}{n}\binom{-2/3}{n} = \frac{(3n)!}{(3^n n!)^3}.$$

Exercise 2. Let $n \in \mathbb{Q}$, $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

(a) Prove that every integer $j \geq a$ satisfies

$$\binom{n}{j}\binom{j}{a}\binom{n-j}{b} = \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{j-a}.$$

(b) Compute the sum $\sum_{j=a}^{n} \binom{n}{j}\binom{j}{a}\binom{n-j}{b}$ for every integer $n \geq a$. (The result should contain no summation signs.)

Recall the concept of lacunar sets, as defined in homework set 1. Recall also the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$ defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Exercise 4 **(c)** on homework set 1 told us that the number $g(n)$ of all lacunar subsets of $[n]$ is f_{n+2}. We shall now see more.

Exercise 3. Let $n \in \mathbb{N}$.

(a) For any $k \in \{0, 1, \ldots, n+1\}$, prove that the number of all lacunar k-element subsets of $[n]$ is $\binom{n-k+1}{k}$.

[Notice that this equals 0 whenever $2k > n+1$. You shouldn't need a separate argument for this case, but make sure you understand why the 0 is not surprising.]

(b) Conclude that

$$f_{n+2} = \sum_{k=0}^{n} \binom{n-k+1}{k}.$$

Recall that if $n \in \mathbb{N}$ and $k \in \mathbb{N}$, then $\operatorname{sur}(n,k)$ denotes the number of surjections $[n] \to [k]$. In class, we have shown the following two recursive formulas:

- We have
$$\operatorname{sur}(n,0) = [n=0] \qquad \text{for all } n \in \mathbb{N},$$
and
$$\operatorname{sur}(n,k) = \sum_{j=0}^{n-1} \binom{n}{j} \operatorname{sur}(j,k-1) \qquad \text{for all } n \in \mathbb{N} \text{ and } k > 0.$$

- We have
$$\operatorname{sur}(n,0) = [n=0] \qquad \text{for all } n \in \mathbb{N},$$
$$\operatorname{sur}(0,k) = [k=0] \qquad \text{for all } k \in \mathbb{N},$$
and
$$\operatorname{sur}(n,k) = k\left(\operatorname{sur}(n-1,k) + \operatorname{sur}(n-1,k-1)\right) \qquad \text{for all } n > 0 \text{ and } k > 0.$$

Exercise 4. Prove that
$$\operatorname{sur}(n,k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$

[**Hint:** Use one of the above recursions in the induction step. You are allowed to use the binomial formula
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$
which holds for any $x, y \in \mathbb{Q}$ and any $n \in \mathbb{N}$.]

Exercise 5. Let $n \in \mathbb{N}$. Let me call a permutation of $[n]$ *oddlike* if it sends every odd element of $[n]$ to an odd element of $[n]$. (For example, the permutation of $[5]$ sending $1,2,3,4,5$ to $3,4,5,2,1$ is oddlike.)

(a) Prove that any oddlike permutation of $[n]$ must also send every even element of $[n]$ to an even element of $[n]$.

(b) Find a formula for the number of oddlike permutations of $[n]$. [**Hint:** The answer may depend on the parity of n.]

Definition 0.1. Let S be a set. Let $f : S \to S$ be a map from S to S. Then, for every $k \in \mathbb{N}$, the map $f^k : S \to S$ is defined to be
$$\underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}.$$

For example, if $f : \mathbb{Z} \to \mathbb{Z}$ is the map $x \mapsto x^2$, then f^k is the map $x \mapsto \underbrace{\left(\left(\left(x^2\right)^2\right)\cdots\right)^2}_{k \text{ squarings}} = x^{\left(2^k\right)}$.

Note that $f^0 = \underbrace{f \circ f \circ \cdots \circ f}_{0 \text{ times}} = \operatorname{id}_S$, since a composition of no maps ("empty composition") is always understood as the identity map.

Exercise 6. Let S be a set. Let $f : S \to S$ be a map.

(a) Prove that $f^n \circ f^m = f^{n+m}$ for each $n, m \in \mathbb{N}$. [Yes, this is a one-liner; you don't need induction.]

(b) Let $g : S \to S$ be a further map such that $f \circ g = g \circ f$. Prove that $(f \circ g)^n = f^n \circ g^n$ for each $n \in \mathbb{N}$.

(c) Find an example in which the claim of **(b)** fails if we drop the assumption that $f \circ g = g \circ f$.

Exercise 7. Let n be a positive integer. Let X be a set.

We define a map $c : X^n \to X^n$ by

$$c\left(x_1, x_2, \ldots, x_n\right) = \left(x_2, x_3, \ldots, x_n, x_1\right) \qquad \text{for all } \left(x_1, x_2, \ldots, x_n\right) \in X^n.$$

(In other words, the map c transforms any n-tuple $\left(x_1, x_2, \ldots, x_n\right) \in X^n$ by "rotating" it one step to the left, or, equivalently, moving its first entry to the last position.)

(a) Prove that

$$c^k\left(x_1, x_2, \ldots, x_n\right) = \left(x_{k+1}, x_{k+2}, \ldots, x_n, x_1, x_2, \ldots, x_k\right)$$

for each $k \in \{0, 1, \ldots, n\}$ and each $\left(x_1, x_2, \ldots, x_n\right) \in X^n$. (Note that $\left(x_{k+1}, x_{k+2}, \ldots, x_n, x_1, x_2, \ldots, x_k\right)$ is to be understood as $\left(x_1, x_2, \ldots, x_n\right)$ if k equals either 0 or n.)

[**Note:** This might be intuitively clear – after all, if c rotates a tuple, then c^k rotates it k times, which causes its first k entries to move to the rightmost spot. But the point is to give a rigorous proof. Induction is recommended.]

(b) Let $\mathbf{x} = \left(x_1, x_2, \ldots, x_n\right)$ be an n-tuple in X^n. A nonnegative integer k is said to be a *period* of $\mathbf{x}$ if it satisfies $c^k\left(\mathbf{x}\right) = \mathbf{x}$. (For example, 0 is always a period of $\mathbf{x}$. For another example, the periods of the 6-tuple $(1, 4, 2, 1, 4, 2)$ are $0, 3, 6, 9, \ldots$.)

Prove that if p and q are two periods of $\mathbf{x}$ satisfying $p \geq q$, then $p - q$ is also a period of $\mathbf{x}$.

(c) Let m be the smallest nonzero period of the n-tuple $\mathbf{x} \in X^n$. Prove that m divides any period of $\mathbf{x}$.

(d) Conclude that m divides n.