

Fall 2017, MATH 4707 (UMN) Homework 1 solutions

Mark Richard

To begin, I want to show the following fact, for the purpose of using it in this homework.

Lemma 1. *Let $n \geq 1$. The Fibonacci number, f_n , counts the tilings of a $2 \times (n - 1)$ board with dominoes.*

Proof of Lemma 1. We shall prove this by induction over n . First, suppose that $n = 1$. Then we have a 2×0 board, and there is exactly 1 way to tile it. Suppose we have $n = 2$. Then, we have a 2×1 board, and again there is exactly one way to tile it: with a single vertical domino. This provides the basis, where $f_1 = f_2 = 1$. Now, we need to show the recurrence relation.

Suppose that f_{n-1} and f_{n-2} count the tilings on a $2 \times (n - 2)$ and $2 \times (n - 3)$ board respectively. Now, consider a $2 \times (n - 1)$ board. For each tiling, we can either have a vertical domino in the first column, or a horizontal domino that covers the first and second columns. In the first case, we just have a $2 \times (n - 2)$ board remaining. In the second case, we have a $2 \times (n - 3)$ board remaining. Thus, the number of ways to tile the $2 \times (n - 1)$ board, f_n , is just $f_{n-1} + f_{n-2}$. Hence, the number of tilings do indeed follow the Fibonacci numbers. \square

I will use Lemma 1 twice: in problem 4 and problem 5.

1) a. Suppose that \mathcal{A} and \mathcal{B} are two equivalent logical statements. Then, if \mathcal{A} is true, so is \mathcal{B} . Similarly, if \mathcal{A} is false, so is \mathcal{B} . Thus, if $[\mathcal{A}] = 1$ we must have $[\mathcal{B}] = 1$; and if $[\mathcal{A}] = 0$, we also have $[\mathcal{B}] = 0$. Hence, $[\mathcal{A}] = [\mathcal{B}]$.

b. Let \mathcal{A} be any logical statement. First, suppose \mathcal{A} is true, so $[\mathcal{A}] = 1$. Then not \mathcal{A} is false, so $[\text{not } \mathcal{A}] = 0 = 1 - [\mathcal{A}]$. Similarly, suppose \mathcal{A} is false, so $[\mathcal{A}] = 0$. Then not \mathcal{A} is true, so $[\text{not } \mathcal{A}] = 1 = 1 - [\mathcal{A}]$. So, $[\text{not } \mathcal{A}] = 1 - [\mathcal{A}]$.

c. For any two logical statements \mathcal{A} and \mathcal{B} , $\mathcal{A} \wedge \mathcal{B}$ is true if and only if both statements are true, otherwise it is false. If both are true, then $[\mathcal{A} \wedge \mathcal{B}] = 1$; otherwise it is 0. Now, $[\mathcal{A}][\mathcal{B}] = 1$ if and only if $[\mathcal{A}] = 1$ and $[\mathcal{B}] = 1$. Otherwise, the expression is 0. But then \mathcal{A} and \mathcal{B} are both true. Hence, $[\mathcal{A} \wedge \mathcal{B}] = [\mathcal{A}][\mathcal{B}]$.

d. We will do this in a brute force way. First, suppose that \mathcal{A} and \mathcal{B} are both false. Then, $[\mathcal{A} \vee \mathcal{B}] = 0$. Also, $[\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}] = 0 + 0 + 0 = 0$. Without loss of generality, suppose \mathcal{A} is true, and \mathcal{B} is false. Then, $[\mathcal{A} \vee \mathcal{B}] = 1$, and $[\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}] = 1 + 0 - 0 = 1$. (This is a symmetric equation, so we get the same for false \mathcal{A} and true \mathcal{B} .) Finally, suppose both \mathcal{A} and \mathcal{B} are true. Then, $[\mathcal{A} \vee \mathcal{B}] = 1$, and we have $[\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}] = 1 + 1 - 1 = 1$. Hence, for any pairing we have that $[\mathcal{A} \vee \mathcal{B}] = [\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}][\mathcal{B}]$.

e. Due to the symmetry in the statement, we just need to prove this for (i) no true statements; (ii) exactly 1 true statement; (iii) exactly 2 true statements; (iv) all true statements. If none of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are true, then the left side evaluates to 0. Also, each term on the right side evaluates to 0, so these are equivalent when there are no true statements.

Now, suppose there is exactly one true statement, such as \mathcal{A} , while the other two are false. Then, the left side evaluates to 1. The right hand side has $[\mathcal{A}] = 1$, but all other terms evaluate to 0. So, the two are equivalent when there is exactly 1 true statement.

Now, suppose there are exactly two true statements, say \mathcal{A} and \mathcal{B} . The left side again evaluates to 1. The right side has $[\mathcal{A}] = [\mathcal{B}] = [\mathcal{A}][\mathcal{B}] = 1$, and all others evaluate to 0. So, the right hand side is equal to $1 + 1 - 1 = 1$, and so the two sides are equivalent.

Finally, suppose all three statements are true. Then, the left hand side gives 1, and each term in the right hand side also evaluates to 1. So, the right hand side is $1 + 1 + 1 - 1 - 1 - 1 + 1 = 1$. So, the two equations are indeed equivalent for any three logical statements.

2) a. Let $n \in \mathbb{Q}$ and $k \in \mathbb{N}$. Then, we have

$$\begin{aligned} (-1)^k \binom{k-n-1}{k} &= (-1)^k \frac{(k-n-1)(k-n-2) \cdots ((k-n-1)-k+2)((k-n-1)-k+1)}{k!} \\ &= \frac{(k-n-1)(k-n-2) \cdots (-n+1)(-n)}{k!} \end{aligned}$$

Now, the product in the numerator has exactly k terms. If we factor out a -1 from each term, we get

$$\begin{aligned} (-1)^k \binom{k-n-1}{k} &= (-1)^k \cdot (-1)^k \frac{(n-k+1)(n-k+2) \cdots (n-1)(n)}{k!} \\ &= \frac{n(n-1) \cdots (n-k+2)(n-k+1)}{k!} = \binom{n}{k} \end{aligned}$$

So, we have equality.

b. Expanding out $n \binom{n-1}{k-1}$, we get

$$\begin{aligned} n \binom{n-1}{k-1} &= \frac{n(n-1)(n-2) \cdots ((n-1)+1-(k-1))}{(k-1)!} \\ &= \frac{n(n-1)(n-2) \cdots (n-k+1)}{(k-1)!} \end{aligned}$$

If we multiply by k/k , we get

$$\begin{aligned} n \binom{n-1}{k-1} &= \frac{n(n-1)(n-2) \cdots (n-k+1)}{(k-1)!} \cdot \frac{k}{k} \\ &= k \frac{n(n-1) \cdots (n-k+1)}{k!} = k \binom{n}{k} \end{aligned}$$

So, we have equality.

c. We first have

$$\begin{aligned} \binom{n}{b} \binom{n-b}{a-b} &= \frac{n(n-1) \cdots (n-b+1)}{b!} \cdot \frac{(n-b)(n-b-1) \cdots (n-b+1-(a-b))}{(a-b)!} \\ &= \frac{n(n-1) \cdots (n-b+1)}{b!} \cdot \frac{(n-b)(n-b-1) \cdots (n-a+1)}{(a-b)!} \end{aligned}$$

We assume that $a \geq b$, which implies that $n-a+1$ is less than all terms before it. Moreover, the two numerators together have all integers from n to $n-a+1$ multiplied together. We can multiply by $a!$ in the numerator and denominator, and we get

$$\begin{aligned} \binom{n}{b} \binom{n-b}{a-b} &= \frac{n(n-1) \cdots (n-b+1)}{b!} \cdot \frac{(n-b)(n-b-1) \cdots (n-a+1)}{(a-b)!} \cdot \frac{a!}{a!} \\ &= \frac{n(n-1) \cdots (n-a+1)}{a!} \cdot \frac{a!}{b!(a-b)!} \\ &= \binom{n}{a} \binom{a}{b} \end{aligned}$$

where the final form of $\binom{a}{b}$ is acceptable since a and b are nonnegative integers. So, we have equality.

3) a. A k digit number has 9 possible digits for the leading digit, then 10 possible for each of the other $k - 1$ spots. In total, this gives $9 \cdot 10^{k-1}$ numbers.

b. Since there are only 10 possible digits, if $k > 10$ there are no numbers that have no repeated digits. So, suppose $k \leq 10$. The first digit has 9 choices, the second digit also has 9 choices, the third digit has 8, and so on. This is given by the expression

$$9 \cdot \frac{9!}{(9 - k + 1)!}, \quad k \leq 10$$

We can quickly check that for $k = 1$, we have 9 different numbers, and for $k = 10$ we have $9 \cdot 9!$ numbers, as expected.

c. We consider an equivalence relation over the set of k -digit numbers, such that $m \sim n$ if they have all digits the same, except for the units digit. For example, if $k = 3$, we can denote an equivalence class that starts with 12 as

$$12X = \{120, 121, 122, \dots, 129\}$$

This equivalence relation partitions the k -digit numbers (there are $9 \cdot 10^{k-1}$ of them) into $9 \cdot 10^{k-2}$ classes. For a given class, the sum of the first $k - 1$ digits is fixed and is either even or odd. If it is even, then the numbers in the class ending with 0, 2, 4, 6, or 8 will have an even digit sum, while those ending in 1, 3, 5, 7, or 9 will have an odd digit sum. Similarly, if the sum of the first $k - 1$ digits is odd, those ending in an odd number will have a total digit sum that is odd, and the rest are even. In each case, we see that exactly 5 of the numbers will have an even digit sum, and 5 will have an odd digit sum. Since half of the numbers in each of the classes have an even digit sum, we must have that half of all k -digit numbers have an even digit sum. So, there are $\frac{9}{2} \cdot 10^{k-1}$ k -digit numbers with an even digit sum. We note that if 0 is not considered a number (since it has a “leading zero”), this only works for $2 \leq k \leq 10$.

d. Suppose that k is even. We will label each digit in numerically increasing order, so the units digit is labeled 1, the tens digit is 2, and so on until the digit labeled k . Then, a number is a palindrome if the digit in the i spot is the same as the digit in the $k + 1 - i$ spot, for $i \leq k/2$. In other words, a palindrome with an even number of digits is uniquely determined by the first $k/2$ digits (and since k is even, we can divide evenly by 2). Thus, the number of k -digit palindromes is the same as the number of $k/2$ -digit numbers; specifically,

$$9 \cdot 10^{k/2-1}, \text{ even } k$$

Now, suppose k is odd. We have the same criteria for a palindrome as before, except i ranges up to $\lfloor k/2 \rfloor = (k - 1)/2$. However, for an odd numbered palindrome we have an extra digit in the $1 + (k - 1)/2$ place, which can be any of 10 digits. So, we need to count the $(k - 1)/2$ digit numbers, but for each of these we have 10 options for an extra digit. So, we have the number of palindromes is

$$90 \cdot 10^{-1+(k-1)/2}, \text{ odd } k$$

We note that if 0 counts as a number, we would need to add 1 more for $k = 1$.

4) a. We do this via brute force. For $n = 1$, we have the sets $\{1\}, \emptyset$, so $g(1) = 2$.

For $n = 2$, we have the sets $\{1\}, \{2\}, \emptyset$, so $g(2) = 3$.

For $n = 3$, we have the sets $\{1\}, \{2\}, \{3\}, \{1, 3\}, \emptyset$, so $g(3) = 5$.

For $n = 4$, we have the sets $\{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \emptyset$, so $g(4) = 8$.

For $n = 5$, we have the 8 sets from $n = 4$, as well as $\{5\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{1, 3, 5\}$. So, $g(5) = 13$.

b./c. I am doing both of these parts as essentially the same problem. The reason being that my proof of the recursive relationship I have for $g(n)$, namely that $g(n) = g(n - 1) + g(n - 2)$, is showing a bijection between $g(n)$ and the tilings of a $2 \times n$ board. This, in turn, shows the relationship between $g(n)$ and f_n (due to Lemma 1).

I claim that $g(n - 1)$ is precisely the number of tilings on a $2 \times n$ board via dominoes. Consider a $2 \times n$ board, such that each column is numbered 1 to n .

1	2	3	4	5

Then, for a given lacunar set, each number i in the set corresponds to a domino lying horizontally on the board, with its left square in the i -th column. We can see this works, since a horizontal domino on the top row requires a horizontal tile on the bottom row. For example, we have $n = 5$ on the board being shown, and so we consider $g(4)$. One subset is $\{2, 4\}$, and this is a valid tiling of the board as shown below.

1	2	3	4	5
	==	==	==	==
	==	==	==	==

More specifically, since tiling is symmetric in the two rows, a tiling just requires us to specify what columns have dominoes across them horizontally (because then, each other column must have a single domino in it vertically). Since a domino lying horizontally takes up exactly two columns, if we specify column i as having a domino horizontally across it (specifically the “left” end of it), this implies that $i + 1$ also has a horizontal domino, and thus cannot have a vertical domino. But, this is specifically what a lacunar subset is: if i is in the set, then $i + 1$ is not. Hence, the lacunar subsets of $[n - 1]$ count the tilings of a $2 \times n$ board (since column n cannot have the left end of a horizontal domino: either it has a vertical domino, or $n - 1$ has the left end of a horizontal domino).

Now, to specifically show the recurrence relation of $g(n) = g(n - 1) + g(n - 2)$, we recall that $g(n)$ counts the number of tilings in a $2 \times (n + 1)$ board. Consider a vertical domino in the column 1. Then, we need to tile the remaining $2 \times n$ board, which can be done in just $g(n - 1)$ ways. The only other possibility is that we have horizontal dominoes in columns 1 and 2; in this case there is a $2 \times (n - 1)$ board remaining, which can be tiled in $g(n - 2)$ ways. So, $g(n) = g(n - 1) + g(n - 2)$.

Lemma 1 shows that the number of tilings of a $2 \times n$ board is given by f_{n+1} ; but this is also given by $g(n - 1)$. Hence, we must have $g(n) = f_{n+2}$ for $n \in \mathbb{N}$.

5) a. By the definition of $a(n)$, we have that $a(n - 1)$ counts all $O < E < O < E \dots$ subsets of $[n - 1]$. Specifically, these are all $O < E < O < E \dots$ subsets of $[n]$ that do *not* contain n . Then, $b(n)$ counts all such subsets of $[n]$ that *do* contain n . So, we partition the $O < E < O < E \dots$ subsets of $[n]$ into these two cases. Their total (by definition) is $a(n)$, and thus we have $a(n) = a(n - 1) + b(n)$.

g. I am doing this out of order, since my proof requires first showing that $a(n) = f_{n+2}$, and this will then imply that $a(n - 2) = b(n)$. We claim there is a bijection between $a(n)$ and the tilings of a $2 \times (n + 1)$ board with dominoes. Specifically, the $O < E < O < E \dots$ sets counted by $a(n)$ denote the dominoes in each tiling that are *vertical* (as opposed to $g(n)$ in the previous problem, which expressed the horizontal dominoes). Suppose we have an $O < E < O < E \dots$ subset of $[n]$, $\{s_1, s_2, \dots, s_k\}$. We place a vertical domino in each column labeled

s_i , and fill in all horizontal dominoes that will fit, moving left to right. If there are single columns remaining, they are filled with vertical dominoes. However, because we fill in on the left, the only column that can remain without a vertical domino is the $n + 1$ column. For example, consider $n = 4$, so we have a 2×5 board. Then, the $O < E < O < E \dots$ subset $\{1, 4\}$ has vertical dominoes in 1 and 4, and a horizontal domino that spans 2 and 3. This leaves column 5 to have a vertical domino.

To explain why this works, we consider what must occur in an $O < E < O < E \dots$ subset. The first element must be odd. If the first element is 1, there is a vertical domino placed there. Otherwise, we have horizontal dominoes until we reach the first odd number in the subset. The second number (if there is one), is even. Thus, it is either the column adjacent to the first number, or there is an even number of spaces between the first vertical domino and the next space. This is because the number of spaces *between* an odd and even number, is even. (Between 1 and 4, there is 2 and 3). Hence, between each column we either have another vertical domino, or horizontal dominoes filling in the space. For the empty set, we either have the entire board filled with horizontal dominoes (if $n + 1$ is even), or we have only the $n + 1$ column with a vertical domino. So, $a(n)$ counts the tilings of a $2 \times (n + 1)$ board, which is also counted by f_{n+2} (by Lemma 1); therefore, $a(n) = f_{n+2}$.

f. Since $a(n) = f_{n+2}$, we know that $a(n)$ must satisfy the recurrence relation $a(n) = a(n - 1) + a(n - 2)$. However, from (a) we know that $a(n) = a(n - 1) + b(n)$. Comparing these two equations, we see that $b(n) = a(n - 2)$.