Math 4707 Fall 2017 (Darij Grinberg): homework set 1 [corrected 24 Sep 2017]
due date: Wednesday 27 Sep 2017 at the beginning of class, or before that by email or moodle
Please solve **at most 5** of the 7 exercises!

Definition 0.1. Let $\mathcal{A}$ be a logical statement. Then, an element $[\mathcal{A}] \in \{0,1\}$ is defined as follows: We set $[\mathcal{A}] = \begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$. This element $[\mathcal{A}]$ is called the *truth value* of $\mathcal{A}$. (For example, $[1+1=2] = 1$ and $[1+1=3] = 0$.) The notation $[\mathcal{A}]$ for the truth value of $\mathcal{A}$ is known as the *Iverson bracket notation*.

Exercise 1. Prove the following rules of truth values:
(a) If $\mathcal{A}$ and $\mathcal{B}$ are two equivalent logical statements, then $[\mathcal{A}] = [\mathcal{B}]$.
(b) If $\mathcal{A}$ is any logical statement, then $[\text{not } \mathcal{A}] = 1 - [\mathcal{A}]$.
(c) If $\mathcal{A}$ and $\mathcal{B}$ are two logical statements, then $[\mathcal{A} \wedge \mathcal{B}] = [\mathcal{A}]\,[\mathcal{B}]$.
(d) If $\mathcal{A}$ and $\mathcal{B}$ are two logical statements, then $[\mathcal{A} \vee \mathcal{B}] = [\mathcal{A}] + [\mathcal{B}] - [\mathcal{A}]\,[\mathcal{B}]$.
(e) If $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are three logical statements, then

$$[\mathcal{A} \vee \mathcal{B} \vee \mathcal{C}] = [\mathcal{A}] + [\mathcal{B}] + [\mathcal{C}] - [\mathcal{A}]\,[\mathcal{B}] - [\mathcal{A}]\,[\mathcal{C}] - [\mathcal{B}]\,[\mathcal{C}] + [\mathcal{A}]\,[\mathcal{B}]\,[\mathcal{C}].$$

Definition 0.2. We define the *binomial coefficient* $\binom{n}{k}$ by

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

for every $n \in \mathbb{Q}$ and $k \in \mathbb{N}$. (Recall that $\mathbb{N} = \{0,1,2,\ldots\}$, and that an empty product is defined to be 1.)

For example, $\binom{-3}{4} = \frac{(-3)(-4)(-5)(-6)}{4!} = 15$ and $\binom{4}{1} = \frac{4}{1!} = 4$ and $\binom{4}{0} = \frac{(\text{empty product})}{0!} = \frac{1}{1} = 1$.

Exercise 2. Prove the following:
(a) We have $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$ for any $n \in \mathbb{Q}$ and $k \in \mathbb{N}$.
(b) We have $k\binom{n}{k} = n\binom{n-1}{k-1}$ for any $n \in \mathbb{Q}$ and any positive integer k.
(c) If $n \in \mathbb{Q}$ and if a and b are two integers such that $a \geq b \geq 0$, then

$$\binom{n}{a}\binom{a}{b} = \binom{n}{b}\binom{n-b}{a-b}.$$

[**Caveat:** You may have seen the formula $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$. But this formula only makes sense when n and k are nonnegative integers and $n \geq k$. Thus it is not general enough to be used in this exercise.]

Exercise 3. Let k be a positive integer.

(a) How many k-digit numbers are there? (A "k-digit number" means a nonnegative integer that has k digits without leading zeroes. For example, 3902 is a 4-digit number, not a 5-digit number. Note that 0 counts as a 0-digit number, not as a 1-digit number.)

(b) How many k-digit numbers are there that have no two equal digits?

(c) How many k-digit numbers have an even sum of digits?

(d) How many k-digit numbers are palindromes? (A "*palindrome*" is a number such that reading its digits from right to left yields the same number. For example, 5 and 1331 and 49094 are palindromes. Your answer may well depend on the parity of k.)

For each $n \in \mathbb{N}$, we set $[n] = \{1, 2, \ldots, n\}$.

Definition 0.3. The *Fibonacci sequence* is the sequence $(f_0, f_1, f_2, \ldots)$ of integers which is defined recursively by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Its first terms are

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 3, \quad f_5 = 5,$$
$$f_6 = 8, \quad f_7 = 13, \quad f_8 = 21, \quad f_9 = 34, \quad f_{10} = 55,$$
$$f_{11} = 89, \quad f_{12} = 144, \quad f_{13} = 233.$$

(Some authors prefer to start the sequence at f_1 rather than f_0; of course, the recursive definition then needs to be modified to require $f_2 = 1$ instead of $f_0 = 0$.)

Exercise 4. A set S of integers is said to be *lacunar* if no two consecutive integers occur in S (that is, there exists no $i \in \mathbb{Z}$ such that both i and $i+1$ belong to S). For example, $\{1, 3, 6\}$ is lacunar, but $\{2, 4, 5\}$ is not. (The empty set and any 1-element set are lacunar, of course.)

For a positive integer n, let $g(n)$ denote the number of all lacunar subsets of $[n]$.

(a) Compute $g(n)$ for all $n \in \{1, 2, 3, 4, 5\}$.

(b) Find and prove a recursive formula for $g(n)$ in terms of $g(n-1)$ and $g(n-2)$.

(c) Prove that $g(n) = f_{n+2}$ for each $n \in \mathbb{N}$.

Recall that if a, b and m are three integers (with $m > 0$), then we write $a \equiv b \mod m$ if and only if $a - b$ is divisible by m. Thus, in particular, $a \equiv b \mod 2$ if and only if a and b have the same parity (i.e., are either both even or both odd).

Exercise 5. A set S of integers is said to be *O<E<O<E<...* (this is an adjective) if it can be written in the form $S = \{s_1, s_2, \ldots, s_k\}$ where

- $s_1 < s_2 < \cdots < s_k$;
- the integer s_i is even whenever i is even;
- the integer s_i is odd whenever i is odd.

(For example, $\{1,4,5,8,11\}$ is an O<E<O<E<... set, while $\{2,3\}$ and $\{1,4,6\}$ are not. Note that k is allowed to be 0, whence $\varnothing$ is an O<E<O<E<... set.)

For each $n \in \mathbb{N}$, we let $a(n)$ denote the number of all O<E<O<E<... subsets of $[n]$, and let $b(n)$ denote the number of all O<E<O<E<... subsets of $[n]$ that contain n.

(a) Show that $a(n) = a(n-1) + b(n)$ for each $n > 0$.

(b) Show that $a(n) = 1 + \sum\limits_{k=0}^{n} b(k)$ for each $n \in \mathbb{N}$.

(c) Show that $b(n) = \sum\limits_{\substack{k \in \{0,1,\ldots,n-1\}; \\ k \equiv n-1 \mod 2}} b(k) + [n \text{ is odd}]$ for each $n \in \mathbb{N}$.

(d) Show that $b(n) + b(n-1) = 1 + \sum\limits_{k=0}^{n-1} b(k)$ for each $n > 0$.

(e) Show that $b(n) = 1 + \sum\limits_{k=0}^{n-2} b(k)$ for each $n > 0$.

(f) Show that $b(n) = a(n-2)$ for each $n \geq 2$.

(g) Show that $a(n) = f_{n+2}$ for each $n \in \mathbb{N}$.

[**Hint:** You may skip parts **(b)–(e)** if you can prove part **(f)** without using any of them.]

Remark 0.4. Comparing Exercise 4 **(c)** with Exercise 5 **(g)** tells us that there are precisely as many lacunar subsets of $[n]$ as there are O<E<O<E<... subsets of $[n]$. Is there a bijection between the former and the latter? I don't know.

Exercise 6. For each $n \in \mathbb{N}$, we let $c(n)$ denote the number of all subsets of $[n]$ that are simultaneously lacunar and O<E<O<E<....

Prove that $c(n) = c(n-2) + c(n-3)$ for all $n \geq 3$.

Remark 0.5. The sequence $(c(0), c(1), c(2), c(3), \ldots)$ from Exercise 6 is the *Padovan sequence* (starting with $1,2,2,3,4,5,7,9,12,16,21,28,37,49$).

Exercise 7. Extend the "twelvefold way" by a new column: counting only the bijective maps $f : N \to K$. Fill in this column (all of its four boxes).

Appendix

Here is a sample exercise (no points for this one...) with a solution. This should give you some idea of what level of detail I expect in your solutions.

> **Exercise 8.** A set S of integers is said to be *self-counting* if the size $|S|$ is an element of S. (For example, $\{1,3,5\}$ is self-counting, since $|\{1,3,5\}| = 3 \in \{1,3,5\}$; but $\{1,2,5\}$ is not self-counting.)
>
> Let n be a positive integer.
>
> **(a)** For each $k \in [n]$, show that the number of self-counting subsets of $[n]$ having size k is $\binom{n-1}{k-1}$.
>
> **(b)** Conclude that the number of self-counting subsets of $[n]$ is $\sum_{k=0}^{n-1} \binom{n-1}{k}$.
>
> **(c)** Find and prove a simpler expression for this number.

Solution to Exercise 8. **(a)** Fix $k \in [n]$. Then, the self-counting subsets of $[n]$ having size k are exactly the subsets of $[n]$ having size k and containing k. Thus, the maps

$$\{\text{self-counting subsets of } [n] \text{ having size } k\} \to \{\text{subsets of } [n] \setminus \{k\} \text{ having size } k-1\},$$
$$S \mapsto S \setminus \{k\}$$

and

$$\{\text{subsets of } [n] \setminus \{k\} \text{ having size } k-1\} \to \{\text{self-counting subsets of } [n] \text{ having size } k\},$$
$$S \mapsto S \cup \{k\}$$

are well-defined[1], mutually inverse[2], and thus are bijections. Hence,

$$\begin{aligned}
& \left|\left\{\text{self-counting subsets of } [n] \text{ having size } k\right\}\right| \\
&= \left|\left\{\text{subsets of } [n] \setminus \{k\} \text{ having size } k-1\right\}\right| \\
&= \binom{\left|[n] \setminus \{k\}\right|}{k-1} \\
&\qquad \left(\begin{array}{c}\text{because for any finite set } Q \text{ and any } m \in \mathbb{N}\text{, we have} \\ \left|\left\{\text{subsets of } Q \text{ having size } m\right\}\right| = \dbinom{|Q|}{m}\end{array}\right) \\
&= \binom{n-1}{k-1} \qquad \left(\text{since } \left|[n] \setminus \{k\}\right| = n-1\right).
\end{aligned}$$

This proves part **(a)**.

(b) Any self-counting subset of $[n]$ must have at least one element (namely, its size); thus, its size must be one of the integers $1, 2, \ldots, n$. Hence,

$$\begin{aligned}
\left|\left\{\text{self-counting subsets of } [n]\right\}\right| &= \sum_{k=1}^{n} \underbrace{\left|\left\{\text{self-counting subsets of } [n] \text{ having size } k\right\}\right|}_{\substack{=\binom{n-1}{k-1} \\ \text{(by part (a))}}} \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k}
\end{aligned}$$

(here, we have substituted k for $k-1$ in the sum). This proves part **(b)**.

[1]This means the following:

- If S is a self-counting subset of $[n]$ having size k, then $S \setminus \{k\}$ is a subset of $[n] \setminus \{k\}$ having size $k-1$.
- If S is a subset of $[n] \setminus \{k\}$ having size $k-1$, then $S \cup \{k\}$ is a self-counting subset of $[n]$ having size k.

Checking this is straightforward; you can do it in your head, but don't forget to do this! If you don't check well-definedness, then it may happen that one of your "maps" does not exist; for example, convince yourself that there is no map

$$\begin{aligned}
\{\text{subsets of } [n]\} &\to \{\text{subsets of } [n]\}, \\
S &\mapsto S \cup \{|S|+1\},
\end{aligned}$$

because the set $S \cup \{|S|+1\}$ is not always a subset of $[n]$ (namely, it fails to be so when $|S| = n$).

[2]For this, you need to show that

- If S is a self-counting subset of $[n]$ having size k, then $(S \setminus \{k\}) \cup \{k\} = S$.
- If S is a subset of $[n] \setminus \{k\}$ having size $k-1$, then $(S \cup \{k\}) \setminus \{k\} = S$.

This is again entirely straightforward, and it is perfectly fine to do this in your head, but you should do it.

(c) This number is 2^{n-1}.

Proof. In light of part **(b)**, it suffices to show that

$$\sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}. \tag{1}$$

In order to do so, it suffices to prove the identity

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m \qquad \text{for all } m \in \mathbb{N} \tag{2}$$

(because we can then apply (2) to $m = n - 1$, and obtain (1)).

The identity (2) is well-known (it says that the sum of all entries in the m-th row of Pascal's triangle is 2^m), but let us sketch a quick combinatorial proof: The number of all subsets of $[m]$ is 2^m (because to choose such a subset means to decide, for each element of $[m]$, whether it goes into the subset or not; thus, we have 2 choices for each element, and m elements, whence there is a total of 2^m possibilities). On the other hand, this number equals

$$\sum_{k=0}^{m} \underbrace{(\text{the number of all } k\text{-element subsets of } [m])}_{=\binom{m}{k}} = \sum_{k=0}^{m} \binom{m}{k}.$$

Comparing the two results, we obtain $\sum_{k=0}^{m} \binom{m}{k} = 2^m$ (because both results are the same number – viz., the number of all subsets of $[m]$). Thus, (2) is proven, and the proof of part **(c)** is thus complete. $\square$