University of Minnesota, School of Mathematics

Math 4707: Combinatorics, Fall 2017
Homework 0

Darij Grinberg

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1 Exercise 1

1.1 Problem

Let $G = (V, E)$ be a simple graph. If $k \in \mathbb{N}$, then a *k-coloring* of G means a map $f : V \to \{1, 2, \ldots, k\}$. If f is a k-coloring of G for some $k \in \mathbb{N}$, then the value $f(v)$ of f at a vertex $v \in V$ is called the *color* of v in the k-coloring f. (It is customary to visualize a k-coloring by pretending that the numbers $1, 2, \ldots, k$ are colors, and so a k-coloring assigns to each vertex a color; i.e., it "colors" the vertices.)

Prove that there exists a 2-coloring of G having the following property: For each vertex $v \in V$, at most $\frac{1}{2} \deg v$ among the neighbors of v have the same color as v.

Remark 1.1. This problem is often restated as follows: You are given a (finite) set of politicians; some politicians are mutual enemies. (No one is their own enemy. If u is an enemy of v, then v is an enemy of u. An enemy of an enemy is not necessarily a friend. So this is just a simple graph.) Prove that it is possible to subdivide this set into two (disjoint) parties such that no politician has more than half of his enemies in his own party.

1.2 Solution

The idea of the solution is fairly simple, particularly using the convenient language of Remark 1.1: We subdivide our set of politicians into two parties in some arbitrary way (e.g., by throwing them all into one party and keeping the other party empty), and then we improve the situation step by step by picking a politician who has more than half his enemies in his own party, and moving him to the opposite party. However, will this procedure necessarily terminate, or will it result in an eternal process of politicians being kicked around back and forth between the two parties? Fortunately, it will terminate, but this needs to be proven.

Let us first make this procedure rigorous:

> **Algorithm 1.2. Input:** a simple graph $G = (V, E)$.
>
> **Output:** a 2-coloring of G having the following property: For each vertex $v \in V$, at most $\dfrac{1}{2} \deg v$ among the neighbors of v have the same color as v.
>
> 1. Define a 2-coloring $f : V \to \{1, 2\}$ arbitrarily (for example, by setting $f(v) = 1$ for all $v \in V$).
> 2. **While** there exists some $v \in V$ such that more than $\dfrac{1}{2} \deg v$ among the neighbors of v have the same color as v, **do** the following:
>   - Flip the color $f(v)$ of this v (that is, change this color to 2 if it is 1, and change it to 1 if it is 2).
> 3. Output the 2-coloring f.

It is clear that if step 3 of this algorithm is ever reached, then the 2-coloring f outputted by the algorithm does have the property that for each vertex $v \in V$, at most $\dfrac{1}{2} \deg v$ among the neighbors of v have the same color as v. (Indeed, this property is saying precisely that we have fallen out of the while-loop in step 2.) But we need to show that step 3 will eventually be reached. This is not obvious, since step 2 contains a while-loop, and a while-loop may go on indefinitely. We need to prove that this while-loop must eventually come to an end.

The most common way to prove such a claim is by exhibiting a *loop monovariant*. In our situation, this means a nonnegative integer defined for each 2-coloring f and which has the property that in every iteration of the while-loop, this integer decreases (strictly). If we can define such an integer, then we can immediately conclude that the while-loop must eventually come to an end (because a nonnegative integer cannot keep decreasing indefinitely while remaining a nonnegative integer).

The role of this nonnegative integer (the loop monovariant) will be played by what I call the "enmity" of a 2-coloring f:

If f is a 2-coloring of G, then I define the *enmity* of f to be the number of f-monochromatic edges of G. Here, an edge e of G is said to be *f-monochromatic* if the two endpoints of e have the same color in the 2-coloring f.

For each 2-coloring f of G, the enmity of f is clearly a nonnegative integer. Now, I claim the following:

> *Claim 1:* Let f be a 2-coloring of G. Let $v \in V$ be a vertex such that more than $\dfrac{1}{2} \deg v$ among the neighbors of v have the same color as v. If we flip the color $f(v)$ of v, then the enmity of f **decreases**.

Proof of Claim 1. Let a be the number of neighbors of v that have the same color as v. Then, $a > \dfrac{1}{2} \deg v$ (since more than $\dfrac{1}{2} \deg v$ among the neighbors of v have the same color as v). Hence, $2a > \deg v$, so that $a > \deg v - a$.

Clearly, an edge e containing v is monochromatic[1] if and only if the neighbor of v contained in e has the same color as v. Thus, the monochromatic edges containing v are in one-to-one correspondence with the neighbors of v that have the same color as v. Since the number of the latter neighbors is a, we thus conclude that the number of the former edges is also a. In other words, among the $\deg v$ edges containing v, exactly a are monochromatic. If we flip the color $f(v)$ of v, then these a monochromatic edges become no longer monochromatic (because they now connect vertices of different color), whereas the remaining $\deg v - a$ edges containing v become monochromatic. As for the edges that do not contain v, their status does not change (i.e., if they are monochromatic before the flipping, then they remain so, and if they are not monochromatic before the flipping, then they do not become monochromatic), because none of their vertices changes its color. Hence, by flipping the color $f(v)$ of v, we lose a monochromatic edges but, at the same time, gain $\deg v - a$ new monochromatic edges. Since $a > \deg v - a$, we thus lose more monochromatic edges than we gain. Therefore, the number of monochromatic edges of G decreases. In other words, the enmity of f decreases (since the enmity of f is defined as the number of monochromatic edges of G). This proves Claim 1. □

Claim 1 shows that the enmity of f decreases in each iteration of the while-loop in Algorithm 1.2. Hence, this while-loop cannot go on indefinitely (since the enmity of f is a nonnegative integer, and thus cannot keep decreasing forever). Thus, Algorithm 1.2 must eventually terminate (since the while-loop in step 2 is the only part in which the algorithm might get stuck). As we already know, the 2-coloring f outputted by the algorithm does have the property that for each vertex $v \in V$, at most $\frac{1}{2} \deg v$ among the neighbors of v have the same color as v. As a consequence, such a 2-coloring exists. This solves the exercise.

1.3 Postscriptum

Remark 1.3. In the 5707 lecture notes, I have stated the fact that if $G = (V, E)$ is a simple graph that has no isolated vertices, then there exist two disjoint dominating subsets A and B of V such that $A \cup B = V$. Do you see why this follows from the above exercise?

[1] We abbreviate the word "f-monochromatic" as "monochromatic".