Combinatorics: a first encounter

Darij Grinberg

Thursday 10^{th} January, 2019at 9:29pm (unfinished draft!)

Contents

1. Preface

These notes (which are work in progress and will remain so for the foreseeable future) are meant as an introduction to combinatorics – the mathematical discipline that studies finite sets (roughly speaking). When finished, they will cover topics such as binomial coefficients, the principles of enumeration, permutations, partitions and graphs. The emphasis falls on *enumerative combinatorics*, meaning the art of computing sizes of finite sets ("counting"), and graph theory.

I have tried to keep the presentation as self-contained and elementary as possible. The reader is assumed to be familiar with some basics such as induction proofs, equivalence relations and summation signs, as well as have some experience with mathematical proofs. One of the best places to catch up on these basics and to gain said experience is the MIT lecture notes [LeLeMe16] (particularly their first five chapters). Two other resources to familiarize oneself with proofs are [Hammac15] and [Day16]. Generally, most good books about "reading and writing mathematics" or "introductions to abstract mathematics" should convey these skills, although the extent to which they actually do so may differ.

These notes are accompanying two classes on combinatorics (Math 4707 and 4990) I am giving at the University of Minneapolis in Fall 2017.

Here is a (subjective and somewhat random) list of recommended texts on the kinds of combinatorics that will be considered in these notes:

- Enumerative combinatorics (aka counting):

- The very basics of the subject can be found in [LeLeMe16, Chapters 14–15].
- Bogart's [Bogart04] is an introductory text to enumerative combinatorics that presents the material as an elaborate series of exercises. (Hints and solutions are available from the "Guided Discovery Archive" on the same website.)
- Loehr's [Loehr11] is a comprehensive textbook on enumerative combinatorics, with a focus on bijections. It goes far beyond these notes, but its first few chapters overlap with what we will do below.
- Galvin's [Galvin17] contains a readable introduction into combinatorics, including enumeration (but also various more exotic subjects).
- Cameron's notes [Camero16], [Camero07], [Camero13] (these have significant overlap) are a neat terse introduction into the subject.
- Stanley's [Stanle12] and [Stanle01] are the most famous treatise on enumerative combinatorics. They are written for graduate students and researchers, so they are not meant as a first introduction; the writing is terse and the exercises often taken from research literature (even following the proofs is challenging at times). We will at most graze this text in the notes below, but you cannot go wrong taking a look at it to see how research-level counting looks like.

- Binomial coefficients:
  - A wonderful introduction to this subject is Graham's, Knuth's and Patashnik's [GrKnPa, Chapter 5].
  - I give a slow and detailed introduction to binomial coefficients in [Grinbe16, Chapter 3]. Beware that this doesn't cover half of what we will do in class.
  - There are whole books (e.g., [Riorda68]) devoted to combinatorial identities (i.e., identities for binomial coefficients and similar kinds of numbers).
- Generating functions:
  - A quick introduction to generating functions, with a few sample applications, appears in [LeLeMe16, Chapter 16].
  - The treatment of generating functions in [Loehr11, Chapters 7–8] is probably one of the best in the literature. (It uses concepts from abstract algebra, but these are introduced earlier on in the book.)
  - Another introduction is [GrKnPa, Chapter 7].
  - Bogart's [Bogart04] has a chapter on generating functions.
- Permutations:

   - Part of Loehr's [Loehr11, Chapter 9] is about permutations and determinants.
   - Zeilberger's paper [Zeilbe85] shows some examples of how properties of determinants can be proven using combinatorial arguments.
   - In [Grinbe16, Chapters 5–6], I cover the theory of permutations necessary to study determinants. Again, I give lots of detail but stay in shallow waters.

- Partitions:
   - Andrews's and Eriksson's [AndEri04] is all about this, and supposedly written for undergraduates.
   - For a quick introduction, see [Stanle12, §1.8].
   - Pak's survey [Pak02] is short on proofs, but gives a deep overview of the subject.
   - Then there is [Wilf09].

- Graphs:
   - [LeLeMe16, Chapter 12] treats the very basics of graph theory.
   - Guichard's [Guicha16, Chapters 4–5] is another introduction to graph theory.
   - Ore's [Ore90] is a classical (somewhat informal) introduction meant for laymen. A more rigorous textbook by Ore (that also goes deeper into the subject) is [Ore74].
   - Diestel's [Dieste16] is a modern graduate textbook on graph theory. It makes no easy reading, but it is rich in content, and I recommend browsing it to get an idea of deep and recent results in graph theory.

- Digraphs:
   - [LeLeMe16, Chapter 10] treats the very basics of digraph (= directed graph) theory.

- Catalan combinatorics:
   - Stanley's book [Stanle15] is dedicated entirely to Catalan numbers and the various combinatorial objects they count. (Much like [Stanle12], this is a challenging read, mainly meant for graduate students.)

- Pólya enumeration:
   - Loehr's [Loehr11, Chapter 9] is partly about this.
   - Bogart's [Bogart04, Chapter 6] is devoted to this.

- Combinatorics on words:
  - There are various texts on this; most famous is [Lothai97].

There are numerous other texts. I have heard people recommend [Aigner07] and [Bona11], for example, but I have no experience with these texts myself. I also have learnt that Brualdi's [Bruald10] has a lot of intersection with what we will do, but I don't have much first-hands experience with that book.

TODO 1.0.1. List other sources on enumerative combinatorics (use my M.SE post) and graph theory (from my 5707 notes).

The notes you are reading are under construction, and will remain so for at least the whole Fall of 2017. Please let me know of any errors and unclarities you encounter (my email address is `dgrinber@umn.edu`). Thank you!

1.1. Acknowledgments

Thanks to Angela Chen for corrections.

[Your name could be in here!]

2. What is combinatorics?

This first chapter is meant to give a taste of combinatorics. Rather than systematically study any specific concept, we shall here [...]

2.1. Notations and conventions

Before we get to anything interesting, let us get some technicalities out of the way. Namely, we shall be using the following notations:

- In the following, we use the symbol $\mathbb{N}$ to denote the set $\{0, 1, 2, \ldots\}$. (Be warned that some other authors use this symbol for $\{1, 2, 3, \ldots\}$ instead.)
- We let $\mathbb{Q}$ denote the set of all rational numbers; we let $\mathbb{R}$ be the set of all real numbers.
- If X and Y are two sets, then we shall use the notation "$X \to Y,\ x \mapsto E$" (where x is some symbol which has no specific meaning in the current context, and where E is some expression which usually involves x) for "the map from X to Y that sends every $x \in X$ to E". For example, "$\mathbb{N} \to \mathbb{N},\ x \mapsto x^2 + x + 6$" means the map from $\mathbb{N}$ to $\mathbb{N}$ that sends every $x \in \mathbb{N}$ to $x^2 + x + 6$. For

another example, "$\mathbb{N} \to \mathbb{Q}$, $x \mapsto \dfrac{x}{1+x}$" denotes the map from $\mathbb{N}$ to $\mathbb{Q}$ that sends every $x \in \mathbb{N}$ to $\dfrac{x}{1+x}$. [1]

- If S is a set, then the *powerset* of S means the set of all subsets of S. This powerset will be denoted by $\mathcal{P}(S)$. For example, the powerset of $\{1,2\}$ is $\mathcal{P}(\{1,2\}) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$.

  Furthermore, if S is a set and k is an integer, then $\mathcal{P}_k(S)$ shall mean the set of all k-element subsets of S. (This is empty if $k < 0$.)

- If $f : X \to Y$ is a map, then the following notations are being used:
  - For any subset U of X, we let $f(U)$ be the subset $\{f(u) \mid u \in U\}$ of Y. This $f(U)$ is called the *image* of U under f.
  - For any subset V of Y, we let $f^{-1}(V)$ be the subset $\{u \in X \mid f(u) \in V\}$ of X. This $f^{-1}(V)$ is called the *preimage* of V under f.

    (Note that in general, $f\left(f^{-1}(V)\right) \neq V$ and $f^{-1}(f(U)) \neq U$. However, $f\left(f^{-1}(V)\right) \subseteq V$ and $U \subseteq f^{-1}(f(U))$.)
  - An *inverse* of f means a map $g : Y \to X$ satisfying $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. It is easy to see that an inverse of f is unique if it exists; it is thus called *the inverse* of f, and is denoted by f^{-1}. The map f is said to be *invertible* if it has an inverse.
  - The map f is said to be *injective* (or *one-to-one*) if it has the following property: Whenever two elements x_1 and x_2 of X satisfy $f(x_1) = f(x_2)$, we must have $x_1 = x_2$. (In other words, no two distinct elements of X are mapped to the same element of Y by f. In other words, any element x of X can be uniquely recovered from its image $f(x)$.)
  - The map f is said to be *surjective* (or *onto*) if $Y = f(X)$. (Equivalently, $f : X \to Y$ is surjective if and only if each element of Y can be written as $f(x)$ for some $x \in X$.)

[1] A word of warning: Of course, the notation "$X \to Y$, $x \mapsto E$" does not always make sense; indeed, the map that it stands for might sometimes not exist. For instance, the notation "$\mathbb{N} \to \mathbb{Q}$, $x \mapsto \dfrac{x}{1-x}$" does not actually define a map, because the map that it is supposed to define (i.e., the map from $\mathbb{N}$ to $\mathbb{Q}$ that sends every $x \in \mathbb{N}$ to $\dfrac{x}{1-x}$) does not exist (since $\dfrac{x}{1-x}$ is not defined for $x = 1$). For another example, the notation "$\mathbb{N} \to \mathbb{Z}$, $x \mapsto \dfrac{x}{1+x}$" does not define a map, because the map that it is supposed to define (i.e., the map from $\mathbb{N}$ to $\mathbb{Z}$ that sends every $x \in \mathbb{N}$ to $\dfrac{x}{1+x}$) does not exist (for $x = 2$, we have $\dfrac{x}{1+x} = \dfrac{2}{1+2} \notin \mathbb{Z}$, which shows that a map from $\mathbb{N}$ to $\mathbb{Z}$ cannot send this x to this $\dfrac{x}{1+x}$). Thus, when defining a map from X to Y (using whatever notation), do not forget to check that it is well-defined (i.e., that your definition specifies precisely one image for each $x \in X$, and that these images all lie in Y). In many cases, this is obvious or very easy to check (I will usually not even mention this check), but in some cases, this is a difficult task.

– The map f is said to be *bijective* (or a *one-to-one correspondence*) if it is both injective and surjective. Equivalently, a map $f : X \to Y$ is bijective if and only if it is invertible (i.e., has an inverse).

[...]

...

References

[Aigner07] Martin Aigner, *A Course in Enumeration*, Graduate Texts in Mathematics #238, Springer 2007.

[AigZie] Martin Aigner, Günter M. Ziegler, *Proofs from the Book*, 4th edition, Springer 2010.

[AndDos] Titu Andreescu, Gabriel Dospinescu, *Problems from the Book*, XYZ Press 2008.

[AndEri04] George E. Andrews, Kimmo Eriksson, *Integer Partitions*, Cambridge University Press 2004.

[AoPS-ISL] Art of Problem Solving (forum), *IMO Shortlist* (collection of threads), `http://www.artofproblemsolving.com/community/c3223_imo_shortlist`

[Bahran15] Cihan Bahran, *Solutions to Math 5707 Spring 2015 homework*. `http://www-users.math.umn.edu/~bahra004/5707.html`

[Balakr97] V. K. Balakrishnan, *Schaum's Outline of Theory and Problems of Graph Theory*, McGraw-Hill 1997.

[BehCha71] Mehdi Behzad, Gary Chartrand, *Introduction to the Theory of Graphs*, Allyn & Bacon, 1971.

[BenWil12] Edward A. Bender and S. Gill Williamson, *Foundations of Combinatorics with Applications*. `http://cseweb.ucsd.edu/~gill/FoundCombSite/`

[BeChZh15] Arthur Benjamin, Gary Chartrand, Ping Zhang, *The fascinating world of graph theory*, Princeton University Press 2015.

[Berge91] Claude Berge, *Graphs*, 3rd edition, North-Holland 1991.

[Bogart04] Kenneth P. Bogart, *Combinatorics Through Guided Discovery*, 2004. `https://www.math.dartmouth.edu/news-resources/electronic/kpbogart/`

[Bogomoln] Alexander Bogomolny, *Cut the Knot* (website devoted to educational applets on various mathematical subjects), `http://www.cut-the-knot.org/Curriculum/index.shtml#combinatorics` .

[Bollob79] Béla Bollobás, *Graph Theory: An Introductory Course*, Graduate Texts in Mathematics #63, 1st edition, Springer 1971.

[Bollob98] Béla Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics #184, 1st edition, Springer 1998.

[Bona11] Miklós Bóna, *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory*, 3rd edition, World Scientific 2011.

[BonMur76] J. A. Bondy and U. S. R. Murty, *Graph theory with Applications*, North-Holland 1976.
`https://www.iro.umontreal.ca/~hahn/IFT3545/GTWA.pdf`.

[BonMur08] J. A. Bondy and U. S. R. Murty, *Graph theory*, Graduate Texts in Mathematics #244, Springer 2008.
`https://www.classes.cs.uchicago.edu/archive/2016/spring/27500-1/hw3.pdf`

[BonTho77] J. A. Bondy, C. Thomassen, *A short proof of Meyniel's theorem*, Discrete Mathematics 19 (1977), pp. 195–197.
`http://www.sciencedirect.com/science/article/pii/0012365X77900346`

[Brouwe09] Andries E. Brouwer, *The number of dominating sets of a finite graph is odd*, `http://www.win.tue.nl/~aeb/preprints/domin2.pdf` .

[Bruald10] Richard A. Brualdi, *Introductory Combinatorics*, 5th edition, Prentice-Hall 2010, `http://emalca.emate.ucr.ac.cr/sites/emalca.emate.ucr.ac.cr/files/combiatoric.pdf` .

[Bruijn12] Nicolaas Govert de Bruijn, J. W. Nienhuys, Ling-Ju Hung, Tom Kloks, *de Bruijn's Combinatorics*, viXra:1208.0223v1.
`http://vixra.org/pdf/1208.0223v1.pdf`

[Camero07] Peter J. Cameron, *Notes on Combinatorics*, 7 December 2007.
`https://cameroncounts.files.wordpress.com/2013/11/comb.pdf`

[Camero13] Peter J. Cameron, *Enumerative Combinatorics: The LTCC lectures*, 3 December 2013.
`https://cameroncounts.files.wordpress.com/2013/12/ec.pdf`

[Camero16] Peter J. Cameron, *St Andrews Notes on Advanced Combinatorics, Part 1: The Art of Counting*, 28 March 2016.
`https://cameroncounts.files.wordpress.com/2016/04/acnotes1.pdf`

[ChaLes15] Gary Chartrand, Linda Lesniak, Ping Zhang, *Graphs & Digraphs*, 6th edition, CRC Press 2016.

[Choo16] David Choo, *4 proofs to Mantel's Theorem*, `http://davinchoo.com/2016/06/13/4-proofs-to-mantels-theorem/` .

[Conrad] Keith Conrad, *Sign of permutations*, `http://www.math.uconn.edu/~kconrad/blurbs/grouptheory/sign.pdf` .

[Day16] Martin V. Day, *An Introduction to Proofs and the Mathematical Vernacular*, 7 December 2016.
`https://www.math.vt.edu/people/day/ProofsBook/IPaMV.pdf` .

[Dieste16] Reinhard Diestel, *Graph Theory*, Graduate Texts in Mathematics #173, 5th edition, Springer 2016.
`http://diestel-graph-theory.com/basic.html` .

[FriFri98] Rudolf Fritsch, Gerda Fritsch, *The Four-Color Theorem*, Springer 1998.

[Galvin17] David Galvin, *Basic Discrete Mathematics*, 2017.
`http://www.cip.ifi.lmu.de/~grinberg/t/17f/60610lectures2017-Galvin.pdf`

[Gessel79] Ira Gessel, *Tournaments and Vandermonde's Determinant*, Journal of Graph Theory, Vol. 3 (1979), pp. 305–307.

[GrKnPa] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics, Second Edition*, Addison-Wesley 1994.

[GrRoSp90] Ronald L. Graham, Bruce L. Rothschild, Joel H. Spencer, *Ramsey Theory*, 2nd edition, Wiley 1990.

[Griffi15] Christopher Griffin, *Graph Theory: Penn State Math 485 Lecture Notes*, version 1.4.2.1 (18 Oct 2015),
`https://sites.google.com/site/cgriffin229/` .

[Grinbe16] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 10 January 2019.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`

[Guicha16] David Guichard, *An Introduction to Combinatorics and Graph Theory*,
`https://www.whitman.edu/mathematics/cgt_online/cgt.pdf`.

[Hammac15] Richard Hammack, *Book of Proof*, 2nd edition, Ingram 2013.
`http://www.people.vcu.edu/~rhammack/BookOfProof/`

[Harary69] Frank Harary, *Graph theory*, Addison-Wesley 1969.
`http://www.dtic.mil/dtic/tr/fulltext/u2/705364.pdf`

[HarPal73] Frank Harary, Edgar M. Palmer, *Graphical enumeration*, Academic Press 1973.

[Harju14] Tero Harju, *Lecture notes on Graph Theory*, 24 April 2014.
`http://users.utu.fi/harju/graphtheory/graphtheory.pdf`

[HaHiMo08] John M. Harris, Jeffry L. Hirst, Michael J. Mossinghoff, *Combinatorics and Graph Theory*, Undergraduate Texts in Mathematics, Springer 2008.

[HeiTit17] Irene Heinrich, Peter Tittmann, *Counting Dominating Sets of Graphs*, `arXiv:1701.03453v1`.

[Jukna11] Stasys Jukna, *Extremal Combinatorics*, 2nd edition, Springer 2011. An early draft is available at `http://www.thi.informatik.uni-frankfurt.de/~jukna/EC_Book_2nd/` .

[Jungni13] Dieter Jungnickel, *Graphs, Networks and Algorithms*, 4th edition, Springer 2013.

[KelTro15] Mitchel T. Keller, William T. Trotter, *Applied Combinatorics*, version 26 May 2015.
`https://people.math.gatech.edu/~trotter/book.pdf`

[Klarre17] Erica Klarreich, *Complexity Theory Problem Strikes Back*, Quanta Magazine, 5 January 2017.
`https://www.quantamagazine.org/20170105-graph-isomorphism-retraction/`

[Knuth97] Donald E. Knuth, *The Art of Computer Programming, Volume 1: Fundamental Algorithms*, 3rd edition, Addison-Wesley 1997.

[Knuth95] Donald E. Knuth, *Overlapping Pfaffians*, Electron. J. Combin. 3 (1996), no. 2, #R5. Also available as arXiv preprint `arXiv:math/9503234v1`.

[LaNaSc16] Isaiah Lankham, Bruno Nachtergaele, Anne Schilling, *Linear Algebra As an Introduction to Abstract Mathematics*, 2016.
`https://www.math.ucdavis.edu/~anne/linear_algebra/mat67_course_notes.pdf`

[LeLeMe16] Eric Lehman, F. Thomson Leighton, Albert R. Meyer, *Mathematics for Computer Science*, revised Monday 11th September, 2017,
`https://courses.csail.mit.edu/6.042/fall17/mcs.pdf` .

[Loehr11] Nicholas A. Loehr, *Bijective Combinatorics*, Chapman & Hall/CRC 2011.

[Lothai97] M. Lothaire, *Combinatorics on Words*, Cambridge University Press, corrected reissue 1997.

[LoPeVe03] László Lovász, József Pelikán, Katalin Vesztergombi, *Discrete Mathematics: Elementary and Beyound*, Springer 2003.

[Martin16] Jeremy L. Martin, *Math 725, Spring 2016 Lecture Notes*, `https://www.math.ku.edu/~jmartin/courses/math725-S16/`

[Oggier14] Frédérique Oggier, *Discrete Mathematics*, 21 November 2014. `http://www1.spms.ntu.edu.sg/~frederique/DiscreteMathws.pdf`

[Ore74] Oystein Ore, *Theory of graphs*, AMS Colloquium Publications #XXXVIII, 4th printing, AMS 1974.

[Ore90] Oystein Ore, *Graphs and their uses*, revised and updated by Robin J. Wilson, Anneli Lax New Mathematical Library #34, MAA 1990.

[Overbe74] Maria Overbeck-Larisch, *Hamiltonian paths in oriented graphs*, Journal of Combinatorial Theory, Series B, Volume 21, Issue 1, August 1976, pp. 76–80. `http://www.sciencedirect.com/science/article/pii/0095895676900307`

[Pak02] Igor Pak, *Partition bijections, a survey*, Ramanujan Journal, vol. 12 (2006), pp. 5–75. `http://www.math.ucla.edu/~pak/papers/psurvey.pdf`

[Petrov15] Fedor Petrov, *mathoverflow post #198679 (Flooding a cycle digraph via chip-firing: $n^{k-1} + n^{k-2} + \cdots + 1$ bound (a Norway 1998-99 problem generalized))*, MathOverflow, `http://mathoverflow.net/q/198679`

[PoTaWo83] George Pólya, Robert E. Tarjan, Donald R. Woods, *Notes on Introductory Combinatorics*, Birkhäuser 1983. See `http://i.stanford.edu/pub/cstr/reports/cs/tr/79/732/CS-TR-79-732.pdf` for a preliminary version.

[Pretzel] Oliver Pretzel, *On reorienting graphs by pushing down maximal vertices*, Order, 1986, Volume 3, Issue 2, pp. 135–153.

[RaWiRa] RationalWiki, *Ramsey theory*, `http://rationalwiki.org/wiki/Ramsey_theory` .

[Riorda68] John Riordan, *Combinatorial Identities*, John Wiley & Sons, 1968.

[RotSot92] Alvin E. Roth, Marilda A. Oliveira Sotomayor, *Two-sided matching: A study in game-theoretic modeling and analysis*, Cambridge University Press 1992.

[Ruohon13] Keijo Ruohonen, *Graph theory*, 2013. `http://math.tut.fi/~ruohonen/GT_English.pdf` .

[Stanle12] Richard P. Stanley, *Enumerative Combinatorics, Volume 1*, 2nd edition, CUP 2012.
See http://math.mit.edu/~rstan/ec/ec1/ for a preliminary version.

[Stanle01] Richard P. Stanley, *Enumerative Combinatorics, Volume 2*, 1st edition 2001.
See http://math.mit.edu/~rstan/ec/ for errata and addenda.

[Stanle15] Richard P. Stanley, *Catalan Numbers*, Cambridge University Press 2015.
See http://math.mit.edu/~rstan/catalan/ for errata and addenda.

[Stanle13] Richard P. Stanley, *Algebraic Combinatorics: Walks, Trees, Tableaux, and More*, Springer 2013.
See http://www-math.mit.edu/~rstan/algcomb/ for errata and a downloadable draft of the book.

[Strick13] Neil P. Strickland, *MAS201 Linear Mathematics for Applications*.
https://neil-strickland.staff.shef.ac.uk/courses/MAS201/

[ThuSwa92] K. Thulasiraman, M. N. S. Swamy, *Graphs: Theory and Algorithms*, Wiley 1992.

[Uecker17] Torsten Ueckerdt, with Maria Axenovich, Stefan Walzer, *Lecture Notes Combinatorics*, Karlsruhe 2017.
http://www.math.kit.edu/iag6/lehre/combinatorics2017s/media/script.pdf

[West01] Douglas B. West, *Introduction to Graph Theory*, 2nd edition, Pearson 2001.
See http://www.math.illinois.edu/~dwest/igt/ for updates and corrections.

[Wilf09] Herbert S. Wilf, *Lectures on Integer Partitions*, 2009.
https://www.math.upenn.edu/~wilf/PIMS/PIMSLectures.pdf

[Wilson96] Robin J. Wilson, *Introduction to Graph Theory*, 4th edition, Addison Wesley 1996.
See https://archive.org/details/IntroductionToGraphTheory for the 3rd edition.

[Zeilbe85] Doron Zeilberger, *A combinatorial approach to matrix algebra*, Discrete Mathematics 56 (1985), pp. 61–72.