Math 4242 Fall 2016 (Darij Grinberg): midterm 3 with solutions
Mon, 12 Dec 2016, in class (75 minutes). Proofs are NOT required.

If you write answers on this sheet, please **use "T" and "F"**, and **SIGN** it with your name and **hand it back to me**!
Your name:

Recall that *to diagonalize an* $n \times n$*-matrix* A means to find an invertible $n \times n$-matrix S and a diagonal $n \times n$-matrix Λ satisfying $A = S\Lambda S^{-1}$, whenever such S and Λ exist. (If S and Λ do not exist, you should state this clearly!) **Explicitly computing** S^{-1} **is not required.**

Exercise 1. **(a)** Diagonalize the matrix $A = \begin{pmatrix} 5 & -2 \\ 2 & 0 \end{pmatrix}$. [6 points]

(b) Diagonalize the matrix $A = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}$. [6 points]

(c) Diagonalize the matrix $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$. [6 points]

Solution to Exercise 1. Each of the three parts can be solved by following Algorithm 0.1 from homework set #8. I shall be sparing with the details here, since you have already seen quite a few examples for this algorithm.

(a) We have

$$\det(A - xI_2) = \det\begin{pmatrix} 5-x & -2 \\ 2 & -x \end{pmatrix} = (5-x)(-x) - (-2)2$$
$$= x^2 - 5x + 4 = (x-1)(x-4).$$

Hence, the eigenvalues of A are 1 and 4. Denote them by $\lambda_1 = 1$ and $\lambda_2 = 4$.

We now need to find bases for $\operatorname{Ker}(A - \lambda_1 I_2)$ and $\operatorname{Ker}(A - \lambda_2 I_2)$.

Computing $\operatorname{Ker}(A - \lambda_1 I_n)$*:* We have

$$\operatorname{Ker}(A - \lambda_1 I_2) = \operatorname{Ker}(A - 1I_2) = \operatorname{Ker}\begin{pmatrix} 5-1 & -2 \\ 2 & -1 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$$
$$= \operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right),$$

with basis $\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$.

Computing $\operatorname{Ker}(A - \lambda_2 I_n)$*:* We have

$$\operatorname{Ker}(A - \lambda_2 I_2) = \operatorname{Ker}(A - 4I_2) = \operatorname{Ker}\begin{pmatrix} 5-4 & -2 \\ 2 & -4 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$$
$$= \operatorname{span}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right),$$

with basis $\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$.

Now, the big list is $(s_1, s_2) = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$, so $s_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mu_1 = 1$, $s_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mu_2 = 4$. Hence,

$$S = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \text{and} \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

(b) We have

$$\det(A - xI_2) = \det\begin{pmatrix} 1-x & 4 \\ -1 & 5-x \end{pmatrix} = (1-x)(5-x) - 4(-1)$$
$$= 5 - 6x + x^2 + 4 = 9 - 6x + x^2 = (x-3)^2,$$

so the only eigenvalue is 3. Denote it by $\lambda_1 = 3$.

Computing $\operatorname{Ker}(A - \lambda_1 I_n)$: We have

$$\operatorname{Ker}(A - \lambda_1 I_2) = \operatorname{Ker}(A - 3I_2) = \operatorname{Ker}\begin{pmatrix} 1-3 & 4 \\ -1 & 5-3 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$$
$$= \operatorname{span}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

Thus, the whole big list is $\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$. This list does not have enough vectors (we have 1, but we need 2). Thus, A is not diagonalizable.

(c) We can do this using Algorithm 0.1 from homework set #8. But we can also get the answer much cheaper: We have to diagonalize a matrix A that is already diagonal. This is particularly easy: just take $S = I_2$ and $\Lambda = A$. □

Exercise 2. Diagonalize the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. [10 points]

Solution to Exercise 2. Again, follow Algorithm 0.1 from homework set #8. We have

$$\det(A - xI_4) = \det\begin{pmatrix} 1-x & 0 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 1-x \end{pmatrix}$$

$$= \det\begin{pmatrix} 1-x & 0 & 0 & 0 \\ 0 & 0 & 1-x^2 & 0 \\ 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 1-x \end{pmatrix}$$

$$\left(\begin{array}{c}\text{here, we have added } x \text{ times the 3-rd row to the 2-nd row;} \\ \text{as we know, this preserves the determinant}\end{array}\right)$$

$$= -\det\begin{pmatrix} 1-x & 0 & 0 & 0 \\ 0 & 1 & -x & 0 \\ 0 & 0 & 1-x^2 & 0 \\ 0 & 0 & 0 & 1-x \end{pmatrix}$$

$$\left(\begin{array}{c}\text{here, we have switched the 2-nd and 3-rd rows;} \\ \text{as we know, this negates the determinant}\end{array}\right)$$

$$= -(1-x)\,1\left(1-x^2\right)(1-x)$$

$$\left(\begin{array}{c}\text{since the determinant of an upper-triangular matrix} \\ \text{is the product of its diagonal entries}\end{array}\right)$$

$$= -(1-x)^3(1+x).$$

Hence, the eigenvalues of A are 1 and -1. Denote them by $\lambda_1 = 1$ and $\lambda_2 = -1$.

We now need to find bases for $\operatorname{Ker}(A - \lambda_1 I_2)$ and $\operatorname{Ker}(A - \lambda_2 I_2)$.

<u>*Computing* $\operatorname{Ker}(A - \lambda_1 I_n)$:</u> We have

$$\operatorname{Ker}(A - \lambda_1 I_2) = \operatorname{Ker}(A - 1I_2) = \operatorname{Ker}\begin{pmatrix} 1-1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1-1 \end{pmatrix}$$

$$= \operatorname{Ker}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Hence, a basis of $\operatorname{Ker}(A - \lambda_1 I_2)$ is $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$.

<u>*Computing* $\operatorname{Ker}(A - \lambda_2 I_n)$:</u> We have

$$\operatorname{Ker}(A - \lambda_2 I_2) = \operatorname{Ker}(A - (-1) I_2) = \operatorname{Ker} \begin{pmatrix} 1-(-1) & 0 & 0 & 0 \\ 0 & -(-1) & 1 & 0 \\ 0 & 1 & -(-1) & 0 \\ 0 & 0 & 0 & 1-(-1) \end{pmatrix}$$
$$= \operatorname{Ker} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \operatorname{span} \left(\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right).$$

Hence, a basis of $\operatorname{Ker}(A - \lambda_2 I_2)$ is $\left(\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right)$.

So the big list is $(s_1, s_2, s_3, s_4) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right)$, and the corresponding eigenvalues are $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$ and $\mu_4 = -1$. Hence,

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

□

As usual, P_n means the vector space of all polynomials of degree $\leq n$ with real coefficients.

Exercise 3. Let $\mathbf{v}$ be the basis $(1, x)$ of the vector space P_1. Let $\mathbf{w}$ be the basis $\left(1, \dfrac{1+x}{2}\right)$ of the same vector space P_1.

(a) Find the matrix $M_{\mathbf{v},\mathbf{w},\mathrm{id}_{P_1}}$. [5 points]

(b) Let $G : P_1 \to P_1$ be the linear map whose representing matrix is $M_{\mathbf{v},\mathbf{w},G} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Find an explicit formula for $G(a + bx)$, where $a, b \in \mathbb{R}$.

[5 points]

Solution to Exercise 3. **(a)** We want to expand $\operatorname{id}_{P_1}(1)$ and $\operatorname{id}_{P_1}(x)$ with respect to the basis $\mathbf{w}$, and pack the resulting coefficients into the columns of a 2×2-matrix.

We have

$$\operatorname{id}_{P_1}(1) = 1 = 1 \cdot 1 + 0 \cdot \frac{1+x}{2},$$

so the coefficients here are 1 and 0.

We have

$$\operatorname{id}_{P_1}(x) = x = -1 \cdot 1 + 2 \cdot \frac{1+x}{2},$$

so the coefficients here are -1 and 2.

Thus,

$$M_{\mathbf{v},\mathbf{w},\operatorname{id}_{P_1}} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

(b) By the definition of $M_{\mathbf{v},\mathbf{w},G}$, the entries in the 1-st column of $M_{\mathbf{v},\mathbf{w},G}$ are the coefficients when $G(1)$ is expanded with respect to the basis $\mathbf{w}$, and the entries in the 2-nd column of $M_{\mathbf{v},\mathbf{w},G}$ are the coefficients when $G(x)$ is expanded with respect to the basis $\mathbf{w}$. Thus, we obtain these expansions immediately:

$$G(1) = 1 \cdot 1 + 1 \cdot \frac{1+x}{2} = \frac{3+x}{2};$$
$$G(x) = 0 \cdot 1 + 1 \cdot \frac{1+x}{2} = \frac{1+x}{2}.$$

Now, for any $a, b \in \mathbb{R}$, we have

$$G(a+bx) = G(a \cdot 1 + b \cdot x) = a \underbrace{G(1)}_{=\frac{3+x}{2}} + b \underbrace{G(x)}_{=\frac{1+x}{2}} \qquad \text{(since the map } G \text{ is linear)}$$
$$= a\frac{3+x}{2} + b\frac{1+x}{2} = \frac{3a+b}{2} + \frac{a+b}{2}x.$$

□

Exercise 4. For each of the following maps, answer whether it is linear, injective, surjective, or any combination of these:

[Please write a "T" into a box to indicate that the answer is "True", and write an "F" to indicate that the answer is "False".]

[2 points per question per part, totalling to $2 \cdot 3 \cdot 7 = 42$ points]

(a) The map $\mathbb{R}^2 \to P_2$ that sends each $(a,b)^T \in \mathbb{R}^2$ to $ax^2 + b \in P_2$.

linear? **T**	injective? **T**	surjective? **F**

(b) The map $\mathbb{R}^2 \to P_2$ that sends each $(a,b)^T \in \mathbb{R}^2$ to $(ax+b)^2 \in P_2$.

linear? **F**	injective? **F**	surjective? **F**

(c) The map $P_2 \to \mathbb{R}^2$ that sends each $f \in P_2$ to $(f(0), f(1))^T$.

linear? **T** | injective? **F** | surjective? **T**

(d) The map $\mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ that sends each $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ to $\begin{pmatrix} a & b \\ a+a' & b+b' \end{pmatrix}$.

linear? **T** | injective? **T** | surjective? **T**

(e) The map $\mathbb{R}^2 \to \mathbb{C}$ that sends each $(a,b)^T \in \mathbb{R}^2$ to $a+bi \in \mathbb{C}$.

linear? **T** | injective? **T** | surjective? **T**

(f) The map $\mathbb{C}^2 \to \mathbb{C}$ that sends each $(a,b)^T \in \mathbb{C}^2$ to $a+bi \in \mathbb{C}$.

linear? **T** | injective? **F** | surjective? **T**

(g) The map $\mathbb{R}^{2\times 2} \to P_2$ that sends each 2×2-matrix A to $\det(A - xI_2)$.

linear? **F** | injective? **F** | surjective? **F**

Solution to Exercise 4. **(a)** The map is linear (this can be checked straightforwardly). It is injective (since $(a,b)^T \in \mathbb{R}^2$ can be reconstructed from ax^2+b; indeed, a is the x^2-coefficient of ax^2+b, whereas b is the x^0-coefficient of ax^2+b). It is **not** surjective (since the polynomial $x \in P_2$ is not the image of any $(a,b)^T \in \mathbb{R}^2$ under this map).

(b) The map is **not** linear (for example, it sends $(1,1)^T$ to $(x+1)^2$, but $2\cdot(1,1)^T = (2,2)^T$ to $(2x+2)^2 \neq 2(x+1)^2$). It is **not** injective (since it sends $(1,0)^T$ and $(-1,0)^T$ to the same polynomial). It is **not** surjective (since the polynomial $-1 \in P_2$ is not the image of any $(a,b)^T \in \mathbb{R}^2$ under this map; indeed, -1 is not the square of any polynomial with real coefficients!).

(c) The map is linear (this is easy to check). It is **not** injective (since it sends the two polynomials x and x^2 to one and the same pair $(0,1)^T$). It is surjective (since every $(a,b)^T \in \mathbb{R}^2$ can be obtained as the image of the polynomial $a+(b-a)x \in P_2$).

(d) The map is linear (again, this is easy to check). It is both injective and surjective. In fact, it is invertible; the inverse map is the map $\mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ that sends each $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ to $\begin{pmatrix} a & b \\ a'-a & b'-b \end{pmatrix}$. (It is clear why these two maps are mutually inverse: The former adds the first row of the matrix to the second, whereas the latter subtracts the first row of the matrix from the second. These two operations obviously undo each other.)

(e) The map is linear. It is both injective and surjective. In fact, it is essentially the identity map, assuming that we are willing to blur the distinction between pairs of real numbers and elements of $\mathbb{R}^2$. More rigorously speaking, the argument proceeds as follows: Recall that complex numbers are pairs of real numbers, and specifically we have $a+bi = (a,b)$ for each $a,b \in \mathbb{R}$. Thus, our map sends each $(a,b)^T \in \mathbb{R}^2$ to $(a,b) \in \mathbb{C}$. Therefore, this map is invertible (and the inverse map sends each $(a,b) \in \mathbb{C}$ to $(a,b)^T \in \mathbb{R}^2$). Of course, an invertible map is injective and surjective.

(f) The map is linear (easy to check). It is **not** injective (since it sends the two pairs $(i,0)^T$ and $(0,1)^T$ to the same complex number i). It is surjective (since every $z \in \mathbb{C}$ is the image of $(z,0)^T$ under this map).

(g) The map is **not** linear (for example, the zero matrix $0_{2\times 2}$ is sent to $\det(0_{2\times 2} - xI_2) = x^2$, which is not zero). It is **not** injective (for example, it sends both matrices $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to one and the same polynomial x^2). It is **not** surjective (because it sends any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the polynomial

$$\det(A - xI_2) = \det \begin{pmatrix} a-x & b \\ c & d-x \end{pmatrix} = (a-x)(d-x) - bc = x^2 - (a+d)x + (ad-bc),$$

whose x^2-coefficient is 1; therefore, no polynomial with an x^2-coefficient different from 1 can be obtained as an image under the map). □

Exercise 5. Which of the following claims are true, and which are false?

[Please write a "T" into a box to indicate that the answer is "True", and write an "F" to indicate that the answer is "False".]

[2 points for each of the 10 claims]

(a) T If A is an $n \times n$-matrix with n distinct eigenvalues, then A is diagonalizable.

(b) F If $f : X \to Y$ and $g : Y \to X$ are two maps such that $f \circ g = \mathrm{id}_Y$, then f is injective and g is surjective.

(c) T If $f : X \to Y$ and $g : Y \to X$ are two maps such that $f \circ g = \mathrm{id}_Y$, then f is surjective and g is injective.

(d) T The eigenvalues of a lower-triangular matrix are its diagonal entries.

(e) F Every lower-triangular matrix is diagonalizable.

(f) T If U, V, W are vector spaces and $g : U \to V$ and $f : V \to W$ are two isomorphisms, then the composition $f \circ g$ is an isomorphism as well.

(g) T A diagonalization of the matrix $A = \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix}$ is given by $A = S\Lambda S^{-1}$, where $S = \begin{pmatrix} 1 & 6 \\ 2 & 3 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$.

(h) T A diagonalization of the matrix $A = \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix}$ is given by $A = S\Lambda S^{-1}$, where $S = \begin{pmatrix} 6 & 1 \\ 3 & 2 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$.

(i) F If $\mathbf{v}$ and $\mathbf{w}$ are two bases of a vector space V, and if $F : V \to V$ is an invertible linear map, then $M_{\mathbf{v},\mathbf{w},F^{-1}} = (M_{\mathbf{v},\mathbf{w},F})^{-1}$.

(j) $\boxed{\text{T}}$ If $\mathbf{v}$ and $\mathbf{w}$ are two bases of a vector space V, and if $F : V \to V$ is an invertible linear map, then $M_{\mathbf{v},\mathbf{w},F^{-1}} = (M_{\mathbf{w},\mathbf{v},F})^{-1}$.

Solution to Exercise 5. **(a) True.**

Proof. We can diagonalize it using Algorithm 0.1 from homework set #8. Let the n distinct eigenvalues be $\lambda_1, \lambda_2, \ldots, \lambda_n$. For each $k \in \{1, 2, \ldots, n\}$, the kernel $\operatorname{Ker}(A - \lambda_k I_n)$ is nonzero, and thus has a basis consisting of at least one vector. Concatenating these n bases, we therefore obtain a list which contains at least n vectors. Thus, our big list contains at least n vectors. Therefore, the dreaded case $m < n$ cannot happen, and A is diagonalizable.

(b) False. For example, take $X = \{1, 2\}$ and $Y = \{1\}$, and let f and g be any maps (there is one choice for f and two choices for g). Then, $f \circ g = \mathrm{id}_Y$ is always true, but f is never injective, and g is never surjective.

(c) True.

Proof. Let $f : X \to Y$ and $g : Y \to X$ be two maps such that $f \circ g = \mathrm{id}_Y$.

1. Let us show that f is surjective. To this aim, we must prove that for each $y \in Y$, there exists some $x \in X$ satisfying $f(x) = y$. Thus, fix $y \in Y$.

   Recall that $f \circ g = \mathrm{id}_Y$. Thus, $(f \circ g)(y) = \mathrm{id}_Y(y) = y$, so that $y = (f \circ g)(y) = f(g(y))$. Hence, there exists some $x \in X$ satisfying $f(x) = y$ (namely, $x = g(y)$). This completes the proof that f is surjective.

2. Let us now show that g is injective. To this aim, we must prove that if x_1 and x_2 are two elements of Y satisfying $g(x_1) = g(x_2)$, then $x_1 = x_2$. Thus, let x_1 and x_2 be two elements of Y satisfying $g(x_1) = g(x_2)$. We have $f(g(x_1)) = \underbrace{(f \circ g)}_{=\mathrm{id}_Y}(x_1) = \mathrm{id}_Y(x_1) = x_1$ and similarly $f(g(x_2)) = x_2$. Now,

   $x_1 = f\left(\underbrace{g(x_1)}_{=g(x_2)}\right) = f(g(x_2)) = x_2$, which is exactly what we wanted to show.

   Hence, we have proven that g is injective.

(d) True.

Proof. Let A be a lower-triangular $n \times n$ matrix. We must show that the eigenvalues of A are $A_{1,1}, A_{2,2}, \ldots, A_{n,n}$.

The matrix $A - xI_n$ is lower-triangular (since both A and xI_n are lower-triangular), with diagonal entries $A_{1,1} - x, A_{2,2} - x, \ldots, A_{n,n} - x$. Hence, its determinant is

$$\det(A - xI_n) = (A_{1,1} - x)(A_{2,2} - x) \cdots (A_{n,n} - x) \qquad (1)$$

(since the determinant of a lower-triangular matrix is the product of its diagonal entries). But the eigenvalues of A are the roots of the polynomial $\det(A - xI_n)$. Because of (1), these roots are precisely $A_{1,1}, A_{2,2}, \ldots, A_{n,n}$. Hence, the eigenvalues of A are $A_{1,1}, A_{2,2}, \ldots, A_{n,n}$.

[*Remark:* Of course, the same claim holds for upper-triangular matrices.]

(e) False. For example, the lower-triangular matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is not diagonalizable.

[*Remark:* Actually, more can be proven: Any **strictly** lower-triangular matrix that is not the zero matrix is not diagonalizable. This is because its only eigenvalue is 0 (by part **(d)** of this problem), but the corresponding kernel $\operatorname{Ker}(A - 0I_n) = \operatorname{Ker} A$ does not have n basis vectors.]

(f) True.

Proof. Let U, V, W be vector spaces and $g : U \to V$ and $f : V \to W$ be two isomorphisms.

The maps f and g are isomorphisms. In other words, they are linear and invertible. Hence, their composition $f \circ g$ is linear (since the composition of two linear maps is linear) and invertible (since the composition of two invertible maps is invertible). In other words, $f \circ g$ is an isomorphism.

(g) True.

The easiest way to check this is to verify that S is invertible (e.g., because $\det S = 1 \cdot 3 - 2 \cdot 6 \neq 0$) and that $AS = S\Lambda$.

(h) True.

You can check this in the same way as part **(g)**.

(i) False. One quick way to convince yourself of the falsehood of the claim is the following: For $F = \mathrm{id}_V$, the claim would say that $M_{\mathbf{v},\mathbf{w},\mathrm{id}_V} = \left(M_{\mathbf{v},\mathbf{w},\mathrm{id}_V}\right)^{-1}$ (since $(\mathrm{id}_V)^{-1} = \mathrm{id}_V$). In other words, it would say that the change-of-basis matrix $M_{\mathbf{v},\mathbf{w},\mathrm{id}_V}$ must always be its own inverse. But any invertible matrix can be a change-of-basis matrix; in particular, it is easy to come up with one that is not its own inverse (e.g., we can take $V = \mathbb{R}^2$, $\mathbf{v} = (e_1, e_2)$ and $\mathbf{w} = (e_1, e_1 + e_2)$.

(j) True.

Proof. Let me derive this from the following two facts

Theorem 0.1. Let U, V and W be three vector spaces with bases $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, respectively. Let $F : U \to V$ and $G : V \to W$ be two linear maps. Then, their composition $G \circ F : U \to W$ is again a linear map, and we have

$$M_{\mathbf{u},\mathbf{w},G\circ F} = M_{\mathbf{v},\mathbf{w},G} M_{\mathbf{u},\mathbf{v},F}.$$

Theorem 0.2. Let $\mathbf{v}$ be a basis of a vector space V. Let n be the size of $\mathbf{v}$ (that is, the number of entries of $\mathbf{v}$). Then, $M_{\mathbf{v},\mathbf{v},\mathrm{id}_V} = I_n$.

Theorem 0.1 is Theorem 0.7 on homework set #7 (where I give a reference to a proof in Lankham/Nachtergaele/Schilling). Theorem 0.2 follows easily from the definition of $M_{\mathbf{v},\mathbf{v},\mathrm{id}_V}$.

Now, let me prove something stronger than the claim of part **(j)**:

Proposition 0.3. Let V and W be two vector spaces. Let $\mathbf{v}$ be a basis of V. Let $\mathbf{w}$ be a basis of W. Let $F : V \to W$ be an invertible linear map. Then, the matrix $M_{\mathbf{v},\mathbf{w},F}$ is invertible, and its inverse is $M_{\mathbf{w},\mathbf{v},F^{-1}} = (M_{\mathbf{v},\mathbf{w},F})^{-1}$.

Proposition 0.3 generalizes Exercise 5 **(j)**, because it allows $\mathbf{v}$ and $\mathbf{w}$ to be bases of two **different** vector spaces. (Attention: The roles of $\mathbf{v}$ and $\mathbf{w}$ in Exercise 5 **(j)** are switched as compared to Proposition 0.3.) Let me now prove Proposition 0.3.

Proof of Proposition 0.3. First of all, F^{-1} is the inverse of a linear map, and thus itself is linear (by Proposition 6.7.3 in Lankham/Nachtergaele/Schilling). Hence, the matrix $M_{\mathbf{v},\mathbf{w},F^{-1}}$ is well-defined.

Let n be the size of $\mathbf{v}$. Let m be the size of $\mathbf{w}$. (It is easy to see that $n = m$, but we will not need this.)

Theorem 0.1 (applied to V, W, V, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v}$, F and F^{-1} instead of U, V, W, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, F and G) yields that the composition $F^{-1} \circ F : V \to V$ is again a linear map, and that we have

$$M_{\mathbf{v},\mathbf{v},F^{-1}\circ F} = M_{\mathbf{w},\mathbf{v},F^{-1}} M_{\mathbf{v},\mathbf{w},F}.$$

Thus,

$$\begin{aligned} M_{\mathbf{w},\mathbf{v},F^{-1}} M_{\mathbf{v},\mathbf{w},F} = M_{\mathbf{v},\mathbf{v},F^{-1}\circ F} &= M_{\mathbf{v},\mathbf{v},\mathrm{id}_V} \qquad \left(\text{since } F^{-1} \circ F = \mathrm{id}_V\right) \\ &= I_n \qquad (\text{by Theorem 0.2}) . \end{aligned} \tag{2}$$

On the other hand, Theorem 0.1 (applied to W, V, W, $\mathbf{w}$, $\mathbf{v}$, $\mathbf{w}$, F^{-1} and F instead of U, V, W, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, F and G) yields that the composition $F \circ F^{-1} : W \to W$ is again a linear map, and that we have

$$M_{\mathbf{w},\mathbf{w},F\circ F^{-1}} = M_{\mathbf{v},\mathbf{w},F} M_{\mathbf{w},\mathbf{v},F^{-1}}.$$

Thus,

$$\begin{aligned} M_{\mathbf{v},\mathbf{w},F} M_{\mathbf{w},\mathbf{v},F^{-1}} = M_{\mathbf{w},\mathbf{w},F\circ F^{-1}} &= M_{\mathbf{w},\mathbf{w},\mathrm{id}_W} \qquad \left(\text{since } F \circ F^{-1} = \mathrm{id}_W\right) \\ &= I_m \qquad \left(\begin{array}{c} \text{by Theorem 0.2, applied to } W, \mathbf{w} \text{ and } m \\ \text{instead of } V, \mathbf{v} \text{ and } n \end{array}\right). \end{aligned} \tag{3}$$

The equalities (2) and (3) (combined) show that the matrix $M_{\mathbf{w},\mathbf{v},F^{-1}}$ is inverse to $M_{\mathbf{v},\mathbf{w},F}$. In other words, the matrix $M_{\mathbf{v},\mathbf{w},F}$ is invertible, and its inverse is $M_{\mathbf{w},\mathbf{v},F^{-1}} = (M_{\mathbf{v},\mathbf{w},F})^{-1}$. This proves Proposition 0.3. □

□

A few reminders on notations and terminology (NOT a complete list of things you will want to use!):

Maps:

- A map $f : X \to Y$ between two sets X and Y is said to be *injective* (or *one-to-one*) if it has the following property:
  - If x_1 and x_2 are two elements of X satisfying $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- A map $f : X \to Y$ between two sets X and Y is said to be *surjective* (or *onto*) if it has the following property:
  - For each $y \in Y$, there exists some $x \in X$ satisfying $f(x) = y$.
- A map $f : X \to Y$ between two sets X and Y is *bijective* if it is both injective and surjective (or, equivalently, if it is invertible).
- A map between vector spaces is said to be an *isomorphism* if and only if it is linear and bijective.
- The composition $f \circ g$ of a map $f : X \to Y$ with a map $g : Z \to X$ is the map $Z \to Y$ that sends each $z \in Z$ to $f(g(z))$.

Determinants:

- Adding a multiple of some row to another row preserves the determinant.
- Scaling a row by λ multiplies the determinant by λ.
- Swapping two rows negates the determinant (i.e., multiplies it by -1).
- The determinant of a lower-triangular or upper-triangular matrix equals the product of its diagonal entries.