**Math 4242 Fall 2016 (Darij Grinberg): midterm 3 pratice problems**

**Exercise 1.** **(a)** Diagonalize the matrix $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$.

**(b)** Diagonalize the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

**(c)** Diagonalize the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

**(d)** Diagonalize the matrix $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$ for arbitrary $n$ nonzero reals $a_1, a_2, \ldots, a_n$ satisfying $a_1 + a_2 + \cdots + a_n \neq 0$.

[Part **(d)** is supposed to be challenging! Things like this won't be on the exam.]

**(e)** Can the matrix $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ be diagonalized? (This is to show that the $a_1 + a_2 + \cdots + a_n \neq 0$ condition in part **(d)** is needed.)

**Exercise 2.** A $3 \times 3$-matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is said to be *magic* if it satisfies the chain of equalities

$$\begin{aligned} & a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = a_3 + b_3 + c_3 \\ & = a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = c_1 + c_2 + c_3 \\ & = a_1 + b_2 + c_3 = c_1 + b_2 + a_3. \end{aligned}$$

(When the entries of the matrix are integers, it is what is commonly called a "magic square".) For example, $\begin{pmatrix} -3 & 8 & -2 \\ 2 & 1 & 0 \\ 4 & -6 & 5 \end{pmatrix}$ is a magic $3 \times 3$-matrix.

The magic $3 \times 3$-matrices form a subspace of $\mathbb{R}^{3 \times 3}$. Denote this subspace by $\mathcal{M}_3$.

**(a)** Find a basis of this space.

**(b)** Consider the map $\operatorname{col}_1 : \mathcal{M}_3 \to \mathbb{R}^3$ that sends any magic $3 \times 3$-matrix to its first column. (Thus, $\operatorname{col}_1 \left( \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \right) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$.)

Is $\operatorname{col}_1$ linear? If so, find the matrix representing it with respect to your chosen basis of $\mathcal{M}_3$ and the standard basis $(e_1, e_2, e_3)$ of $\mathbb{R}^3$.

Is $\operatorname{col}_1$ injective? Surjective? Bijective?

(**Note:** The question whether $\operatorname{col}_1$ is injective is tantamount to asking whether a magic $3 \times 3$-matrix is uniquely determined by its first column. For example, if you know that the first column of a magic $3 \times 3$-matrix is $\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$, then can you reconstruct the whole matrix?

The question whether $\operatorname{col}_1$ is surjective is tantamount to asking whether every column vector of size 3 appears as a first column of a magic $3 \times 3$-matrix. For example, is there a magic $3 \times 3$-matrix with first column $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ ?)

**(c)** Consider the map $\mathrm{NW}_2 : \mathcal{M}_3 \to \mathbb{R}^{2\times 2}$ that sends any magic $3 \times 3$-matrix to its "northwestern $2 \times 2$-submatrix":

$$\mathrm{NW}_2 \left( \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \right) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

Is $\mathrm{NW}_2$ linear? If so, find the matrix representing it with respect to your chosen basis of $\mathcal{M}_3$ and the standard basis $(E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2})$ of $\mathbb{R}^{2\times 2}$.

Is $\mathrm{NW}_2$ injective? Surjective? Bijective?

**(d)** Consider the map $\operatorname{mid} : \mathcal{M}_3 \to \mathbb{R}$ that sends any magic $3 \times 3$-matrix to its "middle entry":

$$\operatorname{mid} \left( \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \right) = b_2.$$

Is mid linear? If so, find the matrix representing it with respect to your chosen basis of $\mathcal{M}_3$ and the standard basis $(1)$ of $\mathbb{R}$.

Is mid injective? Surjective? Bijective?

**(e)** The *Siamese map* $S : \mathbb{R}^2 \to \mathcal{M}_3$ is defined as follows:

$$S\left( (a, b)^T \right) = \begin{pmatrix} a+7b & a & a+5b \\ a+2b & a+4b & a+6b \\ a+3b & a+8b & a+b \end{pmatrix}.$$

(This corresponds to the Siamese method of constructing magic squares.)

Is $S$ linear? If so, find the matrix representing it with respect to the standard basis $(e_1, e_2)$ of $\mathbb{R}^2$ and your chosen basis of $\mathcal{M}_3$.

**(f)** The map $T : \mathcal{M}_3 \to \mathcal{M}_3$ is defined as follows:

$$T(A) = A^T.$$

(In other words, it sends a magic $3 \times 3$-matrix to its transpose. Notice that the transpose is magic, too, as you can easily see.)

Is $T$ linear? If so, find the matrix representing it with respect to your chosen basis of $\mathcal{M}_3$ and your chosen basis of $\mathcal{M}_3$.

Eigenvectors and eigenvalues make sense not only for square matrices, but also for linear maps from a vector space to itself. (This is not that surprising, seeing that square matrices are used to represent such linear maps.) Here is how they are defined:

**Definition 0.1.** Let $V$ be a vector space, and $F : V \to V$ a linear map. (We work with real numbers in this exercise, so all vector spaces are over $\mathbb{R}$, and all scalars are in $\mathbb{R}$. The downside of this is that we miss some eigenvalues; but that's OK for an introduction.)

- Given a scalar $\lambda \in \mathbb{R}$ and a vector $v \in V$, we say that $v$ is an *eigenvector of $F$ for eigenvalue $\lambda$* if and only if we have $F(v) = \lambda v$.
- A scalar $\lambda \in \mathbb{R}$ is said to be an *eigenvalue* of $F$ if and only if there exists a nonzero eigenvector of $F$ for eigenvalue $\lambda$. (Of course, the zero vector $\overrightarrow{0}$ is an eigenvector of $F$ for eigenvalue $\lambda$ for any $\lambda \in \mathbb{R}$; thus, in order to have an interesting notion, we must require "nonzero".)

One way to find eigenvalues and eigenvectors of linear maps is the following: Let $\mathbf{v}$ be any basis of $V$ (assuming that $V$ is finite-dimensional). Then, the eigenvalues of $F$ are precisely the eigenvalues of the matrix $M_{\mathbf{v},\mathbf{v},F}$. Moreover, the eigenvectors of $F$ for a given eigenvalue $\lambda$ are the images of the eigenvectors of $M_{\mathbf{v},\mathbf{v},F}$ for $\lambda$ under the map $L_{\mathbf{v}}$. (See homework set #7 for the definition of $L_{\mathbf{v}}$.)

**Example 0.2.** Let $P_2$ be the vector space of all polynomials of degree $\leq 2$. Let $F : P_2 \to P_2$ be the map that sends every polynomial $f \in P_2$ to $\left(f\left[x^2\right]\right)''$ (that is, the second derivative of $f\left[x^2\right]$). Here, I am again using the notation $f[y]$ for "$f$ evaluated at $y$" (which would usually be denoted by $f(y)$, but that would risk being mistaken for a product).

The map $F$ is well-defined (this is easy to check: evaluating a polynomial $f$ at $x^2$ raises the degree of $f$ to 4 (in the worst case), but then differentiating it twice pulls it back to $\leq 2$) and linear (this is, again, straightforward to check). For example,

$$F\left(x^2\right) = \left(\underbrace{x^2\left[x^2\right]}_{=\left(x^2\right)^2=x^4}\right)'' = \left(x^4\right)'' = \left(4x^3\right)' = 12x^2. \qquad (1)$$

What are the eigenvalues and eigenvectors of $F$ ?

Actually, we have seen one of these already: The equality (1) shows that $x^2$ is an eigenvector of $F$ for eigenvalue 12. But there are more; let's find them all.

We let $\mathbf{v}$ be the basis $\left(1, x, x^2\right)$ of $P_2$. Then, it is straightforward to compute

$$M_{\mathbf{v},\mathbf{v},F} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{pmatrix}.$$

Now, what are the eigenvalues and the eigenvectors of this matrix $M_{\mathbf{v},\mathbf{v},F}$ ?

It is easy to see that $\det(M_{\mathbf{v},\mathbf{v},F} - xI_3) = x^2(12-x)$. Thus, the eigenvalues of $M_{\mathbf{v},\mathbf{v},F}$ are 0 and 12. Therefore, these are also the eigenvalues of $F$. Moreover:

- The eigenvectors of $M_{\mathbf{v},\mathbf{v},F}$ for eigenvalue 0 are the elements of

$$\operatorname{Ker}(M_{\mathbf{v},\mathbf{v},F} - 0I_3) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right),$$

  i.e., the multiples of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

- The eigenvectors of $M_{\mathbf{v},\mathbf{v},F}$ for eigenvalue 12 are the elements of

$$\operatorname{Ker}(M_{\mathbf{v},\mathbf{v},F} - 12I_3) = \operatorname{span}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right),$$

  i.e., the multiples of $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(This, by the way, shows that $M_{\mathbf{v},\mathbf{v},F}$ is not diagonalizable – since it does not have 3 linearly independent eigenvectors.)

Now, recall that the eigenvectors of $F$ for a given eigenvalue $\lambda$ are the images of the eigenvectors of $M_{\mathbf{v},\mathbf{v},F}$ for $\lambda$ under the map $L_{\mathbf{v}}$. Thus:

- The eigenvectors of $F$ for eigenvalue 0 are the elements of

$$L_{\mathbf{v}}\left(\operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)\right),$$

  i.e., the multiples of

$$L_{\mathbf{v}}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 = 1.$$

  In other words, they are the constant polynomials.

- The eigenvectors of $F$ for eigenvalue 12 are the elements of

$$L_{\mathbf{v}}\left(\operatorname{span}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)\right),$$

i.e., the multiples of

$$L_{\mathbf{v}}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 = x^2.$$

**Exercise 3.** **(a)** Find the eigenvalues and the eigenvectors of the map $T_e : P_2 \to P_2$ defined by $T_e(f) = x^2 f\left[\frac{1}{x}\right]$. (This map was discussed in Exercise 2 **(e)** of homework set #6.)

**(b)** Find the eigenvalues and the eigenvectors of the map $T : \mathcal{M}_3 \to \mathcal{M}_3$ from Exercise 2 **(f)**.

**(c)** Consider the map $T' : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ that sends each matrix $A \in \mathbb{R}^{3\times 3}$ to its transpose $A^T$. Find the eigenvalues and the eigenvectors of this map.

[It might be good not to approach part **(c)** this exercise using the $M_{\mathbf{v},\mathbf{v},F}$ method, but rather think in terms of the definition. An eigenvector of the map $T'$ would be a nonzero matrix $A$ whose transpose is a scalar multiple of $A$. How would such a matrix looks like? What relations would have to hold between its entries?]