Math 4242 Fall 2016 (Darij Grinberg): midterm 2 pratice problems

Exercise 1. Consider the vector space $\mathbb{R}^3$.

(a) The list $\mathbf{a} = \left((1,2,-1)^T, (1,1,0)^T, (0,1,-1)^T, (1,1,1)^T \right)$ spans $\mathbb{R}^3$. Shrink this list to a basis of $\mathbb{R}^3$ by removing some redundant elements.

(b) The list $\mathbf{b} = \left((-1,0,1)^T, (2,3,4)^T \right)$ is linearly independent. Extend this list to a basis of $\mathbb{R}^3$ by appending to it some elements from the list $\mathbf{a}$.

Exercise 2. **(a)** Find bases of the four subspaces of the 3×4-matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \end{pmatrix}$.

(b) [Too tricky for a midterm, but worth thinking about!] More generally: Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $A_{n\times m}$ be the $n \times m$-matrix $(\min\{i,j\})_{1\leq i\leq n,\ 1\leq j\leq m}$. (This is the $n \times m$-matrix whose (i,j)-th entry is $\min\{i,j\}$. For example, $A_{3,4}$ is the matrix A from part **(a)** of this exercise.)

Find bases of the four subspaces of $A_{n\times m}$.

If A is an $n \times k$-matrix whose columns are linearly independent, then a *QR decomposition* of A means a way to write A in the form $A = QR$, where:

- Q is an $n \times k$-matrix with orthonormal columns (this is equivalent to saying that Q is an $n \times k$-matrix satisfying $Q^T Q = I_k$);
- R is an upper-triangular $k \times k$-matrix with nonzero diagonal entries.

For example, a QR decomposition of $\begin{pmatrix} 2 & 17 \\ 4 & 13 \\ 8 & 5 \end{pmatrix}$ is

$$\begin{pmatrix} 2 & 17 \\ 4 & 13 \\ 8 & 5 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{21}} & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{21}} & \frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{21}} & \frac{-1}{\sqrt{6}} \end{pmatrix}}_{\text{this is the } Q} \underbrace{\begin{pmatrix} 2\sqrt{21} & 3\sqrt{21} \\ 0 & 7\sqrt{6} \end{pmatrix}}_{\text{this is the } R}.$$

Exercise 3. **(a)** Find a QR decomposition of the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(b) Find a QR decomposition of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

(c) Find a QR decomposition of the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

[**Hint:** Two of the three parts are easy and can be done with no computations whatsoever!]

Exercise 4. **(a)** Apply the Gram-Schmidt process to the two vectors

$$w_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

in $\mathbb{R}^3$.

(b) Let U be the subspace of $\mathbb{R}^3$ spanned by w_1, w_2. Find the projection u of the vector $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ on the subspace U.

Exercise 5. Find the least-squares solution to the equation $Ax = b$, where $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

Exercise 6. Let $p \in \mathbb{N}$. Find the least-squares solution $x \in \mathbb{R}^2$ to the equation $Ax = b$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ -1 \\ 0 \end{pmatrix}$. (The matrix A has $p+2$ rows and 2 columns, and the column vector b has size $p+2$. All entries of A are 1's except for the last two entries of the second column. All entries of b are 1 except for the last two entries.)

(For example, if $p = 3$, then $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, and the least-squares solution is $\begin{pmatrix} \frac{9}{10} \\ -\frac{1}{2} \end{pmatrix}$.)

[Feel free to check your result visually: This exercise is a data-fitting problem, where you are trying to fit a line $y = \alpha t + \beta$ through the $p + 2$ points

$$\underbrace{(1,1)\,,\ (1,1)\,,\ \ldots,\ (1,1)}_{p \text{ times}},\ (2,-1)\,,\ (0,0)\,.$$

Thus, the least-squares solution $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ should lead to a line $y = \alpha + \beta t$ that comes relatively close to all these points, but gets pulled closer and closer to $(1,1)$ when p grows (because with growing p, the point $(1,1)$ gets repeated more often and thus "pulls more weight").]