**Math 4242 Fall 2016 (Darij Grinberg): midterm 1 with solutions**
**Mon, 3 Oct 2016, in class (75 minutes). Proofs are NOT required.**

**Exercise 1.** Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute the matrices $AA^T$ and $A^TA$. [5+5 points]

*Solution.* From $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, we obtain $A^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Now that both $A$ and $A^T$ are known, we can find $AA^T$ and $A^TA$ by matrix multiplication:

$$AA^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix};$$
$$A^TA = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

[*Remark:* The patterns seen here generalize. If $A_n$ denotes the $n \times n$-matrix whose entries on and above the diagonal are 1, and whose entries below the diagonal are 0 (so $A_3$ is our matrix $A$), then $A_n (A_n)^T$ is the $n \times n$-matrix $(n + 1 - \max\{i, j\})_{1 \leq i \leq n,\ 1 \leq j \leq n}$, and $(A_n)^T A_n$ is the $n \times n$-matrix $(\min\{i, j\})_{1 \leq i \leq n,\ 1 \leq j \leq n}$.] □

Elementary matrices are square matrices of the following three kinds:

- The $n \times n$-matrix $A_{u,v}^{\lambda}$ (for an $n \in \mathbb{N}$, two distinct elements $u$ and $v$ of $\{1, 2, \ldots, n\}$, and a number $\lambda$). Its $(u, v)$-th entry is $\lambda$; its diagonal entries are 1; all its other entries are 0.
- The $n \times n$-matrix $S_u^{\lambda}$ (for an $n \in \mathbb{N}$, an element $u \in \{1, 2, \ldots, n\}$, and a number $\lambda \neq 0$). Its $(u, u)$-th entry is $\lambda$; all its other diagonal entries are 1; all its remaining entries are 0.
- The $n \times n$-matrix $T_{u,v}$ (for an $n \in \mathbb{N}$ and two distinct elements $u$ and $v$ of $\{1, 2, \ldots, n\}$). It is the identity matrix $I_n$ with the $u$-th and $v$-th rows swapped.

**Exercise 2.** **($a_1$)** Write the matrix $C = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ as a product of the form $EU$, where $E$ is a product of elementary matrices, and where $U$ is an upper-triangular matrix.

(Do **not** multiply $E$ out! Instead, write $E$ as a product of elementary matrices (possibly of only one factor).)

**(a$_2$)** Do the same with the matrix $C' = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$

**(b$_1$)** Do the same with the matrix $D = \begin{pmatrix} 3 & 1 & 2 \\ 3 & 1 & 3 \\ 0 & 2 & 2 \end{pmatrix}$.

**(b$_2$)** Do the same with the matrix $D' = \begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}$.

[10+10 points]

*Solution.* The solutions given below are probably the shortest, but surely not the only possible.

**(a$_1$)** We perform Gaussian elimination on $C$:

$$\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \underset{A_{2,1}^{-1}}{\overset{A_{2,1}^{1}}{\rightleftarrows}} \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}.$$

Thus, $C = EU$ for $E = A_{2,1}^{-1}$ and $U = \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix}$.

**(a$_2$)** We perform Gaussian elimination on $C'$:

$$\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \underset{A_{2,1}^{-1}}{\overset{A_{2,1}^{1}}{\rightleftarrows}} \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix}.$$

Thus, $C' = EU$ for $E = A_{2,1}^{-1}$ and $U = \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix}$.

**(b$_1$)** We perform Gaussian elimination on $D$:

$$\begin{pmatrix} 3 & 1 & 2 \\ 3 & 1 & 3 \\ 0 & 2 & 2 \end{pmatrix} \underset{A_{2,1}^{1}}{\overset{A_{2,1}^{-1}}{\rightleftarrows}} \begin{pmatrix} 3 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 2 \end{pmatrix} \underset{T_{2,3}}{\overset{T_{2,3}}{\rightleftarrows}} \begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, $D = EU$ for $E = A_{2,1}^{1} T_{2,3}$ and $U = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

**(b$_2$)** We perform Gaussian elimination on $D'$:

$$\begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix} \underset{A_{2,1}^{1}}{\overset{A_{2,1}^{-1}}{\rightleftarrows}} \begin{pmatrix} 3 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix} \underset{T_{2,3}}{\overset{T_{2,3}}{\rightleftarrows}} \begin{pmatrix} 3 & 1 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus, $D' = EU$ for $E = A_{2,1}^{1} T_{2,3}$ and $U = \begin{pmatrix} 3 & 1 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$. □

The next few problems are about determinants. You are allowed to use everything I have told you about determinants, including:

- The row operation $A_{u,v}^{\lambda}$ (adding $\lambda$ times the $v$-th row to the $u$-th row) preserves the determinant (that is, $\det\left(A_{u,v}^{\lambda}C\right) = \det C$ for any $C$).
- The row operation $S_u^{\lambda}$ (scaling the $u$-th row by $\lambda$) multiplies the determinant by $\lambda$ (that is, $\det\left(S_u^{\lambda}C\right) = \lambda \det C$ for any $C$).
- The row operation $T_{u,v}$ (swapping rows $u$ and $v$) negates the determinant (that is, $\det\left(T_{u,v}C\right) = -\det C$ for any $C$).
- We have $\det(AB) = \det A \cdot \det B$ for any two $n \times n$-matrices $A$ and $B$.
- The determinant of a lower-triangular or upper-triangular matrix equals the product of its diagonal entries.

**Exercise 3.** Compute $\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$. [10 points]

*Solution.* We perform row operations on our matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{A_{2,1}^{-1}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{A_{3,1}^{-1}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$
$$\xrightarrow{A_{3,2}^{-1}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix obtained is upper-triangular, and thus its determinant is the product of its diagonal entries, namely $1 \cdot 1 \cdot 1 = 1$. Since all our row operations have preserved the determinant (because each row operation $A_{u,v}^{\lambda}$ preserves the determinant), this yields that 1 is also the determinant of the initial matrix. In other words, $\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} = 1$. □

**Exercise 4.** Compute $\det \begin{pmatrix} 1 & 2 & 7 & 0 \\ 4 & -1 & 3 & 8 \\ 3 & 0 & 0 & 0 \\ 2 & 12 & 0 & 0 \end{pmatrix}$. [10 points]

*Solution.* We perform row operations on our matrix:

$$\begin{pmatrix} 1 & 2 & 7 & 0 \\ 4 & -1 & 3 & 8 \\ 3 & 0 & 0 & 0 \\ 2 & 12 & 0 & 0 \end{pmatrix} \xrightarrow{T_{1,3}} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & -1 & 3 & 8 \\ 1 & 2 & 7 & 0 \\ 2 & 12 & 0 & 0 \end{pmatrix} \xrightarrow{T_{2,4}} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 2 & 12 & 0 & 0 \\ 1 & 2 & 7 & 0 \\ 4 & -1 & 3 & 8 \end{pmatrix}.$$

The matrix obtained is lower-triangular, and thus its determinant is the product of its diagonal entries, namely $3 \cdot 12 \cdot 7 \cdot 8 = 2016$. But we are looking for the determinant of the **initial** matrix, not of the final one. So we need to know what our row operations did to the determinant.

Each of the two row operations $T_{1,3}$ and $T_{2,4}$ has negated the determinant (as each $T_{u,v}$ does). Thus, the determinant was negated twice altogether. As a result, its final value is $(-1)^2$ times its original value. Since $(-1)^2 = 1$, this simply means that its final value is its original value. Since the final value is 2016, we thus conclude that the original value is 2016 as well. In other words, $\det \begin{pmatrix} 1 & 2 & 7 & 0 \\ 4 & -1 & 3 & 8 \\ 3 & 0 & 0 & 0 \\ 2 & 12 & 0 & 0 \end{pmatrix} = 2016$. □

For the purposes of the next two exercises, $\mathbb{R}^3$ shall denote the vector space $\mathbb{R}^{3\times 1}$ of all column vectors of size 3.

**Exercise 5.** Which of the following ten sets is a subspace of $\mathbb{R}^3$ ?

$$\begin{aligned}
A &= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \ \mid \ x_1 + x_2 = 0 \right\}; \\
B &= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \ \mid \ x_1 = x_2 + 1 \text{ and } x_2 = x_3 + 1 \right\}; \\
C &= \varnothing; \\
D &= \mathbb{R}^3; \\
E &= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \ \mid \ x_1 + x_2 = -x_1 - x_2 \right\}; \\
F &= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \ \mid \ x_1 x_2 = 0 \right\}; \\
G &= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \ \mid \ x_1^2 + 2x_1 x_2 + x_2^2 = 0 \right\}; \\
H &= \left\{ (1,1,1)^T \right\}; \\
I &= \left\{ (u, 2u, 3u)^T \ \mid \ u \in \mathbb{R} \right\}; \\
J &= \left\{ (u+1, u+2, u+3)^T \ \mid \ u \in \mathbb{R} \right\}.
\end{aligned}$$

[3+3+3+3+3+3+3+3+3+3 points]

*Solution.* **Short answer:**

| $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ | $I$ | $J$ |
|---|---|---|---|---|---|---|---|---|---|
| Y | N | N | Y | Y | N | Y | N | Y | N |

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**(a)** The set $A$ is a subspace of $\mathbb{R}^3$.

[*Proof.* The zero vector (a.k.a. origin) $\overrightarrow{0} = 0_{3\times 1}$ of $\mathbb{R}^3$ does belong to $A$ (because its entries $0, 0, 0$ do satisfy $0 + 0 = 0$).

Let us next show that $A$ is closed under addition. Indeed, let $v$ and $w$ be two elements of $A$. We must show that $v + w \in A$. Write $v$ as $v = (v_1, v_2, v_3)^T$. Then,

$(v_1, v_2, v_3)^T = v \in A$ yields $v_1 + v_2 = 0$. Write $w$ as $w = (w_1, w_2, w_3)^T$. Then, $(w_1, w_2, w_3)^T = w \in A$ yields $w_1 + w_2 = 0$. We have $(v_1 + w_1, v_2 + w_2, v_3 + w_3)^T \in A$, because

$$(v_1 + w_1) + (v_2 + w_2) = \underbrace{(v_1 + v_2)}_{=0} + \underbrace{(w_1 + w_2)}_{=0} = 0 + 0 = 0.$$

Now,

$$v + w = (v_1, v_2, v_3)^T + (w_1, w_2, w_3)^T = (v_1 + w_1, v_2 + w_2, v_3 + w_3)^T \in A.$$

Thus, we have proven that $A$ is closed under addition.

It remains to prove that $A$ is closed under scaling. Indeed, let $\lambda$ be a real number, and let $v$ be an element of $A$. We must show that $\lambda v \in A$. Write $v$ as $v = (v_1, v_2, v_3)^T$. Then, $(v_1, v_2, v_3)^T = v \in A$ yields $v_1 + v_2 = 0$. We have $(\lambda v_1, \lambda v_2, \lambda v_3)^T \in A$, because

$$\lambda v_1 + \lambda v_2 = \lambda \underbrace{(v_1 + v_2)}_{=0} = \lambda 0 = 0.$$

Now,

$$\lambda v = \lambda (v_1, v_2, v_3)^T = (\lambda v_1, \lambda v_2, \lambda v_3)^T \in A.$$

Thus, we have proven that $A$ is closed under scaling. We now have completed the proof of the fact that $A$ is a subspace of $\mathbb{R}^3$.

The above argument is straightforward and generalizes to the claim that any subset of $\mathbb{R}^{n\times 1}$ "carved out" by a system of linear equations with no constant terms is a subspace. In our situation, the subset $A$ is "carved out" by the system that consists of the single equation $x_1 + x_2 = 0$.]

**(b)** The set $B$ is **not** a subspace of $\mathbb{R}^3$.

[*Proof.* It does not contain the zero vector $\overrightarrow{0}$, since $0 = 0 + 1$ is not satisfied.]

**(c)** The set $C$ is **not** a subspace of $\mathbb{R}^3$.

[*Proof.* It does not contain the zero vector $\overrightarrow{0}$. It is empty!]

**(d)** The set $D$ is a subspace of $\mathbb{R}^3$.

[*Proof.* All three axioms for a subspace are clearly satisfied, because $D$ is the whole $\mathbb{R}^3$.]

**(e)** The set $E$ is a subspace of $\mathbb{R}^3$.

[*Proof.* Analogous to the proof of $A$. Again, this is a subspace "carved out" by a system of linear equations (only that this time, this system consists of the single equation $x_1 + x_2 = -x_1 - x_2$). Note that the equation $x_1 + x_2 = -x_1 - x_2$ can be rewritten as $2x_1 + 2x_2 = 0$, which can be further rewritten as $x_1 + x_2 = 0$. Hence, $E = A$.]

**(f)** The set $F$ is **not** a subspace of $\mathbb{R}^3$.

[*Proof.* It is not closed under addition: In fact, it contains the two vectors $(1, 0, 0)^T$ and $(0, 1, 0)^T$, but does not contain their sum $(1, 1, 0)^T$.]

**(g)** The set $G$ is a subspace of $\mathbb{R}^3$.
[*Proof.* Rewrite $G$ as follows:

$$G = \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid \underbrace{x_1^2 + 2x_1x_2 + x_2^2}_{=(x_1+x_2)^2} = 0 \right\}$$
$$= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid \underbrace{(x_1 + x_2)^2 = 0}_{\substack{\text{this is equivalent to} \\ (x_1+x_2=0)}} \right\}$$
$$= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 + x_2 = 0 \right\}.$$

From here on, this proceeds as the proof for $A$, since $x_1 + x_2 = 0$ is clearly a linear equation with no constant term. (Actually, we see now that $G = E = A$.)

**(h)** The set $H$ is **not** a subspace of $\mathbb{R}^3$.
[*Proof.* It does not contain the zero vector $\overrightarrow{0}$.]

**(i)** The set $I$ is a subspace of $\mathbb{R}^3$.
[*Proof.* We have

$$I = \left\{ \underbrace{(u, 2u, 3u)^T}_{=u(1,2,3)^T} \mid u \in \mathbb{R} \right\}$$
$$= \left\{ u\,(1, 2, 3)^T \mid u \in \mathbb{R} \right\}$$
$$= \left( \text{the set of all linear combinations of the single vector } (1, 2, 3)^T \right)$$
$$= \operatorname{span}\left( (1, 2, 3)^T \right).$$

Thus, $I$ is a span, and thus a subspace (since every span is a subspace).

Of course, we could have also proven this without using the word "span", just by arguing that $(u, 2u, 3u)^T + (v, 2v, 3v)^T = (u + v, 2\,(u + v)\,, 3\,(u + v))^T$ etc.]

**(j)** The set $J$ is **not** a subspace of $\mathbb{R}^3$.
[*Proof.* For the umpteenth time, it does not contain the zero vector $\overrightarrow{0}$. I should have asked some more interesting question :)] □

**Exercise 6.** **(a$_1$)** Find a vector that spans the subspace

$$K = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 + 2x_2 = 0 \right\}$$

of $\mathbb{R}^2$ (where $\mathbb{R}^2$ means $\mathbb{R}^{2\times 1}$).

**($a_2$)** Find a vector that spans the subspace

$$K' = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \mid 2x_1 + x_2 = 0 \right\}$$

of $\mathbb{R}^2$ (where $\mathbb{R}^2$ means $\mathbb{R}^{2\times 1}$).

**(b)** Find a pair of vectors that spans the subspace

$$L = \left\{ (u+v, u+2v, u+3v)^T \mid u \in \mathbb{R} \text{ and } v \in \mathbb{R} \right\}$$

of $\mathbb{R}^3$. [10+10 points]

*Solution.* As usual with problems like this, there are many different correct answers.

**($a_1$)** The set $K$ is the set of all solutions of the system of (one) linear equation $\{x_1 + 2x_2 = 0$ in the two unknowns $x_1, x_2$. Solving this system, we find that its solutions have the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2s \\ s \end{pmatrix}$$

with a free variable $s$. Thus,

$$\begin{aligned} K &= \left\{ \begin{pmatrix} -2s \\ s \end{pmatrix} \mid s \in \mathbb{R} \right\} \\ &= \left\{ s \begin{pmatrix} -2 \\ 1 \end{pmatrix} \mid s \in \mathbb{R} \right\} \qquad \left( \text{since } \begin{pmatrix} -2s \\ s \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right) \\ &= \operatorname{span}\left( \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Thus, the vector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ spans $K$.

**($a_2$)** The set $K'$ is the set of all solutions of the system of (one) linear equation $\{2x_1 + x_2 = 0$ in the two unknowns $x_1, x_2$. Solving this system, we find that its solutions have the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}s \\ s \end{pmatrix}$$

with a free variable $s$. Thus,

$$\begin{aligned} K' &= \left\{ \begin{pmatrix} -\frac{1}{2}s \\ s \end{pmatrix} \mid s \in \mathbb{R} \right\} \\ &= \left\{ s \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \mid s \in \mathbb{R} \right\} \qquad \left( \text{since } \begin{pmatrix} -\frac{1}{2}s \\ s \end{pmatrix} = s \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \right) \\ &= \operatorname{span}\left( \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \right). \end{aligned}$$

Thus, the vector $\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$ spans $K'$.

[*Remark:* Parts **(a$_1$)** and **(a$_2$)** of the problem are clearly identical up to the switching roles of $x_1$ and $x_2$. (We could make this more formal using the notion of "isomorphism", but we will learn that later.) So why are the answers $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$ dissimilar? Because I have been solving the two systems $\{x_1 + 2x_2 = 0$ and $\{2x_1 + x_2 = 0$ by back-substitution. Back-substitution significantly depends on the order of unknown, since it starts with the last unknown and then works its way forward; thus, it is "biased" towards giving a simple expression for the last unknown (which, in our situation, is $x_2$). Of course, if we wanted to give two similar answers, we could give $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ for $K$, and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for $K'$.]

**(b)** We have

$$\begin{aligned} L &= \left\{ \underbrace{(u+v, u+2v, u+3v)^T}_{=u(1,1,1)^T + v(1,2,3)^T} \mid u \in \mathbb{R} \text{ and } v \in \mathbb{R} \right\} \\ &= \left\{ u\,(1,1,1)^T + v\,(1,2,3)^T \mid u \in \mathbb{R} \text{ and } v \in \mathbb{R} \right\} \\ &= \operatorname{span}\left( (1,1,1)^T, (1,2,3)^T \right). \end{aligned}$$

So $\left( (1,1,1)^T, (1,2,3)^T \right)$ is a pair that fits the bill. (Of course, so do many other pairs, for example $\left( (1,1,1)^T, (0,1,2)^T \right)$.) □