Math 4242 Fall 2016 (Darij Grinberg): midterm 1 pratice problems

Rough list of examineable material (bold = recommended)

	lina	L/N/S	O/S	Hefferon
matrix arithmetic	**2**	A.2	1.2	
inverses	**3.2**	(A.2.3)	1.5	
transposes	**2.4, 3.3**		1.6	
triangular matrices	**3.4**		(1.3)	
$A^{\lambda}_{u,v}, S^{\lambda}_{u}, T_{u,v}$	**3.8, 3.13, 3.17**	(A.3.1)	1.3	
Gauss	example in 3.22?	**A.3**	1.3–1.4	One.I.1
determinants	(3.24)	**8**	1.9	Four
vector spaces		**4.1–4.2**	2.1	Two.I.1
subspaces		**4.3**	2.2	Two.I.2
spans		**5.1**	2.3	**Two.I.2**

Exercise 1. **(a)** Let $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $B_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Compute A_3^2, B_3^2, A_3B_3 and B_3A_3.

(b) Let $A_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ and $B_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Compute A_4^2, B_4^2, A_4B_4 and B_4A_4.

Solution. **(a)** $A_3^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $B_3^2 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$, $A_3B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$ and $B_3A_3 = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$.

(b) $A_4^2 = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$, $B_4^2 = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$, $A_4B_4 = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}$ and $B_4A_4 = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}$.

[Computing these products by hand is noticeably simplified by the fact that each of the matrices has only two different rows and only two different columns. This leads to the recognizable patterns in the products.] □

Exercise 2. Recall that we have seen three types of elementary matrices:

- The $n \times n$-matrix $A_{u,v}^{\lambda}$ (for an $n \in \mathbb{N}$, two distinct elements u and v of $\{1, 2, \ldots, n\}$, and a number λ). Its (u, v)-th entry is 1; its diagonal entries are 1; all its other entries are 0.
- The $n \times n$-matrix S_u^{λ} (for an $n \in \mathbb{N}$, an element $u \in \{1, 2, \ldots, n\}$, and a number $\lambda \neq 0$). Its (u, u)-th entry is λ; all its other diagonal entries are 1; all its remaining entries are 0.
- The $n \times n$-matrix $T_{u,v}$ (for an $n \in \mathbb{N}$ and two distinct elements u and v of $\{1, 2, \ldots, n\}$). It is the identity matrix I_n with the u-th and v-th rows swapped.

Write the matrix $C = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 3 & 0 \end{pmatrix}$ as a product of the form EU, where E is a product of elementary matrices, and where U is an upper-triangular matrix.

Solution. We perform Gaussian elimination on C:

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 3 & 0 \end{pmatrix} \underset{T_{1,2}}{\overset{T_{1,2}}{\rightleftarrows}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 3 & 0 \end{pmatrix} \underset{A_{4,1}^{1}}{\overset{A_{4,1}^{-1}}{\rightleftarrows}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\underset{A_{3,2}^{1}}{\overset{A_{3,2}^{-1}}{\rightleftarrows}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} \underset{T_{3,4}}{\overset{T_{3,4}}{\rightleftarrows}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, $C = EU$ for $E = T_{1,2} A_{4,1}^{1} A_{3,2}^{1} T_{3,4}$ and $U = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. □

The next few problems are about determinants. You are allowed to use everything I have told you about determinants, including:

- The row operation $A_{u,v}^{\lambda}$ (adding λ times the v-th row to the u-th row) preserves the determinant (that is, $\det\left(A_{u,v}^{\lambda} C\right) = \det C$ for any C).

- The row operation S_u^λ (scaling the u-th row by λ) multiplies the determinant by λ (that is, $\det\left(S_u^\lambda C\right) = \lambda \det C$ for any C).
- The row operation $T_{u,v}$ (switching rows u and v) negates the determinant (that is, $\det\left(T_{u,v}C\right) = -\det C$ for any C).
- Similarly for column operations.
- We have $\det(AB) = \det A \cdot \det B$ for any two $n \times n$-matrices A and B.
- The determinant of a lower-triangular or upper-triangular matrix equals the product of its diagonal entries.

Try using these tactics to your advantage! The definition of $\det A$ for an $n \times n$-matrix A is as a sum with $n! = 1 \cdot 2 \cdot \cdots \cdot n$ addends; this is too much to compute by hand already for $n = 4$.

(Yes, you can use Laplace expansion too, but make sure you get the signs right. In my experience, it is more error-prone when done by hand than other methods, so it's better to avoid it unless nothing simpler works.)

Exercise 3. Compute $\det \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 14 & 4 \\ 1 & 6 & -1 & 3 \\ 17 & 1 & 2 & 1 \end{pmatrix}$.

Solution. We perform row operations to our matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 14 & 4 \\ 1 & 6 & -1 & 3 \\ 17 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{T_{1,4}} \begin{pmatrix} 17 & 1 & 2 & 1 \\ 0 & 0 & 14 & 4 \\ 1 & 6 & -1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{T_{2,3}} \begin{pmatrix} 17 & 1 & 2 & 1 \\ 1 & 6 & -1 & 3 \\ 0 & 0 & 14 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$
$$\xrightarrow{A_{1,2}^{-17}} \begin{pmatrix} 0 & -101 & 19 & -50 \\ 1 & 6 & -1 & 3 \\ 0 & 0 & 14 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix} \xrightarrow{T_{1,2}} \begin{pmatrix} 1 & 6 & -1 & 3 \\ 0 & -101 & 19 & -50 \\ 0 & 0 & 14 & 4 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

The matrix obtained is upper-triangular, and thus its determinant is the product of its diagonal entries, namely $1 \cdot (-101) \cdot 14 \cdot 3 = -4242$. But we are looking for the determinant of the **initial** matrix, not of the final one. So we need to know what our row operations did to the determinant.

The row operation $A_{1,2}^{-17}$ preserved the determinant (as all row operations $A_{u,v}^\lambda$ do). The row operations $T_{1,4}, T_{2,3}, T_{1,2}$ negated it (as each $T_{u,v}$ does). Thus, the determinant was preserved once and negated thrice. As a result, its final value is $(-1)^3$ times its original value. So $-4242 = (-1)^3 d$, where $d = \det \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 14 & 4 \\ 1 & 6 & -1 & 3 \\ 17 & 1 & 2 & 1 \end{pmatrix}$

denotes the determinant of the initial matrix. Solving this for d, we obtain $d = 4242$. □

Exercise 4. Compute $\det \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$.

Solution. We perform row operations to our matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} \xrightarrow{A_{2,1}^{-2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} \xrightarrow{A_{3,2}^{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow{A_{4,3}^{-2/3}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}.$$

The matrix obtained is upper-triangular, and thus its determinant is the product of its diagonal entries, namely $1 \cdot (-1) \cdot 3 \cdot 1/3 = -1$. Since all row operations that we used have preserved the determinant (because they all were row operations of type $A_{u,v}^{\lambda}$), this means that the determinant of the initial matrix also was -1. □

For the purposes of the next two exercises, $\mathbb{R}^n$ shall denote the vector space $\mathbb{R}^{n \times 1}$ of all column vectors of size n.

Exercise 5. Which of the following eight sets is a subspace of $\mathbb{R}^3$?

$$\begin{aligned}
A &= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 = 2 \right\}; \\
B &= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 = x_2 = -x_3 \right\}; \\
C &= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1^2 = 0 \right\}; \\
D &= \left\{ \left(u, u^2, u^3 \right)^T \mid u \in \mathbb{R} \right\}; \\
E &= \left\{ (1 + u, -1 - u, 0)^T \mid u \in \mathbb{R} \right\}; \\
F &= \left\{ (1, 2, 3)^T \right\}; \\
G &= \operatorname{span}\left((1, 2, 3)^T \right); \\
H &= \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 - x_2 = x_2 - x_3 \right\}.
\end{aligned}$$

Solution. **(a)** The set A is not a subspace, since it does not even contain $\overrightarrow{0}$. (Indeed, $\overrightarrow{0} \in A$ would mean that $0 = 2$, which is absurd.)

(b) The set B is a subspace. Indeed, it is easy to check all three axioms.[1]

(c) The set C is a subspace. Indeed, $x_1^2 = 0$ is equivalent to $x_1 = 0$, and thus C is simply $\left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 = 0 \right\}$, which is easily seen to be a subspace (the proof is just like that for part **(b)**, but simpler).

(d) The set D is not a subspace. Indeed, it violates axiom **(b)**, because if we set $v = (1,1,1)^T$ and $w = (1,1,1)^T$, then $v \in D$ and $w \in D$ (namely, $v = w = (u, u^2, u^3)^T$ for $u = 1$), but $v + w \notin D$ (indeed, $v + w = (2,2,2)^T$ does not have the form $(u, u^2, u^3)^T$ for any $u \in \mathbb{R}$, because any such $u \in \mathbb{R}$ would have to satisfy $u = 2$ and $u^2 = 2$ and $u^3 = 2$ at the same time, which is impossible).

(It also violates axiom **(c)**; but of course, it is enough to find one contradiction.)

(e) The set E is a subspace. It can be rewritten as follows:

$$\begin{aligned} E &= \left\{ \underbrace{(1+u, -1-u, 0)^T}_{=(1+u)(1,-1,0)^T} \mid u \in \mathbb{R} \right\} \\ &= \left\{ (1+u)(1,-1,0)^T \mid u \in \mathbb{R} \right\} \\ &= \left\{ v(1,-1,0)^T \mid v \in \mathbb{R} \right\} \\ &\qquad \text{(here, we have substituted } v \text{ for } 1+u) \\ &= \operatorname{span}\left((1,-1,0)^T \right), \end{aligned}$$

which makes it clear that it is a subspace (because all spans are subspaces).

(f) The set F is not a subspace, since it does not contain $\overrightarrow{0}$.

(g) The set G is a subspsace, since it is a span (and since all spans are subspaces).

(h) The set H is a subspace. The proof is similar to the proof for B. □

Exercise 6. Define four vectors a, b, c, d in $\mathbb{R}^4$ as follows:

$$a = (4,3,2,1)^T, \qquad b = (1,2,3,4)^T,$$
$$c = (2,1,0,-1)^T, \qquad d = (-1,0,1,2)^T.$$

[1] For example, in order to check axiom **(b)**, we need to show that every $v \in B$ and $w \in B$ satisfy $v + w \in B$. Let us show this.

Let $v \in B$ and $w \in B$. Write v as $v = (v_1, v_2, v_3)^T$, and write w as $w = (w_1, w_2, w_3)^T$. Since $v \in B$, we have $v_1 = v_2 = -v_3$ (because this is what it means for v to belong to B). Similarly, $w_1 = w_2 = -w_3$. Adding the equalities $v_1 = v_2 = -v_3$ and $w_1 = w_2 = -w_3$ side by side, we obtain $v_1 + w_1 = v_2 + w_2 = (-v_3) + (-w_3)$. In other words, $v_1 + w_1 = v_2 + w_2 = -(v_3 + w_3)$. In other words, $v + w \in B$ (since $v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)^T$. This completes our proof of axiom **(b)** for B.

Checking axioms **(a)** and **(c)** is similar.

Show that $\operatorname{span}(a,b) = \operatorname{span}(c,d)$ as follows:

(a) Write each of a and b as a linear combination of c and d.

(b) Write $\lambda a + \mu b$ (for any fixed reals λ and μ) as a linear combination of c and d. Conclude that $\lambda a + \mu b \in \operatorname{span}(c,d)$ for each $\lambda, \mu \in \mathbb{R}$, and therefore $\operatorname{span}(a,b) \subseteq \operatorname{span}(c,d)$.

(c) Write each of c and d as a linear combination of a and b.

(d) Write $\lambda c + \mu d$ (for any fixed reals λ and μ) as a linear combination of a and b. Conclude that $\lambda c + \mu d \in \operatorname{span}(a,b)$ for each $\lambda, \mu \in \mathbb{R}$, and therefore $\operatorname{span}(c,d) \subseteq \operatorname{span}(a,b)$.

The results of **(b)** and **(d)** combined yield $\operatorname{span}(a,b) = \operatorname{span}(c,d)$.

Solution. **(a)** Let us first write a as a linear combination of c and d. In other words, we are seeking two real numbers γ and δ such that $a = \gamma c + \delta d$. In other words, we are solving the equation $a = \gamma c + \delta d$ in two real unknowns γ and δ. Since $a = (4,3,2,1)^T$, $c = (2,1,0,-1)^T$ and $d = (-1,0,1,2)^T$, this equation rewrites as $(4,3,2,1)^T = \gamma (2,1,0,-1)^T + \delta (-1,0,1,2)^T$. This is equivalent to the system $\begin{cases} 2\gamma + (-1)\delta = 4; \\ 1\gamma + 0\delta = 3; \\ 0\gamma + 1\delta = 2; \\ (-1)\gamma + 2\delta = 1 \end{cases}$ of linear equations. Solving this system in any way, we find $\gamma = 3$ and $\delta = 2$. Thus, $a = \gamma c + \delta d$ becomes $a = 3c + 2d$.

So we have written a as a linear combination of c and d. Similarly, we can do the same for b, obtaining $b = 2c + 3d$.

(b) For any two reals λ and μ, we have

$$\begin{aligned} \lambda \underbrace{a}_{=3c+2d} + \mu \underbrace{b}_{=2c+3d} &= \lambda (3c + 2d) + \mu (2c + 3d) \\ &= 3\lambda c + 2\lambda d + 2\mu c + 3\mu d \\ &= (3\lambda + 2\mu) c + (2\lambda + 3\mu) d. \end{aligned}$$

This is a representation of $\lambda a + \mu b$ as a linear combination of c and d. Thus, $\lambda a + \mu b \in \operatorname{span}(c,d)$ for each $\lambda, \mu \in \mathbb{R}$. In other words,

$$\{\lambda a + \mu b \mid \lambda, \mu \in \mathbb{R}\} \subseteq \operatorname{span}(c,d).$$

But the definition of $\operatorname{span}(a,b)$ yields $\operatorname{span}(a,b) = \{\lambda a + \mu b \mid \lambda, \mu \in \mathbb{R}\}$. Hence,

$$\operatorname{span}(a,b) = \{\lambda a + \mu b \mid \lambda, \mu \in \mathbb{R}\} \subseteq \operatorname{span}(c,d).$$

This solves part **(b)**.

(c) Similarly to part **(a)**, we find $c = \frac{3}{5} a + \frac{-2}{5} b$ and $d = \frac{-2}{5} a + \frac{3}{5} b$.

(d) Similarly to part **(b)**, we can represent $\lambda c + \mu d$ as a linear combination of a and b as follows:

$$\lambda c + \mu d = \left(\frac{3}{5}\lambda + \frac{-2}{5}\mu \right) a + \left(\frac{-2}{5}\lambda + \frac{3}{5}\mu \right) b.$$

□

Exercise 7. **(a)** Find a list of 3 vectors that spans the subspace

$$K = \left\{ (a,b,c,d)^T \in \mathbb{R}^4 \mid a+b = c+d \right\}$$

of $\mathbb{R}^4$.

(b) Find a list of 4 vectors that spans the subspace

$$L = \left\{ (a_1+a_2, a_2+a_3, a_3+a_4, a_4+a_1)^T \mid a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$$

of $\mathbb{R}^4$.

(c) Find a list of 3 vectors that spans L.

Solution. [**Note:** Throughout the solution of this problem, you have a lot of freedom to make choices. Thus, your answers may be completely different from mine and still correct.]

(a) The set K is the set of all solutions of the system of (one) linear equation $\{a+b = c+d$ in the four unknowns a, b, c, d. Solving this system, we find that its solutions have the form

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} s+t-u \\ u \\ t \\ s \end{pmatrix}$$

with three free variables s, t, u. Thus,

$$\begin{aligned} K &= \left\{ \begin{pmatrix} s+t-u \\ u \\ t \\ s \end{pmatrix} \mid s,t,u \in \mathbb{R} \right\} \\ &= \left\{ s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mid s,t,u \in \mathbb{R} \right\} \\ &\qquad \left(\text{since } \begin{pmatrix} s+t-u \\ u \\ t \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right) \\ &= \operatorname{span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right). \end{aligned}$$

Thus, we have written K as a span of three vectors.

(b) For every $a_1, a_2, a_3, a_4 \in \mathbb{R}$, we have

$$\begin{aligned} &(a_1 + a_2, a_2 + a_3, a_3 + a_4, a_4 + a_1)^T \\ &= \begin{pmatrix} a_1 + a_2 \\ a_2 + a_3 \\ a_3 + a_4 \\ a_4 + a_1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Thus, the definition of L rewrites as follows:

$$\begin{aligned} L &= \left\{ a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \mid a_1, a_2, a_3, a_4 \in \mathbb{R} \right\} \\ &= \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right). \qquad (1) \end{aligned}$$

Hence, we have written L as a span of four vectors.

(c) Set $\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\gamma = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, and $\delta = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Then, (1) rewrites as

$$L = \operatorname{span}(\alpha, \beta, \gamma, \delta). \qquad (2)$$

However, it is easy to observe that $\alpha + \gamma = \beta + \delta$. Hence, $\delta = \alpha + \gamma - \beta$. In particular, δ is a linear combination of α, β, γ. Hence, $\delta \in \operatorname{span}(\alpha, \beta, \gamma)$.

But in class, I mentioned the following fact:

Proposition 0.1. Let V be a vector space. Let $v_1, v_2, \ldots, v_k$ be some vectors in V. Let $W = \operatorname{span}(v_1, v_2, \ldots, v_k)$ (this is a subspace of V). Let $w \in W$. Then, $\operatorname{span}(v_1, v_2, \ldots, v_k, w) = W$.

(Roughly speaking, Proposition 0.1 says that the span of a list of vectors does not change if we append a new vector to the list, as long as this new vector already lies in the span of the old vectors.)

Now, set $V = \mathbb{R}^{4\times 1}$; let $v_1, v_2, \ldots, v_k$ be the vectors α, β, γ (so $k = 3$); let $W = \operatorname{span}(\alpha, \beta, \gamma)$; and let $w = \delta$. Then, the condition of Proposition 0.1 is satisfied (because we have $\delta \in \operatorname{span}(\alpha, \beta, \gamma)$). Thus, Proposition 0.1 yields $\operatorname{span}(\alpha, \beta, \gamma, \delta) = \operatorname{span}(\alpha, \beta, \gamma)$. Hence, (2) becomes $L = \operatorname{span}(\alpha, \beta, \gamma, \delta) = \operatorname{span}(\alpha, \beta, \gamma)$. Thus, we have represented L as a span of three vectors. □