Math 4242 Fall 2016 (Darij Grinberg): homework set 7
due: Wed, 7 Dec 2016

Let me repeat some definitions I gave in class:

Definition 0.1. Let V and W be two vector spaces. Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ be a basis of V. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a basis of W. Let $L : V \to W$ be a linear map.

The *matrix representing* L with respect to $\mathbf{v}$ and $\mathbf{w}$ is the $n \times m$-matrix $M_{\mathbf{v},\mathbf{w},L}$ defined as follows: For every $j \in \{1, 2, \ldots, m\}$, expand the vector $L(v_j)$ with respect to the basis $\mathbf{w}$, say, as follows:

$$L(v_j) = \alpha_{1,j} w_1 + \alpha_{2,j} w_2 + \cdots + \alpha_{n,j} w_n. \qquad (1)$$

Then, $M_{\mathbf{v},\mathbf{w},L}$ is the $n \times m$-matrix whose (i, j)-th entry is $\alpha_{i,j}$.

(I gave some examples for this on homework set 6.)

Definition 0.2. Let V be a vector space. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a list of n vectors in V. Then, $L_{\mathbf{v}}$ is defined to be the map

$$\mathbb{R}^n \to V, \qquad \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \mapsto \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

This map $L_{\mathbf{v}}$ is linear. Moreover, recall that:

(a) The list $\mathbf{v}$ is linearly independent if and only if $L_{\mathbf{v}}$ is injective.

(b) The list $\mathbf{v}$ spans V if and only if $L_{\mathbf{v}}$ is surjective.

(c) The list $\mathbf{v}$ is a basis of V if and only if $L_{\mathbf{v}}$ is bijective.

Let us take a closer look at the case when $\mathbf{v}$ is a basis of V. In this case, the map $L_{\mathbf{v}}$ is bijective, and thus an isomorphism. Hence, in this case, its inverse map $(L_{\mathbf{v}})^{-1}$ is well-defined. This map is called $M_{\mathbf{v}}$. Thus, explicitly, $M_{\mathbf{v}}$ sends a vector $u \in V$ to the unique column vector $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{R}^n$ that satisfies $u = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$. In other words, $M_{\mathbf{v}}$ sends a vector $u \in V$ to the coordinates of u with respect to the basis $\mathbf{v}$ (written as a column vector).

Example 0.3. Recall that P_2 is the vector space of all polynomials of degree ≤ 2.

Let $\mathbf{a}$ be the list $(1, x, x+1, x^2+x+1)$. Then,

$$L_{\mathbf{a}}\left(\begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix}\right) = 1 \cdot 1 + 0 \cdot x + (-2) \cdot (x+1) + 3 \cdot \left(x^2 + x + 1\right) = 3x^2 + x + 2.$$

The reader can easily check that $L_{\mathbf{a}}\left(\left(\begin{array}{c}-1\\-2\\0\\3\end{array}\right)\right)=3x^2+x+2$ as well. Thus, $L_{\mathbf{a}}$ sends two distinct column vectors to one and the same polynomial in P_2. Thus, $L_{\mathbf{a}}$ is not injective. This should not be surprising: after all, $\mathbf{a}$ is not linearly independent.

Conversely, let us compute a vector $u \in \mathbb{R}^4$ satisfying $L_{\mathbf{a}}(u) = x^2 - 2x + 5$. Such a vector should exist, because $\mathbf{a}$ spans P_2 and therefore the map $L_{\mathbf{a}}$ is surjective. How do we find it? Well, we are looking for a vector $u = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ satisfying $L_{\mathbf{a}}(u) = x^2 - 2x + 5$. The definition of $L_{\mathbf{a}}$ shows that

$$\begin{aligned} L_{\mathbf{a}}(u) &= \lambda_1 1 + \lambda_2 x + \lambda_3 (x+1) + \lambda_4 \left(x^2+x+1\right) \\ &= \lambda_4 x^2 + (\lambda_2+\lambda_3+\lambda_4)\, x + (\lambda_1+\lambda_3+\lambda_4)\, 1. \end{aligned}$$

Hence, we want to find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ satisfying the polynomial equation

$$\lambda_4 x^2 + (\lambda_2+\lambda_3+\lambda_4)\, x + (\lambda_1+\lambda_3+\lambda_4)\, 1 = x^2 - 2x + 5 \qquad (\text{for all } x).$$

Comparing coefficients, we translate this polynomial equation into the system

$$\left\{\begin{array}{c} \lambda_4 = 1; \\ \lambda_2+\lambda_3+\lambda_4 = -2; \\ \lambda_1+\lambda_3+\lambda_4 = 5 \end{array}\right. .$$

This system can be solved by Gaussian elimination; the solutions are $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T = (4-r, -3-r, r, 1)^T$ for $r \in \mathbb{R}$. Thus, these are the vectors $u \in \mathbb{R}^4$ satisfying $L_{\mathbf{a}}(u) = x^2 - 2x + 5$. There are infinitely many of them.

Exercise 1. Consider the vector space P_2 of polynomials of degree ≤ 2.

Let $\mathbf{v}$ be the basis $\left(1, x+1, x^2+2x\right)$ of P_2.

(a) Simplify $L_{\mathbf{v}}\left(\left(\begin{array}{c}2\\3\\-1\end{array}\right)\right)$. [5 points]

(b) Find $M_{\mathbf{v}}\left(x^2-3x-7\right)$. (In other words, find the $u \in \mathbb{R}^3$ satisfying $L_{\mathbf{v}}(u) = x^2 - 3x - 7$.) [5 points]

Solution to Exercise 1. **(a)** The definition of $L_{\mathbf{v}}$ yields

$$\begin{aligned} L_{\mathbf{v}}\left(\left(\begin{array}{c}2\\3\\-1\end{array}\right)\right) &= 2\cdot 1 + 3\cdot(x+1) + (-1)\cdot\left(x^2+2x\right) \\ &= -x^2 + x + 5. \end{aligned}$$

(b) The vector $M_{\mathbf{v}}\left(x^2-3x-7\right)$ belongs to $\mathbb{R}^3$; thus, we can write it in the form $(\lambda_1,\lambda_2,\lambda_3)^T$ for some reals $\lambda_1,\lambda_2,\lambda_3$. Consider these $\lambda_1,\lambda_2,\lambda_3$.

We have $(\lambda_1,\lambda_2,\lambda_3)^T = \underbrace{M_{\mathbf{v}}}_{=(L_{\mathbf{v}})^{-1}}\left(x^2-3x-7\right) = (L_{\mathbf{v}})^{-1}\left(x^2-3x-7\right)$, and thus

$$\begin{aligned} x^2-3x-7 &= L_{\mathbf{v}}\left((\lambda_1,\lambda_2,\lambda_3)^T\right) = L_{\mathbf{v}}\left(\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}\right) \\ &= \lambda_1\cdot 1+\lambda_2\cdot(x+1)+\lambda_3\cdot\left(x^2+2x\right) \\ &\qquad \text{(by the definition of } L_{\mathbf{v}}) \\ &= \lambda_3 x^2+(\lambda_2+2\lambda_3)\,x+(\lambda_1+\lambda_2)\,1. \end{aligned}$$

In other words,

$$\lambda_3 x^2+(\lambda_2+2\lambda_3)\,x+(\lambda_1+\lambda_2)\,1 = x^2-3x-7.$$

This is an equality between two polynomials. Comparing coefficients, we translate it into the system

$$\begin{cases} \lambda_3 = 1; \\ \lambda_2+2\lambda_3 = -3; \\ \lambda_1+\lambda_2 = -7 \end{cases}.$$

This system can be solved by Gaussian elimination; there is a unique solution, namely $(\lambda_1,\lambda_2,\lambda_3)^T = (-2,-5,1)^T$. Thus,

$$M_{\mathbf{v}}\left(x^2-3x-7\right) = (\lambda_1,\lambda_2,\lambda_3)^T = (-2,-5,1)^T.$$

□

Exercise 2. Let $\mathcal{A}_3$ be the vector space of all skew-symmetric 3×3-matrices. Recall (from Exercise 1 **(b)** on homework set 4) that $\mathbf{v} = (E_{1,2}-E_{2,1}, E_{1,3}-E_{3,1}, E_{2,3}-E_{3,2})$ is a basis of $\mathcal{A}_3$.

(a) Find $L_{\mathbf{v}}\left(\begin{pmatrix}2\\1\\3\end{pmatrix}\right)$. [5 points]

(b) Find $M_{\mathbf{v}}\left(\begin{pmatrix}0&1&4\\-1&0&-1\\-4&1&0\end{pmatrix}\right)$. [5 points]

Solution to Exercise 2. **(a)** The definition of $L_{\mathbf{v}}$ yields

$$L_{\mathbf{v}}\left(\begin{pmatrix}2\\1\\3\end{pmatrix}\right) = 2\underbrace{(E_{1,2}-E_{2,1})}_{=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}} + 1\underbrace{(E_{1,3}-E_{3,1})}_{=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}} + 3\underbrace{(E_{2,3}-E_{3,2})}_{=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}}$$
$$= 2\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix} + 1\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix} + 3\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}$$
$$= \begin{pmatrix}0&2&1\\-2&0&3\\-1&-3&0\end{pmatrix}.$$

(b) The vector $M_{\mathbf{v}}\left(\begin{pmatrix}0&1&4\\-1&0&-1\\-4&1&0\end{pmatrix}\right)$ belongs to $\mathbb{R}^3$; thus, we can write it in the form $(\lambda_1,\lambda_2,\lambda_3)^T$ for some reals $\lambda_1,\lambda_2,\lambda_3$. Consider these $\lambda_1,\lambda_2,\lambda_3$.

We have $(\lambda_1,\lambda_2,\lambda_3)^T = \underbrace{M_{\mathbf{v}}}_{=(L_{\mathbf{v}})^{-1}}\left(\begin{pmatrix}0&1&4\\-1&0&-1\\-4&1&0\end{pmatrix}\right) = (L_{\mathbf{v}})^{-1}\left(\begin{pmatrix}0&1&4\\-1&0&-1\\-4&1&0\end{pmatrix}\right)$,

and thus

$$\begin{pmatrix}0&1&4\\-1&0&-1\\-4&1&0\end{pmatrix} = L_{\mathbf{v}}\left((\lambda_1,\lambda_2,\lambda_3)^T\right) = L_{\mathbf{v}}\left(\begin{pmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{pmatrix}\right)$$
$$= \lambda_1\underbrace{(E_{1,2}-E_{2,1})}_{=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}} + \lambda_2\underbrace{(E_{1,3}-E_{3,1})}_{=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}} + \lambda_3\underbrace{(E_{2,3}-E_{3,2})}_{=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}}$$
$$\text{(by the definition of } L_{\mathbf{v}}\text{)}$$
$$= \begin{pmatrix}0&\lambda_1&\lambda_2\\-\lambda_1&0&\lambda_3\\-\lambda_2&-\lambda_3&0\end{pmatrix}. \qquad (2)$$

But two matrices are equal if and only if their corresponding entries are equal; thus, the equality (2) entails $1=\lambda_1$, $4=\lambda_2$ and $-1=\lambda_3$ (and further equalities, which don't add anything new to our knowledge). Hence, $(1,4,-1)^T = (\lambda_1,\lambda_2,\lambda_3)^T$. Thus,

$$M_{\mathbf{v}}\left(\begin{pmatrix}0&1&4\\-1&0&-1\\-4&1&0\end{pmatrix}\right) = (\lambda_1,\lambda_2,\lambda_3)^T = (1,4,-1)^T.$$

□

Change-of-basis matrices are a particular case of matrices representing linear maps, only that in this case the linear map is the identity map:

Definition 0.4. Let $\mathbf{v}$ and $\mathbf{w}$ be two bases of a vector space V. Then, the *change-of-basis matrix* from $\mathbf{v}$ to $\mathbf{w}$ is the matrix $M_{\mathbf{v},\mathbf{w},\mathrm{id}_V}$.

Explicitly, it can be computed as follows: Write $\mathbf{v}$ as $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Write $\mathbf{w}$ as $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. For every $j \in \{1, 2, \ldots, n\}$, expand the vector v_j with respect to the basis $\mathbf{w}$, say, as follows:

$$v_j = \alpha_{1,j} w_1 + \alpha_{2,j} w_2 + \cdots + \alpha_{n,j} w_n.$$

Then, the change-of-basis matrix $M_{\mathbf{v},\mathbf{w},\mathrm{id}_V}$ is the $n \times n$-matrix whose (i,j)-th entry is $\alpha_{i,j}$. (This method of computing $M_{\mathbf{v},\mathbf{w},\mathrm{id}_V}$ is, of course, just a particular case of the method for computing $M_{\mathbf{v},\mathbf{w},L}$ shown in Definition 0.1, specialized to the case when $W = V$, $m = n$ and $L = \mathrm{id}_V$.)

It is called the change-of-basis matrix because left multiplication by it transforms coordinates with respect to $\mathbf{v}$ into coordinates with respect to $\mathbf{w}$:

Theorem 0.5. Let $\mathbf{v}$ and $\mathbf{w}$ be two bases of a vector space V. Let $u \in V$. Then, $M_{\mathbf{w}}(u) = M_{\mathbf{v},\mathbf{w},\mathrm{id}_V} M_{\mathbf{v}}(u)$.

Exercise 3. Consider the vector space P_3 of polynomials of degree ≤ 3.

Let $\mathbf{v}$ be the basis $\left(1, x, x^2, x^3\right)$ of P_3. Let $\mathbf{w}$ be the basis $(1, x, x(x-1), x(x-1)(x-2))$ of P_3.

(a) Find the change-of-basis matrix $M_{\mathbf{v},\mathbf{w},\mathrm{id}_{P_3}}$. [5 points]

(b) Find the change-of-basis matrix $M_{\mathbf{w},\mathbf{v},\mathrm{id}_{P_3}}$. [5 points]

(c) Find $M_{\mathbf{w}}\left((x+1)^3\right)$ (that is, the coordinates of $(x+1)^3 \in P_3$ with respect to the basis $\mathbf{w}$). [5 points]

[**Hint:** The matrix $M_{\mathbf{w},\mathbf{v},\mathrm{id}_{P_3}}$ is the inverse of $M_{\mathbf{v},\mathbf{w},\mathrm{id}_{P_3}}$, but you might have an easier time computing it from scratch.]

Solution to Exercise 3. **(a)** Write the basis $\mathbf{v} = \left(1, x, x^2, x^3\right)$ as (v_1, v_2, v_3, v_4). Thus,

$$v_1 = 1, \qquad v_2 = x, \qquad v_3 = x^2, \qquad v_4 = x^3.$$

Write the basis $\mathbf{w} = (1, x, x(x-1), x(x-1)(x-2))$ as (w_1, w_2, w_3, w_4). Thus,

$$w_1 = 1, \qquad w_2 = x, \qquad w_3 = x(x-1), \qquad w_4 = x(x-1)(x-2).$$

We follow the method shown in Definition 0.4. Thus, for every $j \in \{1, 2, 3, 4\}$, we have to expand v_j with respect to the basis $\mathbf{w}$. Let me just give the results of these expansions:

- Expanding the vector v_1 with respect to $\mathbf{w}$ yields

$$\begin{aligned} v_1 &= 1 \cdot 1 + 0 \cdot x + 0 \cdot x(x-1) + 0 \cdot x(x-1)(x-2) \\ &= 1w_1 + 0w_2 + 0w_3 + 0w_4. \end{aligned}$$

- Expanding the vector v_2 with respect to $\mathbf{w}$ yields

$$\begin{aligned} v_2 &= 0 \cdot 1 + 1 \cdot x + 0 \cdot x(x-1) + 0 \cdot x(x-1)(x-2) \\ &= 0w_1 + 1w_2 + 0w_3 + 0w_4. \end{aligned}$$

- Expanding the vector v_3 with respect to $\mathbf{w}$ yields

$$\begin{aligned} v_3 &= 0 \cdot 1 + 1 \cdot x + 1 \cdot x(x-1) + 0 \cdot x(x-1)(x-2) \\ &= 0w_1 + 1w_2 + 1w_3 + 0w_4. \end{aligned}$$

- Expanding the vector v_4 with respect to $\mathbf{w}$ yields

$$\begin{aligned} v_4 &= 0 \cdot 1 + 1 \cdot x + 3 \cdot x(x-1) + 1 \cdot x(x-1)(x-2) \\ &= 0w_1 + 1w_2 + 3w_3 + 1w_4. \end{aligned}$$

(All of these expansions can be obtained by solving systems of linear equations[1]. That said, the first three of them can also be found quickly by educated guessing.)

[1] Here is (as an example) how to obtain the fourth expansion: We want to expand v_4 with respect to $\mathbf{w}$. In other words, we want to find reals $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ that satisfy the equation

$$v_4 = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 + \lambda_4 w_4.$$

Since $v_4 = x^3$ and

$$\begin{aligned} &\lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 + \lambda_4 w_4 \\ &= \lambda_1 1 + \lambda_2 x + \lambda_3 x(x-1) + \lambda_4 x(x-1)(x-2) \\ &= \lambda_4 x^3 + (\lambda_3 - 3\lambda_4) x^2 + (\lambda_2 - \lambda_3 + 2\lambda_4) x + \lambda_1, \end{aligned}$$

this equality rewrites as

$$x^3 = \lambda_4 x^3 + (\lambda_3 - 3\lambda_4) x^2 + (\lambda_2 - \lambda_3 + 2\lambda_4) x + \lambda_1.$$

But this latter equality of polynomials is equivalent to the system of linear equations $\begin{cases} 1 = \lambda_4; \\ 0 = \lambda_3 - 3\lambda_4; \\ 0 = \lambda_2 - \lambda_3 + 2\lambda_4; \\ 0 = \lambda_1 \end{cases}$ (because two polynomials are equal if and only if their respective coefficients are equal). And the latter system has the unique solution $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T = (0, 1, 3, 1)^T$ (this can be easily found by back-substitution). Thus, we have found the four reals $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ that we wanted. The expansion $v_4 = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 + \lambda_4 w_4$ thus takes the form $v_4 = 0w_1 + 1w_2 + 3w_3 + 1w_4$.

To build the matrix $M_{\mathbf{v},\mathbf{w},\operatorname{id}_{P_3}}$ out of these expansions, we proceed as in Definition 0.4:

$$M_{\mathbf{v},\mathbf{w},\operatorname{id}_{P_3}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Solving part **(b)** is completely analogous to part **(a)**, except that the roles of **v** and **w** are switched (and that the computations become easier because expanding a polynomial in the basis $\mathbf{v} = (1, x, x^2, x^3)$ is just a matter of combining like powers). The result is

$$M_{\mathbf{w},\mathbf{v},\operatorname{id}_{P_3}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) This is just asking for the expansion of $(x+1)^3$ with respect to the basis **w**. We have seen how to compute such an expansion already; let me again just give the result:

$$(x+1)^3 = 1w_0 + 7w_1 + 6w_2 + 1w_3.$$

Thus,

$$M_{\mathbf{w}}\left((x+1)^3\right) = (1,7,6,1)^T.$$

[*Remark:* There are other solutions to part **(c)**. For example, you can argue that $M_{\mathbf{v}}\left((x+1)^3\right) = (1,3,3,1)^T$ (since $(x+1)^3 = 1 \cdot 1 + 3 \cdot x + 3 \cdot x^2 + 1 \cdot x^3 = 1v_0 + 3v_1 + 3v_2 + 1v_3$), and this allows you to compute $M_{\mathbf{w}}\left((x+1)^3\right)$ using Theorem 0.5. But I find the above easier.] □

Exercise 4. A 2×3-matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$ is said to be *zero-sum* if it satisfies the equalities

$$a_1 + b_1 + c_1 = 0, \qquad a_2 + b_2 + c_2 = 0, \tag{3}$$
$$a_1 + a_2 = 0, \qquad b_1 + b_2 = 0, \qquad c_1 + c_2 = 0 \tag{4}$$

(in other words: each row sums to 0, and each column sums to 0).

The zero-sum 2×3-matrices form a subspace $\mathcal{Z}$ of $\mathbb{R}^{2\times 3}$. Here are two bases of $\mathcal{Z}$:

- the basis $\mathbf{v} = \left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \right)$;
- the basis $\mathbf{w} = \left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right)$.

(a) Find the change-of-basis matrix $M_{\mathbf{v},\mathbf{w},\operatorname{id}_{\mathcal{Z}}}$. [5 points]
(b) Find the change-of-basis matrix $M_{\mathbf{w},\mathbf{v},\operatorname{id}_{\mathcal{Z}}}$. [5 points]

Solution to Exercise 4. **(a)** Write the basis $\mathbf{v} = \left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \right)$ as (v_1, v_2). Thus,

$$v_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Write the basis $\mathbf{w} = \left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right)$ as (w_1, w_2). Thus,

$$w_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

We follow the method shown in Definition 0.4. Thus, for every $j \in \{1, 2\}$, we have to expand v_j with respect to the basis $\mathbf{w}$. Let me just give the results of these expansions:

- Expanding the vector v_1 with respect to $\mathbf{w}$ yields

$$\begin{aligned} v_1 &= 1 \cdot \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \\ &= 1w_1 + 0w_2. \end{aligned}$$

- Expanding the vector v_2 with respect to $\mathbf{w}$ yields

$$\begin{aligned} v_2 &= 1 \cdot \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \\ &= 1w_1 + 1w_2. \end{aligned}$$

(All of these expansions can be obtained by solving systems of linear equations[2]. The first one is also obvious because $v_1 = w_1$.)

[2] Again, let me show (as an example) how to compute the second one: We want to expand v_2 with respect to $\mathbf{w}$. In other words, we want to find reals λ_1, λ_2 that satisfy the equation

$$v_2 = \lambda_1 w_1 + \lambda_2 w_2.$$

Since $v_2 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}$ and

$$\begin{aligned} \lambda_1 w_1 + \lambda_2 w_2 &= \lambda_1 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & -\lambda_1 + \lambda_2 & -\lambda_2 \\ -\lambda_1 & \lambda_1 - \lambda_2 & \lambda_2 \end{pmatrix} \end{aligned}$$

this equality rewrites as

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & -\lambda_1 + \lambda_2 & -\lambda_2 \\ -\lambda_1 & \lambda_1 - \lambda_2 & \lambda_2 \end{pmatrix}$$

To build the matrix Again, let me show (as an example) how to compute the second one: We want to expand v_2 with respect to $\mathbf{w}$. In other words, we want to find reals λ_1, λ_2 that satisfy the equation

$$v_2 = \lambda_1 w_1 + \lambda_2 w_2.$$

Since $v_2 = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}$ and

$$\begin{aligned} \lambda_1 w_1 + \lambda_2 w_2 &= \lambda_1 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & -\lambda_1 + \lambda_2 & -\lambda_2 \\ -\lambda_1 & \lambda_1 - \lambda_2 & \lambda_2 \end{pmatrix} \end{aligned}$$

this equality rewrites as

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & -\lambda_1 + \lambda_2 & -\lambda_2 \\ -\lambda_1 & \lambda_1 - \lambda_2 & \lambda_2 \end{pmatrix}$$

But this latter equality of matrices is equivalent to the system of linear equations $\begin{cases} 1 = \lambda_1; \\ -1 = -\lambda_1; \\ 0 = -\lambda_1 + \lambda_2; \\ 0 = \lambda_1 - \lambda_2; \\ -1 = -\lambda_2; \\ 1 = \lambda_2 \end{cases}$ (because two matrices are equal if and only if their respective entries are equal). And the latter system has the unique solution $(\lambda_1, \lambda_2)^T = (1, 1)^T$ (this can be easily found by Gaussian elimination, but should also be clear by inspection). Thus, we have found the two reals λ_1, λ_2 that we wanted. The expansion $v_2 = \lambda_1 w_1 + \lambda_2 w_2$ thus takes the form $v_2 = 1w_1 + 1w_2$. out of these expansions, we proceed as in Definition 0.4:

$$M_{\mathbf{v},\mathbf{w},\mathrm{id}_{\mathcal{Z}}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

But this latter equality of matrices is equivalent to the system of linear equations $\begin{cases} 1 = \lambda_1; \\ -1 = -\lambda_1; \\ 0 = -\lambda_1 + \lambda_2; \\ 0 = \lambda_1 - \lambda_2; \\ -1 = -\lambda_2; \\ 1 = \lambda_2 \end{cases}$ (because two matrices are equal if and only if their respective entries are equal). And the latter system has the unique solution $(\lambda_1, \lambda_2)^T = (1, 1)^T$ (this can be easily found by Gaussian elimination, but should also be clear by inspection). Thus, we have found the two reals λ_1, λ_2 that we wanted. The expansion $v_2 = \lambda_1 w_1 + \lambda_2 w_2$ thus takes the form $v_2 = 1w_1 + 1w_2$.

(b) Solving part **(b)** is completely analogous to part **(a)**, except that the roles of $\mathbf{v}$ and $\mathbf{w}$ are switched. The result is

$$M_{\mathbf{w},\mathbf{v},\mathrm{id}_{\mathcal{Z}}} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

□

Exercise 5. Let A be the 2×2-matrix $\begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$. Our goal is to find an invertible 2×2-matrix S and a diagonal 2×2-matrix Λ such that $A = S\Lambda S^{-1}$.

We first assume that these S and Λ exist, and try to identify them. (We can afterwards check whether the ones we have found actually work.)

We denote the two columns of S by s_1 and s_2. We denote the two diagonal entries of Λ by λ_1 and λ_2 (so that $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$). Clearly, knowing S and Λ is tantamount to knowing s_1, s_2, λ_1 and λ_2.

Let us first try to find λ_1 and λ_2. We have $Se_1 = (\text{the first column of } S) = s_1$ and $\Lambda e_1 = (\text{the first column of } \Lambda) = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} = \lambda_1 e_1$.

Now, $A = S\Lambda S^{-1}$, so that $AS = S\Lambda$ and thus $ASe_1 = S \underbrace{\Lambda e_1}_{=\lambda_1 e_1} = S\lambda_1 e_1 = \lambda_1 Se_1$. Since $Se_1 = s_1$, this rewrites as $As_1 = \lambda_1 s_1$. Hence, $(A - \lambda_1 I_2) s_1 = As_1 - \lambda_1 s_1 = \overrightarrow{0}$ (since $As_1 = \lambda_1 s_1$). In other words, $s_1 \in \operatorname{Ker}(A - \lambda_1 I_2)$. But s_1 is a column of the invertible matrix S, and thus nonzero. Hence, $\operatorname{Ker}(A - \lambda_1 I_2) \neq \left\{ \overrightarrow{0} \right\}$, so that $\det(A - \lambda_1 I_2) = 0$ (because a square matrix whose kernel is $\neq \left\{ \overrightarrow{0} \right\}$ must have determinant 0). Similarly, $s_2 \in \operatorname{Ker}(A - \lambda_2 I_2)$ and $\det(A - \lambda_2 I_2) = 0$.

(a) Compute $\det(A - xI_2)$ as a polynomial in the variable x. It has two roots r_- and r_+, with $r_- < r_+$. Find them. [5 points]

From $\det(A - \lambda_1 I_2) = 0$, we know that λ_1 must be one of these roots. Similarly, λ_2 also is one of these roots.

[**Hint:** Check your answer for **(a)** before going on! The roots should come out as integers for this particular A.]

(b) Set $\lambda_1 = r_-$ and $\lambda_2 = r_-$, and try to construct S (by setting s_1 to be a nonzero vector in $\operatorname{Ker}(A - \lambda_1 I_2)$, and setting s_2 to be a nonzero vector in $\operatorname{Ker}(A - \lambda_2 I_2)$). Do you get an invertible matrix S ? [5 points]

(c) Same question if you set $\lambda_1 = r_+$ and $\lambda_2 = r_+$. [5 points]

(d) Same question if you set $\lambda_1 = r_-$ and $\lambda_2 = r_+$. [5 points]

(e) Same question if you set $\lambda_1 = r_+$ and $\lambda_2 = r_-$. [5 points]

(f) Check that the answers you found actually work! (i.e., that S is invertible and $A = S\Lambda S^{-1}$). [5 points]

Solution to Exercise 5. **(a)** From $A = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$ and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain

$$A - xI_2 = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix} - x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-x & 5 \\ 4 & 2-x \end{pmatrix}$$

and thus

$$\det(A - xI_2) = \det \begin{pmatrix} 3-x & 5 \\ 4 & 2-x \end{pmatrix} = (3-x)(2-x) - 5 \cdot 4 = x^2 - 5x - 14 = (x+2)(x-7).$$

Hence, the roots of the polynomial $\det(A - xI_2)$ are -2 and 7 (because the roots of the polynomial $(x+2)(x-7)$ are clearly -2 and 7). The smaller of these is -2, while the larger is 7. Thus, $r_- = -2$ and $r_+ = 7$.

[*Remark:* A good way to double-check that these roots are correct is to verify that the matrices $A - r_- I_2$ and $A - r_+ I_2$ actually have a nonzero kernel. If they do, then your roots are correct. Besides, finding the kernels of $A - r_- I_2$ and $A - r_+ I_2$ is needed in the next parts of the exercises, so it is not labor lost.]

(b) Let us find the kernels of $A - r_- I_2$ and $A - r_+ I_2$; these will be used several times in the rest of the problem.

From $A = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$, $r_- = -2$ and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain

$$A - r_- I_2 = A = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix} - (-2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 4 & 4 \end{pmatrix}.$$

Since we know how to compute the kernel of a matrix, we thus can find $\operatorname{Ker}(A - r_- I_2)$. What we get is

$$\operatorname{Ker}(A - r_- I_2) = \operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right).$$

A similar computation (but using r_+) shows that

$$\operatorname{Ker}(A - r_+ I_2) = \operatorname{span}\left(\begin{pmatrix} 5 \\ 4 \end{pmatrix} \right).$$

Now, let us do what the exercise asks us to do. We set $\lambda_1 = r_-$ and $\lambda_2 = r_+$. We are to construct a matrix S by setting s_1 to be a nonzero vector in $\operatorname{Ker}(A - \lambda_1 I_2)$, and setting s_2 to be a nonzero vector in $\operatorname{Ker}(A - \lambda_2 I_2)$. What are our options here?

The column s_1 has to be a nonzero vector in $\operatorname{Ker}(A - \lambda_1 I_2)$. Since $\operatorname{Ker}\left(A - \underbrace{\lambda_1}_{= r_-} I_2 \right) = \operatorname{Ker}(A - r_- I_2) = \operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$, this means that s_1 has to be a nonzero vector in $\operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$. In other words, s_1 has to be a vector of the form $\alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for some nonzero $\alpha \in \mathbb{R}$. Consider this α. Thus, $s_1 = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$.

Similarly, s_2 has to be a nonzero vector in $\operatorname{span}\left(\left(\begin{array}{c} 1 \\ -1 \end{array}\right)\right)$. In other words, s_2 has to be a vector of the form $\beta\left(\begin{array}{c} 1 \\ -1 \end{array}\right)$ for some nonzero $\beta \in \mathbb{R}$. (We cannot use the letter α here, since it already stands for something fixed.) Consider this β. Thus, $s_2 = \beta\left(\begin{array}{c} 1 \\ -1 \end{array}\right) = \left(\begin{array}{c} \beta \\ -\beta \end{array}\right)$.

Now, the two columns s_1 and s_2 of our matrix S are $s_1 = \left(\begin{array}{c} \alpha \\ -\alpha \end{array}\right)$ and $s_2 = \left(\begin{array}{c} \beta \\ -\beta \end{array}\right)$. Hence, the whole matrix is $S = \left(\begin{array}{cc} \alpha & \beta \\ -\alpha & -\beta \end{array}\right)$. This matrix S has determinant $\det S = \det\left(\begin{array}{cc} \alpha & \beta \\ -\alpha & -\beta \end{array}\right) = \alpha(-\beta) - \beta(-\alpha) = 0$, and thus is not invertible (because a square matrix is invertible if and only if its determinant is nonzero). This solves part **(b)**.

[*Remark:* This was the straightforward argument. There is a shortcut: Once you know that both s_1 and s_2 must lie in $\operatorname{Ker}(A - r_- I_2)$, you can immediately tell that the vectors s_1, s_2 are linearly dependent (since they are 2 vectors in the 1-dimensional vector space $\operatorname{Ker}(A - r_- I_2)$, but any 2 vectors in a 1-dimensional vector space are linearly dependent), and thus the matrix S has linearly dependent columns; but this means that S is not invertible.]

(c) The solution to part **(c)** is analogous to the solution to part **(b)**. (This time, S will be $\left(\begin{array}{cc} 5\alpha & 5\beta \\ 4\alpha & 4\beta \end{array}\right)$ instead of $\left(\begin{array}{cc} \alpha & \beta \\ -\alpha & -\beta \end{array}\right)$; but this new matrix S is just as non-invertible as the old one.)

(d) This time we do get an invertible matrix S (and not just one – we get infinitely many options). Indeed, proceed as before. Then, we find $S = \left(\begin{array}{cc} \alpha & 5\beta \\ -\alpha & 4\beta \end{array}\right)$. This matrix S has determinant $\det S = \det\left(\begin{array}{cc} \alpha & 5\beta \\ -\alpha & 4\beta \end{array}\right) = \alpha(4\beta) - 5\beta(-\alpha) = 9\alpha\beta$, which is always nonzero (since α and β are nonzero). To get a specific value of S (as opposed to a general form), we can (for example) set $\alpha = 1$ and $\beta = 1$; then we obtain $S = \left(\begin{array}{cc} 1 & 5 \\ -1 & 4 \end{array}\right)$. But, of course, other choices of values for α and β work just as well.

(e) Again, we do get an invertible matrix S. The general form is $S = \left(\begin{array}{cc} 5\alpha & \beta \\ 4\alpha & -\beta \end{array}\right)$.

(f) Only the answers for parts **(d)** and **(e)** must be checked (because in parts **(b)** and **(c)**, we did not find any invertible matrices S). Let me check the answer for **(d)** in its general form: We want to prove that $A = S\Lambda S^{-1}$ where $S = \left(\begin{array}{cc} \alpha & 5\beta \\ -\alpha & 4\beta \end{array}\right)$ and $\Lambda = \left(\begin{array}{cc} -2 & 0 \\ 0 & 7 \end{array}\right)$ (since $\lambda_1 = r_- = -2$ and $\lambda_2 = r_+ = 7$). We can, of course, can check this directly by computation (it is straightforward to compute S^{-1}). But we can also make our life easier and check the equivalent equality $AS = S\Lambda$ (since we already know that S is invertible). In light of $A = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$, $S = \begin{pmatrix} \alpha & 5\beta \\ -\alpha & 4\beta \end{pmatrix}$ and $\Lambda = \begin{pmatrix} -2 & 0 \\ 0 & 7 \end{pmatrix}$, this equality rewrites as

$$\begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \alpha & 5\beta \\ -\alpha & 4\beta \end{pmatrix} = \begin{pmatrix} \alpha & 5\beta \\ -\alpha & 4\beta \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 7 \end{pmatrix}.$$

But this is really easy to check (both sides equal $\begin{pmatrix} -2\alpha & 35\beta \\ 2\alpha & 28\beta \end{pmatrix}$).

Checking the answer for **(e)** is analogous. □

0.1. Appendix: some lecture material

Theorem 0.6. Let V and W be two vector spaces. Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ be a basis of V. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a basis of W. Let $F : V \to W$ be a linear map. Let A be the $n \times m$-matrix $M_{\mathbf{v},\mathbf{w},F}$.

(a) The diagram

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{L_A} & \mathbb{R}^n \\ {\scriptstyle L_{\mathbf{v}}}\downarrow & & \downarrow{\scriptstyle L_{\mathbf{w}}} \\ V & \xrightarrow[F]{} & W \end{array}$$

is commutative; in other words, we have $L_{\mathbf{w}} \circ L_A = F \circ L_{\mathbf{v}}$.

(b) The diagram

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{L_A} & \mathbb{R}^n \\ {\scriptstyle M_{\mathbf{v}}}\uparrow & & \uparrow{\scriptstyle M_{\mathbf{w}}} \\ V & \xrightarrow[F]{} & W \end{array}$$

is commutative; in other words, we have $L_A \circ M_{\mathbf{v}} = M_{\mathbf{w}} \circ F$.

In words, Theorem 0.6 says that "as a vector in V undergoes the map F, the corresponding column vector in $\mathbb{R}^m$ gets left-multiplied by A". The meaning of "corresponding" is formalized by the inverse bijections $L_{\mathbf{v}}$ and $M_{\mathbf{v}}$ (for vectors in V) and $L_{\mathbf{w}}$ and $M_{\mathbf{w}}$ (for vectors in W).

Proof of Theorem 0.6. We have $A = M_{\mathbf{v},\mathbf{w},F}$. Thus, for each $j \in \{1, 2, \ldots, m\}$, the j-th column of the matrix A consists of the coordinates of $F(v_j)$ with respect to the basis $\mathbf{w}$ (since this is how the matrix $M_{\mathbf{v},\mathbf{w},F}$ was defined). In other words, for each $j \in \{1, 2, \ldots, m\}$, we have

$$F(v_j) = A_{1,j}w_1 + A_{2,j}w_2 + \cdots + A_{n,j}w_n. \tag{5}$$

Let $g \in \mathbb{R}^m$ be a column vector. Write g in the form $g = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$. Then,

$$\begin{aligned} L_A(g) &= Ag \qquad \text{(since the map } L_A \text{ is just left multiplication by } A) \\ &= A(\lambda_1, \lambda_2, \ldots, \lambda_m)^T \qquad \left(\text{since } g = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T\right) \\ &= \begin{pmatrix} A_{1,1}\lambda_1 + A_{1,2}\lambda_2 + \cdots + A_{1,m}\lambda_m \\ A_{2,1}\lambda_1 + A_{2,2}\lambda_2 + \cdots + A_{2,m}\lambda_m \\ \vdots \\ A_{n,1}\lambda_1 + A_{n,2}\lambda_2 + \cdots + A_{n,m}\lambda_m \end{pmatrix} \\ &\qquad \text{(by the rule for multiplying matrices)}, \end{aligned}$$

so that

$$
\begin{aligned}
L_{\mathbf{w}}\left(L_A\left(g\right)\right) &= L_{\mathbf{w}}\left(\left(\begin{array}{c} A_{1,1}\lambda_1 + A_{1,2}\lambda_2 + \cdots + A_{1,m}\lambda_m \\ A_{2,1}\lambda_1 + A_{2,2}\lambda_2 + \cdots + A_{2,m}\lambda_m \\ \vdots \\ A_{n,1}\lambda_1 + A_{n,2}\lambda_2 + \cdots + A_{n,m}\lambda_m \end{array}\right)\right) \\
&= \left(A_{1,1}\lambda_1 + A_{1,2}\lambda_2 + \cdots + A_{1,m}\lambda_m\right) w_1 \\
&\qquad + \left(A_{2,1}\lambda_1 + A_{2,2}\lambda_2 + \cdots + A_{2,m}\lambda_m\right) w_2 \\
&\qquad + \cdots + \left(A_{n,1}\lambda_1 + A_{n,2}\lambda_2 + \cdots + A_{n,m}\lambda_m\right) w_n \\
&= A_{1,1}\lambda_1 w_1 + A_{1,2}\lambda_2 w_1 + \cdots + A_{1,m}\lambda_m w_1 \\
&\qquad + A_{2,1}\lambda_1 w_2 + A_{2,2}\lambda_2 w_2 + \cdots + A_{2,m}\lambda_m w_2 \\
&\qquad + \cdots + A_{n,1}\lambda_1 w_n + A_{n,2}\lambda_2 w_n + \cdots + A_{n,m}\lambda_m w_n. \qquad (6)
\end{aligned}
$$

On the other hand, from $g = \left(\lambda_1, \lambda_2, \ldots, \lambda_m\right)^T$, we obtain

$$
\begin{aligned}
L_{\mathbf{v}}\left(g\right) = L_{\mathbf{v}}\left(\left(\lambda_1, \lambda_2, \ldots, \lambda_m\right)^T\right) &= \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m \\
&\qquad \left(\text{by the definition of } L_{\mathbf{v}}\right),
\end{aligned}
$$

and thus

$$
\begin{aligned}
F\left(L_{\mathbf{v}}\left(g\right)\right) &= F\left(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m\right) \\
&= \lambda_1 \underbrace{F\left(v_1\right)}_{\substack{= A_{1,1}w_1 + A_{2,1}w_2 + \cdots + A_{n,1}w_n \\ \text{(by (5))}}} + \lambda_2 \underbrace{F\left(v_2\right)}_{\substack{= A_{1,2}w_1 + A_{2,2}w_2 + \cdots + A_{n,2}w_n \\ \text{(by (5))}}} \\
&\qquad + \cdots + \lambda_m \underbrace{F\left(v_m\right)}_{\substack{= A_{1,m}w_1 + A_{2,m}w_2 + \cdots + A_{n,m}w_n \\ \text{(by (5))}}} \\
&= \lambda_1 \left(A_{1,1}w_1 + A_{2,1}w_2 + \cdots + A_{n,1}w_n\right) \\
&\qquad + \lambda_2 \left(A_{1,2}w_1 + A_{2,2}w_2 + \cdots + A_{n,2}w_n\right) \\
&\qquad + \cdots + \lambda_m \left(A_{1,m}w_1 + A_{2,m}w_2 + \cdots + A_{n,m}w_n\right) \\
&= A_{1,1}\lambda_1 w_1 + A_{2,1}\lambda_1 w_2 + \cdots + A_{n,1}\lambda_1 w_n \\
&\qquad + A_{1,2}\lambda_2 w_1 + A_{2,2}\lambda_2 w_2 + \cdots + A_{n,2}\lambda_2 w_n \\
&\qquad + \cdots + A_{1,m}\lambda_m w_1 + A_{2,m}\lambda_m w_2 + \cdots + A_{n,m}\lambda_m w_n. \qquad (7)
\end{aligned}
$$

Now, the sum on the right hand side of (6) and the sum on the right hand side of (7) consist of the same addends, just in a different order. Hence, these two sums are equal. In other words, the right hand sides of (6) and (7) are equal. Thus, the left hand sides of (6) and (7) are equal as well. In other words, $L_{\mathbf{w}}\left(L_A\left(g\right)\right) = F\left(L_{\mathbf{v}}\left(g\right)\right)$. In other words, $\left(L_{\mathbf{w}} \circ L_A\right)\left(g\right) = \left(F \circ L_{\mathbf{v}}\right)\left(g\right)$.

So we have proven that $\left(L_{\mathbf{w}} \circ L_A\right)\left(g\right) = \left(F \circ L_{\mathbf{v}}\right)\left(g\right)$ for every $g \in \mathbb{R}^m$. In other words, $L_{\mathbf{w}} \circ L_A = F \circ L_{\mathbf{v}}$. This proves Theorem 0.6 **(a)**.

(b) We have

$$\underbrace{L_{\mathbf{w}} \circ L_A}_{=F \circ L_{\mathbf{v}}} \circ M_{\mathbf{v}} = F \circ \underbrace{L_{\mathbf{v}} \circ M_{\mathbf{v}}}_{\substack{=\operatorname{id}_V \\ (\text{since } M_{\mathbf{v}} = (L_{\mathbf{v}})^{-1})}} = F \circ \operatorname{id}_V = F,$$

so that

$$M_{\mathbf{w}} \circ \underbrace{F}_{=L_{\mathbf{w}} \circ L_A \circ M_{\mathbf{v}}} = \underbrace{M_{\mathbf{w}} \circ L_{\mathbf{w}}}_{\substack{=\operatorname{id}_{\mathbb{R}^n} \\ (\text{since } M_{\mathbf{w}} = (L_{\mathbf{w}})^{-1})}} \circ L_A \circ M_{\mathbf{v}} = \operatorname{id}_{\mathbb{R}^n} \circ L_A \circ M_{\mathbf{v}} = L_A \circ M_{\mathbf{v}}.$$

This proves Theorem 0.6 **(b)**. □

Another fact, whose proof I won't show (see, e.g., Proposition 6.6.5 in Lankham/Nachtergaele/Schilling, but keep in mind that their notation for $M_{\mathbf{v},\mathbf{w},F}$ is $M(F)$, with the bases **v** and **w** being hidden), shows what happens to the matrices representing two linear maps when said maps are composed:

Theorem 0.7. Let U, V and W be three vector spaces with bases **u**, **v** and **w**, respectively. Let $F : U \to V$ and $G : V \to W$ be two linear maps. Then, their composition $G \circ F : U \to W$ is again a linear map, and we have

$$M_{\mathbf{u},\mathbf{w},G\circ F} = M_{\mathbf{v},\mathbf{w},G} M_{\mathbf{u},\mathbf{v},F}. \tag{8}$$

In other words, the matrix representing the composition $G \circ F$ is the product of the matrix representing G with the matrix representing F. (Fine print: the bases have to "match", i.e., the basis for the domain for $G \circ F$ has to be the basis for the domain for F, and so on. In case of doubt, look at (8).)