Math 4242 Fall 2016 (Darij Grinberg): homework set 7
due: Wed, 7 Dec 2016, in class
(or **earlier** by moodle)

Let me repeat some definitions I gave in class:

Definition 0.1. Let V and W be two vector spaces. Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ be a basis of V. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a basis of W. Let $L : V \to W$ be a linear map.

The *matrix representing* L with respect to $\mathbf{v}$ and $\mathbf{w}$ is the $n \times m$-matrix $M_{\mathbf{v},\mathbf{w},L}$ defined as follows: For every $j \in \{1, 2, \ldots, m\}$, expand the vector $L(v_j)$ with respect to the basis $\mathbf{w}$, say, as follows:

$$L(v_j) = \alpha_{1,j} w_1 + \alpha_{2,j} w_2 + \cdots + \alpha_{n,j} w_n. \tag{1}$$

Then, $M_{\mathbf{v},\mathbf{w},L}$ is the $n \times m$-matrix whose (i, j)-th entry is $\alpha_{i,j}$.

(I gave some examples for this on homework set 6.)

Definition 0.2. Let V be a vector space. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be a list of n vectors in V. Then, $L_{\mathbf{v}}$ is defined to be the map

$$\mathbb{R}^n \to V, \qquad \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \mapsto \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

This map $L_{\mathbf{v}}$ is linear. Moreover, recall that:

(a) The list $\mathbf{v}$ is linearly independent if and only if $L_{\mathbf{v}}$ is injective.

(b) The list $\mathbf{v}$ spans V if and only if $L_{\mathbf{v}}$ is surjective.

(c) The list $\mathbf{v}$ is a basis of V if and only if $L_{\mathbf{v}}$ is bijective.

Let us take a closer look at the case when $\mathbf{v}$ is a basis of V. In this case, the map $L_{\mathbf{v}}$ is bijective, and thus an isomorphism. Hence, in this case, its inverse map $(L_{\mathbf{v}})^{-1}$ is well-defined. This map is called $M_{\mathbf{v}}$. Thus, explicitly, $M_{\mathbf{v}}$ sends a vector $u \in V$ to the unique column vector $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{R}^n$ that satisfies $u = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$. In other words, $M_{\mathbf{v}}$ sends a vector $u \in V$ to the coordinates of u with respect to the basis $\mathbf{v}$ (written as a column vector).

Example 0.3. Recall that P_2 is the vector space of all polynomials of degree ≤ 2.

Let $\mathbf{a}$ be the list $(1, x, x+1, x^2+x+1)$. Then,

$$L_{\mathbf{a}}\left(\begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix}\right) = 1 \cdot 1 + 0 \cdot x + (-2) \cdot (x+1) + 3 \cdot \left(x^2 + x + 1\right) = 3x^2 + x + 2.$$

The reader can easily check that $L_{\mathbf{a}}\left(\begin{pmatrix} -1 \\ -2 \\ 0 \\ 3 \end{pmatrix}\right) = 3x^2 + x + 2$ as well. Thus, $L_{\mathbf{a}}$ sends two distinct column vectors to one and the same polynomial in P_2. Thus, $L_{\mathbf{a}}$ is not injective. This should not be surprising: after all, $\mathbf{a}$ is not linearly independent.

Conversely, let us compute a vector $u \in \mathbb{R}^4$ satisfying $L_{\mathbf{a}}(u) = x^2 - 2x + 5$. Such a vector should exist, because $\mathbf{a}$ spans P_2 and therefore the map $L_{\mathbf{a}}$ is surjective. How do we find it? Well, we are looking for a vector $u = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$ satisfying $L_{\mathbf{a}}(u) = x^2 - 2x + 5$. The definition of $L_{\mathbf{a}}$ shows that

$$\begin{aligned} L_{\mathbf{a}}(u) &= \lambda_1 1 + \lambda_2 x + \lambda_3 (x+1) + \lambda_4 \left(x^2 + x + 1\right) \\ &= \lambda_4 x^2 + (\lambda_2 + \lambda_3 + \lambda_4) x + (\lambda_1 + \lambda_3 + \lambda_4) 1. \end{aligned}$$

Hence, we want to find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ satisfying the polynomial equation

$$\lambda_4 x^2 + (\lambda_2 + \lambda_3 + \lambda_4) x + (\lambda_1 + \lambda_3 + \lambda_4) 1 = x^2 - 2x + 5 \qquad \text{(for all } x\text{)}.$$

Comparing coefficients, we translate this polynomial equation into the system

$$\begin{cases} \lambda_4 = 1; \\ \lambda_2 + \lambda_3 + \lambda_4 = -2; \\ \lambda_1 + \lambda_3 + \lambda_4 = 5 \end{cases}.$$

This system can be solved by Gaussian elimination; the solutions are $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T = (4 - r, -3 - r, r, 1)^T$ for $r \in \mathbb{R}$. Thus, these are the vectors $u \in \mathbb{R}^4$ satisfying $L_{\mathbf{a}}(u) = x^2 - 2x + 5$. There are infinitely many of them.

Exercise 1. Consider the vector space P_2 of polynomials of degree ≤ 2.

Let $\mathbf{v}$ be the basis $\left(1, x+1, x^2 + 2x\right)$ of P_2.

(a) Simplify $L_{\mathbf{v}}\left(\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}\right)$. [5 points]

(b) Find $M_{\mathbf{v}}\left(x^2 - 3x - 7\right)$. (In other words, find the $u \in \mathbb{R}^3$ satisfying $L_{\mathbf{v}}(u) = x^2 - 3x - 7$.) [5 points]

Exercise 2. Let $\mathcal{A}_3$ be the vector space of all skew-symmetric 3×3-matrices. Recall (from Exercise 1 **(b)** on homework set 4) that $\mathbf{v} = (E_{1,2} - E_{2,1}, E_{1,3} - E_{3,1}, E_{2,3} - E_{3,2})$ is a basis of $\mathcal{A}_3$.

(a) Find $L_{\mathbf{v}}\left(\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}\right)$. [5 points]

(b) Find $M_{\mathbf{v}}\left(\left(\begin{array}{ccc} 0 & 1 & 4 \\ -1 & 0 & -1 \\ -4 & 1 & 0 \end{array}\right)\right)$. [5 points]

Change-of-basis matrices are a particular case of matrices representing linear maps, only that in this case the linear map is the identity map:

Definition 0.4. Let $\mathbf{v}$ and $\mathbf{w}$ be two bases of a vector space V. Then, the *change-of-basis matrix* from $\mathbf{v}$ to $\mathbf{w}$ is the matrix $M_{\mathbf{v},\mathbf{w},\mathrm{id}_V}$.

Explicitly, it can be computed as follows: Write $\mathbf{v}$ as $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Write $\mathbf{w}$ as $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. For every $j \in \{1, 2, \ldots, n\}$, expand the vector v_j with respect to the basis $\mathbf{w}$, say, as follows:

$$v_j = \alpha_{1,j} w_1 + \alpha_{2,j} w_2 + \cdots + \alpha_{n,j} w_n.$$

Then, the change-of-basis matrix $M_{\mathbf{v},\mathbf{w},\mathrm{id}_V}$ is the $n \times n$-matrix whose (i, j)-th entry is $\alpha_{i,j}$.

It is called the change-of-basis matrix because left multiplication by it transforms coordinates with respect to $\mathbf{v}$ into coordinates with respect to $\mathbf{w}$:

Theorem 0.5. Let $\mathbf{v}$ and $\mathbf{w}$ be two bases of a vector space V. Let $u \in V$. Then, $M_{\mathbf{w}}(u) = M_{\mathbf{v},\mathbf{w},\mathrm{id}_V} M_{\mathbf{v}}(u)$.

Exercise 3. Consider the vector space P_3 of polynomials of degree ≤ 3.

Let $\mathbf{v}$ be the basis $\left(1, x, x^2, x^3\right)$ of P_3. Let $\mathbf{w}$ be the basis $(1, x, x(x-1), x(x-1)(x-2))$ of P_3.

(a) Find the change-of-basis matrix $M_{\mathbf{v},\mathbf{w},\mathrm{id}_{P_3}}$. [5 points]

(b) Find the change-of-basis matrix $M_{\mathbf{w},\mathbf{v},\mathrm{id}_{P_3}}$. [5 points]

(c) Find $M_{\mathbf{w}}\left((x+1)^3\right)$ (that is, the coordinates of $(x+1)^3 \in P_3$ with respect to the basis $\mathbf{w}$). [5 points]

[**Hint:** The matrix $M_{\mathbf{w},\mathbf{v},\mathrm{id}_{P_3}}$ is the inverse of $M_{\mathbf{v},\mathbf{w},\mathrm{id}_{P_3}}$, but you might have an easier time computing it from scratch.]

Exercise 4. A 2×3-matrix $\left(\begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right)$ is said to be *zero-sum* if it satisfies the equalities

$$a_1 + b_1 + c_1 = 0, \qquad a_2 + b_2 + c_2 = 0, \tag{2}$$
$$a_1 + a_2 = 0, \qquad b_1 + b_2 = 0, \qquad c_1 + c_2 = 0 \tag{3}$$

(in other words: each row sums to 0, and each column sums to 0).

The zero-sum 2×3-matrices form a subspace $\mathcal{Z}$ of $\mathbb{R}^{2\times 3}$. Here are two bases of $\mathcal{Z}$:

- the basis $\mathbf{v} = \left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \right)$;
- the basis $\mathbf{w} = \left(\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right)$.

(a) Find the change-of-basis matrix $M_{\mathbf{v},\mathbf{w},\mathrm{id}_{\mathcal{Z}}}$. [5 points]
(b) Find the change-of-basis matrix $M_{\mathbf{w},\mathbf{v},\mathrm{id}_{\mathcal{Z}}}$. [5 points]

Exercise 5. Let A be the 2×2-matrix $\begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$. Our goal is to find an invertible 2×2-matrix S and a diagonal 2×2-matrix Λ such that $A = S\Lambda S^{-1}$.

We first assume that these S and Λ exist, and try to identify them. (We can afterwards check whether the ones we have found actually work.)

We denote the two columns of S by s_1 and s_2. We denote the two diagonal entries of Λ by λ_1 and λ_2 (so that $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$). Clearly, knowing S and Λ is tantamount to knowing s_1, s_2, λ_1 and λ_2.

Let us first try to find λ_1 and λ_2. We have $Se_1 = (\text{the first column of } S) = s_1$ and $\Lambda e_1 = (\text{the first column of } \Lambda) = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} = \lambda_1 e_1$.

Now, $A = S\Lambda S^{-1}$, so that $AS = S\Lambda$ and thus $ASe_1 = S \underbrace{\Lambda e_1}_{=\lambda_1 e_1} = S\lambda_1 e_1 = \lambda_1 Se_1$. Since $Se_1 = s_1$, this rewrites as $As_1 = \lambda_1 s_1$. Hence, $(A - \lambda_1 I_2) s_1 = As_1 - \lambda_1 s_1 = \overrightarrow{0}$ (since $As_1 = \lambda_1 s_1$). In other words, $s_1 \in \operatorname{Ker}(A - \lambda_1 I_2)$. But s_1 is a column of the invertible matrix S, and thus nonzero. Hence, $\operatorname{Ker}(A - \lambda_1 I_2) \neq \left\{ \overrightarrow{0} \right\}$, so that $\det(A - \lambda_1 I_2) = 0$ (because a square matrix whose kernel is $\neq \left\{ \overrightarrow{0} \right\}$ must have determinant 0). Similarly, $s_2 \in \operatorname{Ker}(A - \lambda_2 I_2)$ and $\det(A - \lambda_2 I_2) = 0$.

(a) Compute $\det(A - xI_2)$ as a polynomial in the variable x. It has two roots r_- and r_+, with $r_- < r_+$. Find them. [5 points]

From $\det(A - \lambda_1 I_2) = 0$, we know that λ_1 must be one of these roots. Similarly, λ_2 also is one of these roots.

[**Hint:** Check your answer for **(a)** before going on! The roots should come out as integers for this particular A.]

(b) Set $\lambda_1 = r_-$ and $\lambda_2 = r_-$, and try to construct S (by setting s_1 to be a nonzero vector in $\operatorname{Ker}(A - \lambda_1 I_2)$, and setting s_2 to be a nonzero vector in $\operatorname{Ker}(A - \lambda_2 I_2)$). Do you get an invertible matrix S ? [5 points]
(c) Same question if you set $\lambda_1 = r_+$ and $\lambda_2 = r_+$. [5 points]
(d) Same question if you set $\lambda_1 = r_-$ and $\lambda_2 = r_+$. [5 points]
(e) Same question if you set $\lambda_1 = r_+$ and $\lambda_2 = r_-$. [5 points]
(f) Check that the answers you found actually work! (i.e., that S is invertible and $A = S\Lambda S^{-1}$). [5 points]

0.1. Appendix: some lecture material

Theorem 0.6. Let V and W be two vector spaces. Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ be a basis of V. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a basis of W. Let $F : V \to W$ be a linear map. Let A be the $n \times m$-matrix $M_{\mathbf{v},\mathbf{w},F}$.

(a) The diagram

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{L_A} & \mathbb{R}^n \\ {\scriptstyle L_{\mathbf{v}}}\downarrow & & \downarrow{\scriptstyle L_{\mathbf{w}}} \\ V & \xrightarrow[F]{} & W \end{array}$$

is commutative; in other words, we have $L_{\mathbf{w}} \circ L_A = F \circ L_{\mathbf{v}}$.

(b) The diagram

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{L_A} & \mathbb{R}^n \\ {\scriptstyle M_{\mathbf{v}}}\uparrow & & \uparrow{\scriptstyle M_{\mathbf{w}}} \\ V & \xrightarrow[F]{} & W \end{array}$$

is commutative; in other words, we have $L_A \circ M_{\mathbf{v}} = M_{\mathbf{w}} \circ F$.

In words, Theorem 0.6 says that "as a vector in V undergoes the map F, the corresponding column vector in $\mathbb{R}^m$ gets left-multiplied by A". The meaning of "corresponding" is formalized by the inverse bijections $L_{\mathbf{v}}$ and $M_{\mathbf{v}}$ (for vectors in V) and $L_{\mathbf{w}}$ and $M_{\mathbf{w}}$ (for vectors in W).

Proof of Theorem 0.6. We have $A = M_{\mathbf{v},\mathbf{w},F}$. Thus, for each $j \in \{1, 2, \ldots, m\}$, the j-th column of the matrix A consists of the coordinates of $F(v_j)$ with respect to the basis $\mathbf{w}$ (since this is how the matrix $M_{\mathbf{v},\mathbf{w},F}$ was defined). In other words, for each $j \in \{1, 2, \ldots, m\}$, we have

$$F(v_j) = A_{1,j}w_1 + A_{2,j}w_2 + \cdots + A_{n,j}w_n. \tag{4}$$

Let $g \in \mathbb{R}^m$ be a column vector. Write g in the form $g = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$. Then,

$$\begin{aligned} L_A(g) &= Ag \qquad \text{(since the map } L_A \text{ is just left multiplication by } A\text{)} \\ &= A(\lambda_1, \lambda_2, \ldots, \lambda_m)^T \qquad \left(\text{since } g = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T\right) \\ &= \begin{pmatrix} A_{1,1}\lambda_1 + A_{1,2}\lambda_2 + \cdots + A_{1,m}\lambda_m \\ A_{2,1}\lambda_1 + A_{2,2}\lambda_2 + \cdots + A_{2,m}\lambda_m \\ \vdots \\ A_{n,1}\lambda_1 + A_{n,2}\lambda_2 + \cdots + A_{n,m}\lambda_m \end{pmatrix} \\ &\qquad \text{(by the rule for multiplying matrices)}, \end{aligned}$$

so that

$$\begin{aligned}
L_{\mathbf{w}}\left(L_A\left(g\right)\right) &= L_{\mathbf{w}}\left(\left(\begin{array}{c} A_{1,1}\lambda_1 + A_{1,2}\lambda_2 + \cdots + A_{1,m}\lambda_m \\ A_{2,1}\lambda_1 + A_{2,2}\lambda_2 + \cdots + A_{2,m}\lambda_m \\ \vdots \\ A_{n,1}\lambda_1 + A_{n,2}\lambda_2 + \cdots + A_{n,m}\lambda_m \end{array}\right)\right) \\
&= \left(A_{1,1}\lambda_1 + A_{1,2}\lambda_2 + \cdots + A_{1,m}\lambda_m\right) w_1 \\
&\qquad + \left(A_{2,1}\lambda_1 + A_{2,2}\lambda_2 + \cdots + A_{2,m}\lambda_m\right) w_2 \\
&\qquad + \cdots + \left(A_{n,1}\lambda_1 + A_{n,2}\lambda_2 + \cdots + A_{n,m}\lambda_m\right) w_n \\
&= A_{1,1}\lambda_1 w_1 + A_{1,2}\lambda_2 w_1 + \cdots + A_{1,m}\lambda_m w_1 \\
&\qquad + A_{2,1}\lambda_1 w_2 + A_{2,2}\lambda_2 w_2 + \cdots + A_{2,m}\lambda_m w_2 \\
&\qquad + \cdots + A_{n,1}\lambda_1 w_n + A_{n,2}\lambda_2 w_n + \cdots + A_{n,m}\lambda_m w_n. \qquad (5)
\end{aligned}$$

On the other hand, from $g = \left(\lambda_1, \lambda_2, \ldots, \lambda_m\right)^T$, we obtain

$$\begin{aligned}
L_{\mathbf{v}}\left(g\right) = L_{\mathbf{v}}\left(\left(\lambda_1, \lambda_2, \ldots, \lambda_m\right)^T\right) &= \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m \\
&\qquad \left(\text{by the definition of } L_{\mathbf{v}}\right),
\end{aligned}$$

and thus

$$\begin{aligned}
F\left(L_{\mathbf{v}}\left(g\right)\right) &= F\left(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m\right) \\
&= \lambda_1 \underbrace{F\left(v_1\right)}_{\substack{= A_{1,1}w_1 + A_{2,1}w_2 + \cdots + A_{n,1}w_n \\ \text{(by (4))}}} + \lambda_2 \underbrace{F\left(v_2\right)}_{\substack{= A_{1,2}w_1 + A_{2,2}w_2 + \cdots + A_{n,2}w_n \\ \text{(by (4))}}} \\
&\qquad + \cdots + \lambda_m \underbrace{F\left(v_m\right)}_{\substack{= A_{1,m}w_1 + A_{2,m}w_2 + \cdots + A_{n,m}w_n \\ \text{(by (4))}}} \\
&= \lambda_1\left(A_{1,1}w_1 + A_{2,1}w_2 + \cdots + A_{n,1}w_n\right) \\
&\qquad + \lambda_2\left(A_{1,2}w_1 + A_{2,2}w_2 + \cdots + A_{n,2}w_n\right) \\
&\qquad + \cdots + \lambda_m\left(A_{1,m}w_1 + A_{2,m}w_2 + \cdots + A_{n,m}w_n\right) \\
&= A_{1,1}\lambda_1 w_1 + A_{2,1}\lambda_1 w_2 + \cdots + A_{n,1}\lambda_1 w_n \\
&\qquad + A_{1,2}\lambda_2 w_1 + A_{2,2}\lambda_2 w_2 + \cdots + A_{n,2}\lambda_2 w_n \\
&\qquad + \cdots + A_{1,m}\lambda_m w_1 + A_{2,m}\lambda_m w_2 + \cdots + A_{n,m}\lambda_m w_n. \qquad (6)
\end{aligned}$$

Now, the sum on the right hand side of (5) and the sum on the right hand side of (6) consist of the same addends, just in a different order. Hence, these two sums are equal. In other words, the right hand sides of (5) and (6) are equal. Thus, the left hand sides of (5) and (6) are equal as well. In other words, $L_{\mathbf{w}}\left(L_A\left(g\right)\right) = F\left(L_{\mathbf{v}}\left(g\right)\right)$. In other words, $\left(L_{\mathbf{w}} \circ L_A\right)\left(g\right) = \left(F \circ L_{\mathbf{v}}\right)\left(g\right)$.

So we have proven that $\left(L_{\mathbf{w}} \circ L_A\right)\left(g\right) = \left(F \circ L_{\mathbf{v}}\right)\left(g\right)$ for every $g \in \mathbb{R}^m$. In other words, $L_{\mathbf{w}} \circ L_A = F \circ L_{\mathbf{v}}$. This proves Theorem 0.6 **(a)**.

(b) We have

$$\underbrace{L_{\mathbf{w}} \circ L_A}_{=F \circ L_{\mathbf{v}}} \circ M_{\mathbf{v}} = F \circ \underbrace{L_{\mathbf{v}} \circ M_{\mathbf{v}}}_{\substack{=\operatorname{id}_V \\ (\text{since } M_{\mathbf{v}} = (L_{\mathbf{v}})^{-1})}} = F \circ \operatorname{id}_V = F,$$

so that

$$M_{\mathbf{w}} \circ \underbrace{F}_{=L_{\mathbf{w}} \circ L_A \circ M_{\mathbf{v}}} = \underbrace{M_{\mathbf{w}} \circ L_{\mathbf{w}}}_{\substack{=\operatorname{id}_{\mathbb{R}^n} \\ (\text{since } M_{\mathbf{w}} = (L_{\mathbf{w}})^{-1})}} \circ L_A \circ M_{\mathbf{v}} = \operatorname{id}_{\mathbb{R}^n} \circ L_A \circ M_{\mathbf{v}} = L_A \circ M_{\mathbf{v}}.$$

This proves Theorem 0.6 **(b)**. □

Another fact, whose proof I won't show (see, e.g., Proposition 6.6.5 in Lankham/Nachtergaele/Schilling, but keep in mind that their notation for $M_{\mathbf{v},\mathbf{w},F}$ is $M(F)$, with the bases $\mathbf{v}$ and $\mathbf{w}$ being hidden), shows what happens to the matrices representing two linear maps when said maps are composed:

Theorem 0.7. Let U, V and W be three vector spaces with bases $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, respectively. Let $F : U \to V$ and $G : V \to W$ be two linear maps. Then, their composition $G \circ F : U \to W$ is again a linear map, and we have

$$M_{\mathbf{u},\mathbf{w},G\circ F} = M_{\mathbf{v},\mathbf{w},G} M_{\mathbf{u},\mathbf{v},F}. \tag{7}$$

In other words, the matrix representing the composition $G \circ F$ is the product of the matrix representing G with the matrix representing F. (Fine print: the bases have to "match", i.e., the basis for the domain for $G \circ F$ has to be the basis for the domain for F, and so on. In case of doubt, look at (7).)