Math 4242 Fall 2016 (Darij Grinberg): homework set 6

Let me first recall a definition.

Definition 0.1. Let V and W be two vector spaces. Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ be a basis of V. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a basis of W. Let $L : V \to W$ be a linear map.

The *matrix representing* L with respect to $\mathbf{v}$ and $\mathbf{w}$ is the $n \times m$-matrix $M_{\mathbf{v},\mathbf{w},L}$ defined as follows: For every $j \in \{1, 2, \ldots, m\}$, expand the vector $L(v_j)$ with respect to the basis $\mathbf{w}$, say, as follows:

$$L(v_j) = \alpha_{1,j} w_1 + \alpha_{2,j} w_2 + \cdots + \alpha_{n,j} w_n. \tag{1}$$

Then, $M_{\mathbf{v},\mathbf{w},L}$ is the $n \times m$-matrix whose (i,j)-th entry is $\alpha_{i,j}$.

For example, if $n = 3$ and $m = 2$, then

$$M_{\mathbf{v},\mathbf{w},L} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \\ \alpha_{3,1} & \alpha_{3,2} \end{pmatrix},$$

where

$$L(v_1) = \alpha_{1,1} w_1 + \alpha_{2,1} w_2 + \alpha_{3,1} w_3;$$
$$L(v_2) = \alpha_{1,2} w_1 + \alpha_{2,2} w_2 + \alpha_{3,2} w_3.$$

The purpose of this matrix $M_{\mathbf{v},\mathbf{w},L}$ is to allow easily expanding $L(v)$ in the basis $(w_1, w_2, \ldots, w_n)$ of W if v is a vector in V whose expansion in the basis $(v_1, v_2, \ldots, v_m)$ of V is known. For instance, if v is one of the basis vectors v_j, then the expansion of $L(v_j)$ can be simply read off from the j-th column of $M_{\mathbf{v},\mathbf{w},L}$; otherwise, it is an appropriate linear combination:

$$L(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m) = \lambda_1 L(v_1) + \lambda_2 L(v_2) + \cdots + \lambda_m L(v_m)$$

(where the $L(v_j)$ can be computed by (1)).

You can abbreviate $M_{\mathbf{v},\mathbf{w},L}$ as M_L, but it's your job to ensure that you know what $\mathbf{v}$ and $\mathbf{w}$ are (and they aren't changing midway through your work).

Example 0.2. Let A be the 2×2-matrix $\begin{pmatrix} 5 & 7 \\ -2 & 9 \end{pmatrix}$. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map L_A. (Recall that this is the map $\mathbb{R}^2 \to \mathbb{R}^2$ that sends every vector $v \in \mathbb{R}^2$ to Av.)

Consider the following basis $\mathbf{v} = (v_1, v_2)$ of the vector space $\mathbb{R}^2$:

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Consider the following basis $\mathbf{w} = (w_1, w_2)$ of the vector space $\mathbb{R}^2$:

$$w_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

What is the matrix $M_{\mathbf{v},\mathbf{w},L}$ representing L with respect to these two bases $\mathbf{v}$ and $\mathbf{w}$?

First, let me notice that it is not A (or at least it doesn't have to be A a priori), because our two bases $\mathbf{v}$ and $\mathbf{w}$ are not the standard basis of $\mathbb{R}^2$. Only if we pick both $\mathbf{v}$ and $\mathbf{w}$ to be the standard bases of the respective spaces we can guarantee that $M_{\mathbf{v},\mathbf{w},L}$ will be A.

Without having this shortcut, we must resort to the definition of $M_{\mathbf{v},\mathbf{w},L}$. It tells us to expand $L(v_1)$ and $L(v_2)$ in the basis $\mathbf{w}$ of $\mathbb{R}^2$, and to place the resulting coefficients in a 2×2-matrix. Let's do this. We begin with $L(v_1)$:

$$L(v_1) = L_A(v_1) = Av_1 = \begin{pmatrix} 5 & 7 \\ -2 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -11 \end{pmatrix}.$$

How do we expand this in the basis $\mathbf{w}$? This is a typical exercise in Gaussian elimination (we just need to solve the equation $L(v_1) = \lambda_1 w_1 + \lambda_2 w_2$ in the two unknowns λ_1 and λ_2), and the result is

$$L(v_1) = \frac{7}{3} w_1 + \frac{-20}{3} w_2.$$

Similarly, we take care of $L(v_2)$, obtaining

$$L(v_2) = 13 w_1 + 5 w_2.$$

Thus, the required matrix is

$$M_{\mathbf{v},\mathbf{w},L} = \begin{pmatrix} \frac{7}{3} & 13 \\ \frac{-20}{3} & 5 \end{pmatrix}.$$

Exercise 1. Let A be the 3×2-matrix $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map L_A. (Recall that this is the map $\mathbb{R}^3 \to \mathbb{R}^2$ that sends every vector $v \in \mathbb{R}^3$ to Av.)

Consider the following basis $\mathbf{v} = (v_1, v_2, v_3)$ of the vector space $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Consider the following basis $\mathbf{w} = (w_1, w_2)$ of the vector space $\mathbb{R}^2$:

$$w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(a) Find the matrix $M_{\mathbf{v},\mathbf{w},L}$ representing L with respect to these two bases $\mathbf{v}$ and $\mathbf{w}$. [15 points]

(b) Let $\mathbf{v}'$ be the basis (v_3, v_2, v_1) of $\mathbb{R}^3$. Let $\mathbf{w}'$ be the basis (w_2, w_1) of $\mathbb{R}^2$. Find the matrix $M_{\mathbf{v}',\mathbf{w}',L}$. [5 points]

Solution. **(a)** We follow Definition 0.1. Thus, we compute the three vectors $L(v_1), L(v_2), L(v_3)$, and expand them in the basis $\mathbf{w}$:

We have

$$\begin{aligned}
\underbrace{L}_{=L_A}(v_1) &= L_A(v_1) = Av_1 \qquad \text{(by the definition of } L_A\text{)} \\
&= \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} \\
&= 12w_1 + (-3)\, w_2. \qquad (2)
\end{aligned}$$

(We found the coefficients 12 and -3 by solving the equation $\begin{pmatrix} 9 \\ 12 \end{pmatrix} = \lambda_1 w_1 + \lambda_2 w_2$ in the unknowns λ_1 and λ_2. This is a straightforward exercise in Gaussian elimination.) Thus, we have expanded $L(v_1)$ in the basis $\mathbf{w}$. Similarly, we can expand $L(v_2)$ and $L(v_3)$ in $\mathbf{w}$; the results are

$$L(v_2) = 10w_1 + (-2)\, w_2; \qquad (3)$$
$$L(v_3) = 6w_1 + (-1)\, w_2. \qquad (4)$$

The matrix $M_{\mathbf{v},\mathbf{w},L}$ now can be built from the coefficients of the right hand sides of (2), (3) and (4): Namely, we get

$$M_{\mathbf{v},\mathbf{w},L} = \begin{pmatrix} 12 & 10 & 6 \\ -3 & -2 & -1 \end{pmatrix}. \qquad (5)$$

(b) Let us introduce new names for the vectors v_3, v_2, v_1 and w_2, w_1. Namely:

- Denote the list (v_3, v_2, v_1) by (v_1', v_2', v_3'); thus, $v_1' = v_3$, $v_2' = v_2$ and $v_3' = v_1$. Thus, $\mathbf{v}' = (v_3, v_2, v_1) = (v_1', v_2', v_3')$.
- Denote the list (w_2, w_1) by (w_1', w_2'); thus, $w_1' = w_2$ and $w_2' = w_1$. Thus, $\mathbf{w}' = (w_2, w_1) = (w_1', w_2')$.

(You do not have to do this; but I am doing this in order not to get confused with subscripts playing unusual roles. The advantage of writing $\mathbf{v}'$ as (v_1', v_2', v_3') instead of (v_3, v_2, v_1) is that the 1-st entry has a subscript "1", etc.)

Now, we have a basis $\mathbf{v}' = (v_1', v_2', v_3')$ of V and a basis $\mathbf{w}' = (w_1', w_2')$ of W (and we know the entries of these bases explicitly: for example, $v_1' = v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$), and we want to find the matrix $M_{\mathbf{v}',\mathbf{w}',L}$. We can do this using the same method as in part **(a)** of this exercise; the final result is

$$M_{\mathbf{v}',\mathbf{w}',L} = \begin{pmatrix} -1 & -2 & -3 \\ 6 & 10 & 12 \end{pmatrix}.$$

(Notice that this matrix can be obtained from the matrix $M_{\mathbf{v},\mathbf{w},L}$ shown in (5) by reversing the order of the rows and reversing the order of columns. This should not be a surprise, since the bases $\mathbf{v}'$ and $\mathbf{w}'$ are obtained from the bases $\mathbf{v}$ and $\mathbf{w}$ by reversing the order of the vectors. If you make this observation, you can of course find the matrix $M_{\mathbf{v}',\mathbf{w}',L}$ right away without computing anything, provided you have solved part **(a)**.) □

For every $n \in \mathbb{N}$, we let P_n denote the vector space of all polynomial functions (with real coefficients) of degree $\leq n$ in one variable x. This vector space has dimension $n+1$, and its simplest basis is $\left(1, x, x^2, \ldots, x^n\right)$. We call this basis the *monomial basis* of P_n.

If f is a polynomial in one variable x, then I shall use the notation $f[y]$ for "substitute y for x into f". (For example, if $f = x^3 + 7x + 2$, then $f[5] = 5^3 + 7 \cdot 5 + 2 = 162$.) This would normally be denoted by $f(y)$, but this is somewhat ambiguous, since the notation $x(x+1)$ could then stand for two different things (namely, "substitute $x+1$ into the polynomial function x" or "multiply x by $x+1$"), whereas the notation $f[y]$ removes this ambiguity.

Example 0.3. (a) Define a map $S_a : P_2 \to \mathbb{R}$ by $S_a(f) = f[2] + f[3]$. Then, S_a is linear, because:

1. If f and g are two elements of P_2, then

$$\begin{aligned} S_a(f+g) &= \underbrace{(f+g)[2]}_{=f[2]+g[2]} + \underbrace{(f+g)[3]}_{=f[3]+g[3]} = (f[2]+g[2]) + (f[3]+g[3]) \\ &= \underbrace{(f[2]+f[3])}_{=S_a(f)} + \underbrace{(g[2]+g[3])}_{=S_a(g)} = S_a(f) + S_a(g). \end{aligned}$$

2. If $f \in P_2$ and $\lambda \in \mathbb{R}$, then

$$S_a(\lambda f) = (\lambda f)[2] + (\lambda f)[3] = \lambda f[2] + \lambda f[3] = \lambda \underbrace{(f[2]+f[3])}_{=S_a(f)} = \lambda S_a(f).$$

Let $\mathbf{v}$ be the monomial basis $\left(1, x, x^2\right)$ of P_2, and let $\mathbf{w}$ be the one-element basis (1) of $\mathbb{R}$. What is the matrix $M_{\mathbf{v},\mathbf{w},S_a}$?

Again, follow the definition of $M_{\mathbf{v},\mathbf{w},S_a}$. It tells us to expand $S_a(1)$, $S_a(x)$ and $S_a\left(x^2\right)$ in the basis $\mathbf{w}$ of $\mathbb{R}$, and to place the resulting coefficients in a 1×3-matrix. Expanding things in the basis $\mathbf{w}$ is particularly simple, since $\mathbf{w}$ is a one-element list; specifically, we obtain the expansions

$$\begin{aligned} S_a(1) &= \underbrace{1[2]}_{=1} + \underbrace{1[3]}_{=1} = 1 + 1 = 2 = 2 \cdot 1; \\ S_a(x) &= \underbrace{x[2]}_{=2} + \underbrace{x[3]}_{=3} = 2 + 3 = 5 = 5 \cdot 1; \\ S_a\left(x^2\right) &= \underbrace{x^2[2]}_{=2^2} + \underbrace{x^2[3]}_{=3^2} = 2^2 + 3^2 = 13 = 13 \cdot 1. \end{aligned}$$

Thus, the required matrix is

$$M_{\mathbf{v},\mathbf{w},S_a} = \begin{pmatrix} 2 & 5 & 13 \end{pmatrix}.$$

(b) Define a map $S_b : P_2 \to P_4$ by $S_b(f) = f\left[x^2\right]$. (Notice that we chose P_4 as the target space, because substituting x^2 for x will double the degree of a polynomial.) The map S_b is linear (for reasons that are similar to the ones that convinced us that S_a is linear).

Let $\mathbf{v}$ be the monomial basis $\left(1, x, x^2\right)$ of P_2, and let $\mathbf{w}$ be the monomial basis $\left(1, x, x^2, x^3, x^4\right)$ of P_4. What is the matrix $M_{\mathbf{v},\mathbf{w},S_b}$?

Again, follow the definition of $M_{\mathbf{v},\mathbf{w},S_b}$. It tells us to expand $S_b(1)$, $S_b(x)$ and $S_b\left(x^2\right)$ in the basis $\mathbf{w}$ of P_4, and to place the resulting coefficients in a 5×3-matrix. The expansions are as follows:

$$\begin{aligned} S_b(1) &= 1\left[x^2\right] = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4; \\ S_b(x) &= x\left[x^2\right] = x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4; \\ S_b\left(x^2\right) &= x^2\left[x^2\right] = \left(x^2\right)^2 = x^4 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 1 \cdot x^4. \end{aligned}$$

Thus, the required matrix is

$$M_{\mathbf{v},\mathbf{w},S_b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 2. Which of the following maps are linear? For every one that is, represent it as a matrix with respect to the monomial bases of its domain and its target. [6 points for each part, split into 2+4 if the map is linear]

(a) The map $T_a : P_2 \to P_2$ given by $T_a(f) = f[x+1]$. (Thus, T_a is the map that substitutes $x+1$ for x into f. Thus, $T_a(x^n) = (x+1)^n$ for every $n \in \{0,1,2\}$.)

(b) The map $T_b : P_2 \to P_3$ given by $T_b(f) = xf[x]$. (Notice that $f[x]$ is the same as f, because substituting x for x changes nothing. I am just writing $f[x]$ to stress that f is a function of x.)

(c) The map $T_c : P_2 \to P_4$ given by $T_c(f) = f[1]\, f[x]$.

(d) The map $T_d : P_2 \to P_4$ given by $T_d(f) = f\left[x^2+1\right]$.

(e) The map $T_e : P_2 \to P_2$ given by $T_e(f) = x^2 f\left[\dfrac{1}{x}\right]$.

(g) The map $T_g : P_3 \to P_3$ given by $T_g(f) = xf'[x]$.

[There is no part **(f)** because I want to avoid calling a map "T_f" while the letter f stands for a polynomial.]

[**Note:** Proofs are not required.]

Solution. **(a)** The map T_a is linear, because:

1. If f and g are two elements of P_2, then
$$T_a(f+g) = (f+g)[x+1] = \underbrace{f[x+1]}_{=T_a(f)} + \underbrace{g[x+1]}_{=T_a(g)} = T_a(f) + T_a(g).$$

2. If $f \in P_2$ and $\lambda \in \mathbb{R}$, then
$$T_a(\lambda f) = (\lambda f)[x+1] = \lambda \underbrace{f[x+1]}_{=T_a(f)} = \lambda T_a(f).$$

Let $\mathbf{v}$ be the monomial basis $\left(1, x, x^2\right)$ of P_2. Let $\mathbf{w}$ be the monomial basis $\left(1, x, x^2\right)$ of P_2.

To find the matrix $M_{\mathbf{v},\mathbf{w},T_a}$ that represents T_a with respect to the bases $\mathbf{v}$ and $\mathbf{w}$, we proceed as in Example 0.3. Thus, we expand $T_a(1)$, $T_a(x)$ and $T_a\left(x^2\right)$ in the basis $\mathbf{w}$ of P_2, and we place the resulting coefficients in a 3×3-matrix. The expansions are as follows:

$$\begin{aligned}
T_a(1) &= 1[x+1] = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2; \\
T_a(x) &= x[x+1] = x+1 = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2; \\
T_a\left(x^2\right) &= x^2[x+1] = (x+1)^2 = x^2 + 2x + 1 = 1 \cdot 1 + 2 \cdot x + 1 \cdot x^2.
\end{aligned}$$

Thus, the required matrix is

$$M_{\mathbf{v},\mathbf{w},T_a} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) The map T_b is linear, because:

1. If f and g are two elements of P_2, then

$$T_b(f+g) = x \underbrace{(f+g)[x]}_{=f[x]+g[x]} = x(f[x]+g[x]) = \underbrace{xf[x]}_{=T_b(f)} + \underbrace{xg[x]}_{=T_b(g)} = T_b(f) + T_b(g).$$

2. If $f \in P_2$ and $\lambda \in \mathbb{R}$, then

$$T_b(\lambda f) = x \underbrace{(\lambda f)[x]}_{=\lambda f[x]} = \lambda \underbrace{xf[x]}_{=T_b(f)} = \lambda T_b(f).$$

Let $\mathbf{v}$ be the monomial basis $(1, x, x^2)$ of P_2. Let $\mathbf{w}$ be the monomial basis $(1, x, x^2, x^3)$ of P_3.

To find the matrix $M_{\mathbf{v},\mathbf{w},T_b}$ that represents T_b with respect to the bases $\mathbf{v}$ and $\mathbf{w}$, we proceed as in Example 0.3. Thus, we expand $T_b(1)$, $T_b(x)$ and $T_b(x^2)$ in the basis $\mathbf{w}$ of P_3, and we place the resulting coefficients in a 4×3-matrix. The expansions are as follows:

$$T_b(1) = x \underbrace{1[x]}_{=1} = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3;$$
$$T_b(x) = x \underbrace{x[x]}_{=x} = xx = x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3;$$
$$T_b\left(x^2\right) = x \underbrace{x^2[x]}_{=x^2} = xx^2 = x^3 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 1 \cdot x^3.$$

Thus, the required matrix is

$$M_{\mathbf{v},\mathbf{w},T_b} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) The map T_c is **not** linear.[1]

(d) The map T_d is linear, because:

1. If f and g are two elements of P_2, then

$$T_d(f+g) = (f+g)\left[x^2+1\right] = \underbrace{f\left[x^2+1\right]}_{=T_d(f)} + \underbrace{g\left[x^2+1\right]}_{=T_d(g)} = T_d(f) + T_d(g).$$

[1] For example, it fails to satisfy the "if $f \in P_2$ and $\lambda \in \mathbb{R}$, then $T_c(\lambda f) = \lambda T_c(f)$" condition if we pick $f = 1$ and $\lambda = 2$, because in this case we have $T_c(\lambda f) = \underbrace{(\lambda f)}_{=2}[1] \cdot \underbrace{(\lambda f)}_{=2}[x] = \underbrace{2[1]}_{=2} \cdot \underbrace{2[x]}_{=2} = 2 \cdot 2 = 4$ and $\underbrace{\lambda}_{=2} \underbrace{T_c(f)}_{=f[1]f[x]} = 2 \underbrace{f}_{=1}[1] \underbrace{f}_{=1}[x] = 2 \cdot \underbrace{1[1]}_{=1} \cdot \underbrace{1[x]}_{=1} = 2.$

2. If $f \in P_2$ and $\lambda \in \mathbb{R}$, then
$$T_d(\lambda f) = (\lambda f)\left[x^2+1\right] = \lambda \underbrace{f\left[x^2+1\right]}_{=T_d(f)} = \lambda T_d(f).$$

Let $\mathbf{v}$ be the monomial basis $\left(1, x, x^2\right)$ of P_2. Let $\mathbf{w}$ be the monomial basis $\left(1, x, x^2, x^3, x^4\right)$ of P_4.

To find the matrix $M_{\mathbf{v},\mathbf{w},T_d}$ that represents T_d with respect to the bases $\mathbf{v}$ and $\mathbf{w}$, we proceed as in Example 0.3. Thus, we expand $T_d(1)$, $T_d(x)$ and $T_d\left(x^2\right)$ in the basis $\mathbf{w}$ of P_4, and we place the resulting coefficients in a 5×3-matrix. The expansions are as follows:

$$T_d(1) = 1\left[x^2+1\right] = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 + 0\cdot x^4;$$
$$T_d(x) = x\left[x^2+1\right] = x^2+1 = 1\cdot 1 + 0\cdot x + 1\cdot x^2 + 0\cdot x^3 + 0\cdot x^4;$$
$$T_d\left(x^2\right) = x^2\left[x^2+1\right] = \left(x^2+1\right)^2 = x^4+2x^2+1 = 1\cdot 1 + 0\cdot x + 2\cdot x^2 + 0\cdot x^3 + 1\cdot x^4.$$

Thus, the required matrix is
$$M_{\mathbf{v},\mathbf{w},T_d} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(e) The map T_e is linear, because:

1. If f and g are two elements of P_2, then
$$T_e(f+g) = x^2 \underbrace{(f+g)\left[\frac{1}{x}\right]}_{=f\left[\frac{1}{x}\right]+g\left[\frac{1}{x}\right]} = x^2\left(f\left[\frac{1}{x}\right] + g\left[\frac{1}{x}\right]\right)$$
$$= \underbrace{x^2 f\left[\frac{1}{x}\right]}_{=T_e(f)} + \underbrace{x^2 g\left[\frac{1}{x}\right]}_{=T_e(g)} = T_e(f) + T_e(g).$$

2. If $f \in P_2$ and $\lambda \in \mathbb{R}$, then
$$T_e(\lambda f) = x^2 \underbrace{(\lambda f)\left[\frac{1}{x}\right]}_{=\lambda f\left[\frac{1}{x}\right]} = \lambda \underbrace{x^2 f\left[\frac{1}{x}\right]}_{=T_e(f)} = \lambda T_e(f).$$

Let $\mathbf{v}$ be the monomial basis $(1, x, x^2)$ of P_2. Let $\mathbf{w}$ be the monomial basis $(1, x, x^2)$ of P_2.

To find the matrix $M_{\mathbf{v},\mathbf{w},T_e}$ that represents T_e with respect to the bases $\mathbf{v}$ and $\mathbf{w}$, we proceed as in Example 0.3. Thus, we expand $T_e(1)$, $T_e(x)$ and $T_e(x^2)$ in the basis $\mathbf{w}$ of P_2, and we place the resulting coefficients in a 3×3-matrix. The expansions are as follows:

$$T_e(1) = x^2 \underbrace{1\left[\frac{1}{x}\right]}_{=1} = x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2;$$

$$T_e(x) = x^2 \underbrace{x\left[\frac{1}{x}\right]}_{=\frac{1}{x}} = x^2 \cdot \frac{1}{x} = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2;$$

$$T_e\left(x^2\right) = x^2 \underbrace{x^2\left[\frac{1}{x}\right]}_{=\left(\frac{1}{x}\right)^2} = x^2 \left(\frac{1}{x}\right)^2 = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2.$$

Thus, the required matrix is

$$M_{\mathbf{v},\mathbf{w},T_e} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(g) The map T_g is linear, because:

1. If f and g are two elements of P_3, then

$$T_g(f+g) = x \underbrace{(f+g)'[x]}_{=f'[x]+g'[x]} = x\left(f'[x] + g'[x]\right)$$
$$= \underbrace{xf'[x]}_{=T_g(f)} + \underbrace{xg'[x]}_{=T_g(g)} = T_g(f) + T_g(g).$$

2. If $f \in P_3$ and $\lambda \in \mathbb{R}$, then

$$T_g(\lambda f) = x \underbrace{(\lambda f)'[x]}_{=\lambda f'[x]} = \lambda \underbrace{xf'[x]}_{=T_g(f)} = \lambda T_g(f).$$

Let $\mathbf{v}$ be the monomial basis $(1, x, x^2, x^3)$ of P_3. Let $\mathbf{w}$ be the monomial basis $(1, x, x^2, x^3)$ of P_3.

To find the matrix $M_{\mathbf{v},\mathbf{w},T_g}$ that represents T_g with respect to the bases $\mathbf{v}$ and $\mathbf{w}$, we proceed as in Example 0.3. Thus, we expand $T_g(1)$, $T_g(x)$, $T_g(x^2)$ and $T_g(x^3)$ in the basis $\mathbf{w}$ of P_3, and we place the resulting coefficients in a 4×4-matrix. The expansions are as follows:

$$T_g(1) = x \underbrace{1'[x]}_{=0} = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3;$$
$$T_g(x) = x \underbrace{x'[x]}_{=1} = x = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3;$$
$$T_g\left(x^2\right) = x \underbrace{\left(x^2\right)'[x]}_{=2x} = x \cdot 2x = 2x^2 = 0 \cdot 1 + 0 \cdot x + 2 \cdot x^2 + 0 \cdot x^3;$$
$$T_g\left(x^3\right) = x \underbrace{\left(x^3\right)'[x]}_{=3x^2} = x \cdot 3x^2 = 3x^3 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 3 \cdot x^3.$$

Thus, the required matrix is

$$M_{\mathbf{v},\mathbf{w},T_g} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

□

See the beginning of §3.21 of my notes, the Wikipedia, or various other sources, for examples of injective, surjective and bijective maps.

Exercise 3. **(a)** Which of the six maps in Exercise 2 are injective? [2 points per map]

(b) Which of them are surjective? [2 points per map]

[**Note:** Proofs are not required.]

Solution. As we have seen in the solution to Exercise 2, five of our six maps (more precisely, all the maps apart from T_c) are linear. Hence, it will be useful to know how to determine whether a linear map is injective. There is a method for that; to find it, we combine two results. First here is a result from homework set 4:

Proposition 0.4. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let A be an $n \times m$-matrix.

(a) The matrix A is right-invertible if and only if $\operatorname{rank} A = n$.

(b) The matrix A is left-invertible if and only if $\operatorname{rank} A = m$.

(c) The matrix A is invertible if and only if $\operatorname{rank} A = n = m$. (In particular, only square matrices can be invertible!)

Next, we recall a result from class:

Proposition 0.5. Let V and W be two vector spaces with bases $\mathbf{v}$ and $\mathbf{w}$, respectively. Let $L : V \to W$ be a linear map.
(a) The map L is injective if and only if the matrix $M_{\mathbf{v},\mathbf{w},L}$ is left-invertible.
(b) The map L is surjective if and only if the matrix $M_{\mathbf{v},\mathbf{w},L}$ is right-invertible.
(c) The map L is bijective if and only if the matrix $M_{\mathbf{v},\mathbf{w},L}$ is invertible.

Combining these two propositions, we obtain the following:

Corollary 0.6. Let V and W be two vector spaces with bases $\mathbf{v}$ and $\mathbf{w}$, respectively. Let $m = \dim V$ and $n = \dim W$. Let $L : V \to W$ be a linear map.
(a) The map L is injective if and only if $\operatorname{rank}(M_{\mathbf{v},\mathbf{w},L}) = m$.
(b) The map L is surjective if and only if $\operatorname{rank}(M_{\mathbf{v},\mathbf{w},L}) = n$.
(c) The map L is bijective if and only if $\operatorname{rank}(M_{\mathbf{v},\mathbf{w},L}) = n = m$.

Proof of Corollary 0.6. The basis $\mathbf{v}$ of V has size $\dim V = m$, and the basis $\mathbf{w}$ of W has size $\dim W = n$. Thus, $M_{\mathbf{v},\mathbf{w},L}$ is an $n \times m$-matrix.

(a) Proposition 0.5 **(a)** shows that the map L is injective if and only if the matrix $M_{\mathbf{v},\mathbf{w},L}$ is left-invertible. But Proposition 0.4 **(b)** (applied to $A = M_{\mathbf{v},\mathbf{w},L}$) shows that the matrix $M_{\mathbf{v},\mathbf{w},L}$ is left-invertible if and only if $\operatorname{rank}(M_{\mathbf{v},\mathbf{w},L}) = m$. Combining these two "if and only if" statements, we conclude that the map L is injective if and only if $\operatorname{rank}(M_{\mathbf{v},\mathbf{w},L}) = m$. This proves Corollary 0.6 **(a)**.

We have thus derived Corollary 0.6 **(a)** from Proposition 0.5 **(a)** and Proposition 0.4 **(b)**. Similarly, one can derive Corollary 0.6 **(b)** from Proposition 0.5 **(b)** and Proposition 0.4 **(a)**, and derive Corollary 0.6 **(c)** from Proposition 0.5 **(c)** and Proposition 0.4 **(c)**. □

Now, let us solve the actual exercise.

(a) Let us find out whether T_a is injective.

Let $\mathbf{v}$ be the monomial basis $(1, x, x^2)$ of P_2. Let $\mathbf{w}$ be the monomial basis $(1, x, x^2)$ of P_2.

From our solution of Exercise 2 **(a)**, we remember that the map $T_a : P_2 \to P_2$ is linear, and is represented by the matrix $M_{\mathbf{v},\mathbf{w},T_a} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. Clearly, $\dim(P_2) = 3$ and $\dim(P_2) = 3$. Hence, Corollary 0.6 **(a)** (applied to $V = P_2$, $W = P_2$, $m = 3$, $n = 3$ and $L = T_a$) shows that the map T_a is injective if and only if $\operatorname{rank}(M_{\mathbf{v},\mathbf{w},T_a}) = 3$. Since $\operatorname{rank}(M_{\mathbf{v},\mathbf{w},T_a}) = 3$ holds (indeed, this can be checked straightforwardly, since we know the matrix $M_{\mathbf{v},\mathbf{w},T_a}$), we thus conclude that the map T_a is injective.

By the same method, we can conclude that the maps T_b, T_d and T_e are injective, but the map T_g is not. (Each time, we need to apply Corollary 0.6 **(a)**, but of course the vector spaces V and W and their dimensions m and n change from map to map – thus, the rank of the matrix isn't always being compared to 3. For example, to tell whether T_g is injective, we have to determine whether $\operatorname{rank}\left(M_{\mathbf{v},\mathbf{w},T_g}\right) = 4$; but this is not satisfied because $\operatorname{rank}\left(M_{\mathbf{v},\mathbf{w},T_g}\right) = 3 \neq 4$.)

It remains to check whether the map T_c is injective. The method we have showed for T_a does not apply to T_c because T_c is not linear. Instead, we use the definition of the word "injective". Recall that a map $F : X \to Y$ (from a set X to a set Y) is injective if and only if it satisfies the following statement:

> If u_1 and u_2 are two elements of X satisfying $F(u_1) = F(u_2)$, then $u_1 = u_2$.

Applying this to the map $T_c : P_2 \to P_4$, we see that the map $T_c : P_2 \to P_4$ is injective if and only if it satisfies the following statement:

> If u_1 and u_2 are two elements of P_2 satisfying $T_c(u_1) = T_c(u_2)$, then $u_1 = u_2$.

But this statement is not true: For example, the two elements $u_1 = x - 1$ and $u_2 = 0$ of P_2 satisfy $T_c(u_1) = T_c(u_2)$ (because $T_c(u_1) = \underbrace{u_1[1]}_{=1-1=0} u_1[x] = 0$ and $T_c(u_2) = \underbrace{u_2[1]}_{=0} u_2[x] = 0$) but not $u_1 = u_2$.

Thus, the map T_c is not injective.

(b) The surjectivity of a linear map can be checked exactly in the same way as its injectivity, except that we have to use Corollary 0.6 **(b)** instead of Corollary 0.6 **(a)**. Thus, let me only summarize the results: The maps T_a and T_e are surjective, whereas the maps T_b, T_d and T_g are not. Again, the map T_c needs to be treated separately, since it is not linear. For this, we use the definition of the word "surjective". Recall that a map $F : X \to Y$ (from a set X to a set Y) is surjective if and only if it satisfies the following statement:

> For each $v \in Y$, there exists some $u \in X$ such that $v = F(u)$.

Applying this to the map $T_c : P_2 \to P_4$, we see that the map $T_c : P_2 \to P_4$ is surjective if and only if it satisfies the following statement:

> For each $v \in P_4$, there exists some $u \in P_2$ such that $v = T_c(u)$.

But this statement is not true: For example, for $v = x^4 \in P_4$, there exists no $u \in P_2$ such that $v = T_c(u)$ (because $T_c(u) = \underbrace{u[1]}_{\text{a constant}} \underbrace{u[x]}_{\substack{\text{a polynomial} \\ \text{of degree } \leq 2}}$ would have to be a polynomial of degree ≤ 2, but $v = x^4$ is not such a polynomial).

Hence, the map T_c is not surjective.

[*Remark:* We might wonder whether the map T_c becomes surjective if we restrict its target to P_2. In other words, is the map $T_{c2} : P_2 \to P_2$ given by $T_c(f) = f[1] f[x]$ surjective? We can no longer rule this out using $v = x^4$, because x^4 does not belong to P_2.

However, this new map T_{c2} is still not surjective. Indeed, for $v = x - 1 \in P_2$, there exist no $u \in P_2$ such that $v = T_{c2}(u)$. In order to prove this, we assume the contrary. Thus, there exists some $u \in P_2$ such that $v = T_{c2}(u)$. Consider this u. We have $x - 1 = v = T_{c2}[u] = u[1]\, u[x]$. Substituting 1 for x in this equality, we obtain $1 - 1 = u[1]\, u[1] = u[1]^2$. In other words, $0 = u[1]^2$. Hence, $u[1] = 0$. Now, $v = T_{c2}[u] = \underbrace{u[1]}_{=0} u[x] = 0$. But this contradicts $v = x - 1$. This contradiction shows that our assumption was wrong, qed.] □