Math 4242 Fall 2016 (Darij Grinberg): homework set 6
due: Mon, 21 Nov 2016, in class
(or **earlier** by moodle)

Let me first recall a definition.

Definition 0.1. Let V and W be two vector spaces. Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ be a basis of V. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a basis of W. Let $L : V \to W$ be a linear map.

The *matrix representing* L with respect to $\mathbf{v}$ and $\mathbf{w}$ is the $n \times m$-matrix $M_{\mathbf{v},\mathbf{w},L}$ defined as follows: For every $j \in \{1, 2, \ldots, m\}$, expand the vector $L(v_j)$ with respect to the basis $\mathbf{w}$, say, as follows:

$$L(v_j) = \alpha_{1,j} w_1 + \alpha_{2,j} w_2 + \cdots + \alpha_{n,j} w_n. \quad (1)$$

Then, $M_{\mathbf{v},\mathbf{w},L}$ is the $n \times m$-matrix whose (i,j)-th entry is $\alpha_{i,j}$.

For example, if $n = 3$ and $m = 2$, then

$$M_{\mathbf{v},\mathbf{w},L} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \\ \alpha_{3,1} & \alpha_{3,2} \end{pmatrix},$$

where

$$L(v_1) = \alpha_{1,1} w_1 + \alpha_{2,1} w_2 + \alpha_{3,1} w_3;$$
$$L(v_2) = \alpha_{1,2} w_1 + \alpha_{2,2} w_2 + \alpha_{3,2} w_3.$$

The purpose of this matrix $M_{\mathbf{v},\mathbf{w},L}$ is to allow easily expanding $L(v)$ in the basis $(w_1, w_2, \ldots, w_n)$ of W if v is a vector in V whose expansion in the basis $(v_1, v_2, \ldots, v_m)$ of V is known. For instance, if v is one of the basis vectors v_j, then the expansion of $L(v_j)$ can be simply read off from the j-th column of $M_{\mathbf{v},\mathbf{w},L}$; otherwise, it is an appropriate linear combination:

$$L(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m) = \lambda_1 L(v_1) + \lambda_2 L(v_2) + \cdots + \lambda_m L(v_m)$$

(where the $L(v_j)$ can be computed by (1)).

You can abbreviate $M_{\mathbf{v},\mathbf{w},L}$ as M_L, but it's your job to ensure that you know what $\mathbf{v}$ and $\mathbf{w}$ are (and they aren't changing midway through your work).

Example 0.2. Let A be the 2×2-matrix $\begin{pmatrix} 5 & 7 \\ -2 & 9 \end{pmatrix}$. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map L_A. (Recall that this is the map $\mathbb{R}^2 \to \mathbb{R}^2$ that sends every vector $v \in \mathbb{R}^2$ to Av.)

Consider the following basis $\mathbf{v} = (v_1, v_2)$ of the vector space $\mathbb{R}^2$:

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Consider the following basis $\mathbf{w} = (w_1, w_2)$ of the vector space $\mathbb{R}^2$:

$$w_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

What is the matrix $M_{\mathbf{v},\mathbf{w},L}$ representing L with respect to these two bases $\mathbf{v}$ and $\mathbf{w}$?

First, let me notice that it is not A (or at least it doesn't have to be A a priori), because our two bases $\mathbf{v}$ and $\mathbf{w}$ are not the standard basis of $\mathbb{R}^2$. Only if we pick both $\mathbf{v}$ and $\mathbf{w}$ to be the standard bases of the respective spaces we can guarantee that $M_{\mathbf{v},\mathbf{w},L}$ will be A.

Without having this shortcut, we must resort to the definition of $M_{\mathbf{v},\mathbf{w},L}$. It tells us to expand $L(v_1)$ and $L(v_2)$ in the basis $\mathbf{w}$ of $\mathbb{R}^2$, and to place the resulting coefficients in a 2×2-matrix. Let's do this. We begin with $L(v_1)$:

$$L(v_1) = L_A(v_1) = Av_1 = \begin{pmatrix} 5 & 7 \\ -2 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -11 \end{pmatrix}.$$

How do we expand this in the basis $\mathbf{w}$? This is a typical exercise in Gaussian elimination (we just need to solve the equation $L(v_1) = \lambda_1 w_1 + \lambda_2 w_2$ in the two unknowns λ_1 and λ_2), and the result is

$$L(v_1) = \frac{7}{3} w_1 + \frac{-20}{3} w_2.$$

Similarly, we take care of $L(v_2)$, obtaining

$$L(v_2) = 13 w_1 + 5 w_2.$$

Thus, the required matrix is

$$M_{\mathbf{v},\mathbf{w},L} = \begin{pmatrix} \frac{7}{3} & 13 \\ \frac{-20}{3} & 5 \end{pmatrix}.$$

Exercise 1. Let A be the 3×2-matrix $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map L_A. (Recall that this is the map $\mathbb{R}^3 \to \mathbb{R}^2$ that sends every vector $v \in \mathbb{R}^3$ to Av.)

Consider the following basis $\mathbf{v} = (v_1, v_2, v_3)$ of the vector space $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Consider the following basis $\mathbf{w} = (w_1, w_2)$ of the vector space $\mathbb{R}^2$:

$$w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(a) Find the matrix $M_{\mathbf{v},\mathbf{w},L}$ representing L with respect to these two bases $\mathbf{v}$ and $\mathbf{w}$. [15 points]

(b) Let $\mathbf{v}'$ be the basis (v_3, v_2, v_1) of $\mathbb{R}^3$. Let $\mathbf{w}'$ be the basis (w_2, w_1) of $\mathbb{R}^2$. Find the matrix $M_{\mathbf{v}',\mathbf{w}',L}$. [5 points]

For every $n \in \mathbb{N}$, we let P_n denote the vector space of all polynomial functions (with real coefficients) of degree $\leq n$ in one variable x. This vector space has dimension $n+1$, and its simplest basis is $\left(1, x, x^2, \ldots, x^n\right)$. We call this basis the *monomial basis* of P_n.

If f is a polynomial in one variable x, then I shall use the notation $f[y]$ for "substitute y for x into f". (For example, if $f = x^3 + 7x + 2$, then $f[5] = 5^3 + 7 \cdot 5 + 2 = 162$.) This would normally be denoted by $f(y)$, but this is somewhat ambiguous, since the notation $x(x+1)$ could then stand for two different things (namely, "substitute $x+1$ into the polynomial function x" or "multiply x by $x+1$"), whereas the notation $f[y]$ removes this ambiguity.

Example 0.3. (a) Define a map $S_a : P_2 \to \mathbb{R}$ by $S_a(f) = f[2] + f[3]$. Then, S_a is linear, because:

1. If f and g are two elements of P_2, then

$$\begin{aligned} S_a(f+g) &= \underbrace{(f+g)[2]}_{=f[2]+g[2]} + \underbrace{(f+g)[3]}_{=f[3]+g[3]} = (f[2]+g[2]) + (f[3]+g[3]) \\ &= \underbrace{(f[2]+f[3])}_{=S_a(f)} + \underbrace{(g[2]+g[3])}_{=S_a(g)} = S_a(f) + S_a(g). \end{aligned}$$

2. If $f \in P_2$ and $\lambda \in \mathbb{R}$, then

$$S_a(\lambda f) = (\lambda f)[2] + (\lambda f)[3] = \lambda f[2] + \lambda f[3] = \lambda \underbrace{(f[2]+f[3])}_{=S_a(f)} = \lambda S_a(f).$$

Let $\mathbf{v}$ be the monomial basis $\left(1, x, x^2\right)$ of P_2, and let $\mathbf{w}$ be the one-element basis (1) of $\mathbb{R}$. What is the matrix $M_{\mathbf{v},\mathbf{w},S_a}$?

Again, follow the definition of $M_{\mathbf{v},\mathbf{w},S_a}$. It tells us to expand $S_a(1)$, $S_a(x)$ and $S_a(x^2)$ in the basis $\mathbf{w}$ of $\mathbb{R}$, and to place the resulting coefficients in a 1×3-matrix. Expanding things in the basis $\mathbf{w}$ is particularly simple, since $\mathbf{w}$ is a one-element list; specifically, we obtain the expansions

$$S_a(1) = \underbrace{1[2]}_{=1} + \underbrace{1[3]}_{=1} = 1 + 1 = 2 = 2 \cdot 1;$$
$$S_a(x) = \underbrace{x[2]}_{=2} + \underbrace{x[3]}_{=3} = 2 + 3 = 5 = 5 \cdot 1;$$
$$S_a\left(x^2\right) = \underbrace{x^2[2]}_{=2^2} + \underbrace{x^2[3]}_{=3^2} = 2^2 + 3^2 = 13 = 13 \cdot 1.$$

Thus, the required matrix is

$$M_{\mathbf{v},\mathbf{w},S_a} = \begin{pmatrix} 2 & 5 & 13 \end{pmatrix}.$$

(b) Define a map $S_b : P_2 \to P_4$ by $S_b(f) = f\left[x^2\right]$. (Notice that we chose P_4 as the target space, because substituting x^2 for x will double the degree of a polynomial.) The map S_b is linear (for reasons that are similar to the ones that convinced us that S_a is linear).

Let $\mathbf{v}$ be the monomial basis $(1, x, x^2)$ of P_2, and let $\mathbf{w}$ be the monomial basis $(1, x, x^2, x^3, x^4)$ of P_4. What is the matrix $M_{\mathbf{v},\mathbf{w},S_b}$?

Again, follow the definition of $M_{\mathbf{v},\mathbf{w},S_b}$. It tells us to expand $S_b(1)$, $S_b(x)$ and $S_b(x^2)$ in the basis $\mathbf{w}$ of P_4, and to place the resulting coefficients in a 5×3-matrix. The expansions are as follows:

$$S_b(1) = 1\left[x^2\right] = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4;$$
$$S_b(x) = x\left[x^2\right] = x^2 = 0 \cdot 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4;$$
$$S_b\left(x^2\right) = x^2\left[x^2\right] = \left(x^2\right)^2 = x^4 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + 1 \cdot x^4.$$

Thus, the required matrix is

$$M_{\mathbf{v},\mathbf{w},S_b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 2. Which of the following maps are linear? For every one that is, represent it as a matrix with respect to the monomial bases of its domain and its target. [6 points for each part, split into 2+4 if the map is linear]

(a) The map $T_a : P_2 \to P_2$ given by $T_a(f) = f[x+1]$. (Thus, T_a is the map that substitutes $x+1$ for x into f. Thus, $T_a(x^n) = (x+1)^n$ for every $n \in \{0,1,2\}$.)

(b) The map $T_b : P_2 \to P_3$ given by $T_b(f) = xf[x]$. (Notice that $f[x]$ is the same as f, because substituting x for x changes nothing. I am just writing $f[x]$ to stress that f is a function of x.)

(c) The map $T_c : P_2 \to P_4$ given by $T_c(f) = f[1]\,f[x]$.

(d) The map $T_d : P_2 \to P_4$ given by $T_d(f) = f\left[x^2+1\right]$.

(e) The map $T_e : P_2 \to P_2$ given by $T_e(f) = x^2 f\left[\frac{1}{x}\right]$.

(g) The map $T_g : P_3 \to P_3$ given by $T_g(f) = xf'[x]$.

[There is no part **(f)** because I want to avoid calling a map "T_f" while the letter f stands for a polynomial.]

[**Note:** Proofs are not required.]

See the beginning of §3.21 of my notes, the Wikipedia, or various other sources, for examples of injective, surjective and bijective maps.

Exercise 3. **(a)** Which of the six maps in Exercise 2 are injective? [2 points per map]

(b) Which of them are surjective? [2 points per map]

[**Note:** Proofs are not required.]