Math 4242, Section 070

Homework 5

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1) a. We have that $\{(1,1,1)^T, (1,2,3)^T\}$ is a basis of U. So set $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$. Then we know that $U^\perp = \ker(A^T)$. Well, $\ker(A^T) = \{x \mid A^T x = \mathbf{0}\}$. We can then set up the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

Setting $z = r$, we have that $y = -2r$ and $x = r$. So $U^\perp = \operatorname{span}\left((1,-2,1)^T\right)$.

b. Since we are in $\mathbb{R}^3$, we know that two orthogonal spaces will only meet at $\mathbf{0}$. Hence we have that $U \cap U^\perp = \{\mathbf{0}\}$. Thus, $U \cap U^\perp$ is the span of the empty list.

c. Once again we are in $\mathbb{R}^3$, so two perpendicular subspaces will create all of $\mathbb{R}^3$. Hence $U + U^\perp = \mathbb{R}^3$. Thus, $U \cap U^\perp = \operatorname{span}(e_1, e_2, e_3)$, where e_1, e_2, e_3 are the three standard basis vectors.

d. We will use the Gram-Schmidt process. First we set $u_1 = w_1 = (1,1,1)^T$. Then

$$u_2 = w_2 - \lambda_{2,1} u_1 = (1,2,3)^T - \frac{\langle (1,2,3)^T, (1,1,1)^T \rangle}{3} (1,1,1)^T = (-1,0,1)^T$$

Hence an orthogonal basis of U is $\{(1,1,1)^T, (-1,0,1)^T\}$.

e. Since we already found a basis for $U^\perp$ and it only contains one element, we have that an orthogonal basis for it is just $\{(1,-2,1)^T\}$.

f. We know that $\left(U^\perp\right)^\perp = U$, so set $W = U^\perp$. Then $U = \left(U^\perp\right)^\perp = W^\perp$.

2) a. We know that $||Ax - b||$ is minimized when $x = Kf = (A^T A)^{-1}(A^T b)$.

$$A^T A = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} \implies K = \left(A^T A\right)^{-1} = \frac{1}{3} \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{pmatrix}$$

$$f = A^T b = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$$

Hence we have that

$$x = Kf = \begin{pmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 6 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{4}{3} \end{pmatrix}$$

b. We use the same approach, but now we have that

$$A^T A = \begin{pmatrix} 3 & 5 \\ 5 & 10 \end{pmatrix} \implies K = \left(A^T A\right)^{-1} = \frac{1}{5} \begin{pmatrix} 10 & -5 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & \frac{3}{5} \end{pmatrix}$$

$$f = A^T b = \begin{pmatrix} 10 \\ 19 \end{pmatrix}$$

Hence we have that

$$x = Kf = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 19 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{7}{5} \end{pmatrix}$$

3) a. Looking at the column vectors of A, we have that the third column is the sum of the first two columns, while the first two columns are linearly independent. Hence a basis for Col A is $\{(1,0,1,1)^T, (0,1,1,2)^T\}$.

b. We have that $A' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$.

c. Since A' has columns which are a basis of the column space of A, it is clear that $\text{Col}(A') = \text{Col } A$.

d. Since $\text{Col}(A') = \text{Col}(A)$, this is tantamount to finding the projection u of b on $\text{Col}(A')$. We can do this using the general projection equation, denoting the two columns of A' by u_1 and u_2. The result is

$$u = \frac{\langle b, u_1 \rangle}{||u_1||^2} u_1 + \frac{\langle b, u_2 \rangle}{||u_2||^2} u_2 = \frac{8}{3}(1,0,1,1)^T + \frac{13}{6}(0,1,1,2)^T = \left(\frac{8}{3}, \frac{13}{6}, \frac{29}{6}, 7 \right)^T$$

e. We wish to solve $A\mathbf{x} = u$. This can be done using Gaussian elimination. For example, we can set up and reduce the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 8/3 \\ 0 & 1 & 1 & 13/6 \\ 1 & 1 & 2 & 29/6 \\ 1 & 2 & 3 & 7 \end{array} \right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 8/3 \\ 0 & 1 & 1 & 13/6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

If we have $\mathbf{x} = (x, y, z)^T$, then we can set $z = r$. Then we have that $y = 13/6 - r$ and $x = 8/3 - r$. In other words

$$\mathbf{x} = (8/3 - r, 13/6 - r, r)^T \qquad \text{for } r \in \mathbb{R}.$$

4) Solution omitted.

5) *1st part:* From $\underline{\vec{0} \in U_1}$ and $\underline{z \in U_2}$, we obtain $\underline{\vec{0} + z \in U_1 + U_2}$ (by the definition of $U_1 + U_2$). In other words, $\underline{z \in U_1 + U_2}$ (since $\underline{\vec{0} + z = z}$). Hence, (1) (applied to $\underline{x = z}$) yields $\underline{v \perp z}$.

2nd part: By $\underline{(2)}$, applied to $\underline{x = u_1}$.

6) a. Essentially we just need to find $P_1^\perp$ since $\left(P_1^\perp\right)^\perp = P_1$. So we put the basis vectors of P_1 into a matrix as its columns, then transpose this matrix and find its kernel. Specifically, we want the kernel of:

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Finding its kernel, we set up the system equal to zero. Doing so we get that $x = -z = -r$ and that $y = 0$. Hence $U_1 = \text{span}\left((1, 0, -1)^T\right)$.

b. Using a similar approach as in (a), we just want to find $P_2^\perp$. So we are aiming to find the kernel of the matrix

$$B^T = \begin{pmatrix} -2 & 1 & 3 \\ 0 & 4 & 5 \end{pmatrix}$$

Here we see that given $z = r$, we have $y = -\frac{5}{4}r$, and solving through for x we get that $x = \frac{7}{8}r$. So our basis vector can be $(7/8, -5/4, 1)^T$. Scaling by 8 to get integers, we see that $U_2 = \operatorname{span}\left((7, -10, 8)^T\right)$.

c. In general: If P and Q are two subspaces of a vector space V, written as spans as $P = \operatorname{span}(p_1, p_2, \ldots, p_k)$ and $Q = \operatorname{span}(q_1, q_2, \ldots, q_\ell)$, then $P + Q$ can be written as a span as follows:

$$P + Q = \operatorname{span}(p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_\ell)$$

(because the elements of $P+Q$ are the sums of linear combinations of $p_1, p_2, \ldots, p_k$ with linear combinations of $q_1, q_2, \ldots, q_\ell$; but such sums are precisely the linear combinations of $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_\ell$). In other words, we can obtain a spanning list of $P + Q$ by concatenating a spanning list of P with a spanning list of Q. (Of course, this spanning list will not necessarily be a basis, even if we started with two bases.)

Thus, we can obtain a spanning list of $U_1 + U_2$ by concatenating a spanning list of U_1 with a spanning list of U_2. By doing this (using the spanning list $((1, 0, -1)^T)$ of U_1, and the spanning list $((7, -10, 8)^T)$ for U_2), we obtain $U_1 + U_2 = \operatorname{span}\left((1, 0, -1)^T, (7, -10, 8)^T\right)$.

d. We know that $P_1 \cap P_2 = U_1^\perp \cap U_2^\perp = (U_1 + U_2)^\perp$ by proposition 0.1. Hence we need to find $(U_1 + U_2)^\perp$. Using a similar approach as before, we find the kernel of:

$$D^T = \begin{pmatrix} 1 & 0 & -1 \\ 7 & -10 & 8 \end{pmatrix}$$

We can see that $x = z = r$, and hence $y = \frac{3}{2}r$. Hence:

$$(U_1 + U_2)^\perp = P_1 \cap P_2 = \operatorname{span}\left(\left(1, \frac{3}{2}, 1\right)^T\right)$$

7) We aim to find the kernel of the matrix $A^T = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$. We use Gaussian elimination first, assuming (for the time being!) that all expressions we encounter can be divided by unless they expand to 0. Thus, Gaussian elimination tells us to subtract a_2/a_1 times the first row from the second. This yields, adding in the system for the kernel:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 - \frac{a_2 b_1}{a_1} & c_2 - \frac{a_2 c_1}{a_1} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now we can solve this system in general. Letting $z = r$ be our free variable, we can find that

$$y = r\frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}.$$

And after a bit of messy algebra, we get that

$$x = r\left(-\frac{c_1}{a_1} + \frac{b_1(c_2 a_1 - a_2 c_1)}{a_1(a_1 b_2 - a_2 b_1)}\right) = r\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}.$$

Thus,

$$U^\perp = \operatorname{span}\left(\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}, 1\right)^T\right).$$

We can simplify this expression by multiplying through by $a_1 b_2 - a_2 b_1$. Thus, we obtain

$$U^\perp = \operatorname{span}\left((b_1 c_2 - b_2 c_1, c_1 a_2 - c_2 a_1, a_1 b_2 - a_2 b_1)^T\right). \tag{1}$$

We have obtained the equality (1) using the assumption that some numbers (namely, a_1 and $a_1b_2 - a_2b_1$) are nonzero. Does it still hold without this assumption? The answer is "yes". One way to prove this is by painstakingly checking each possible case that can appear in the Gaussian elimination algorithm. This is not much harder than what we have done already, but it is laborious. Another (better) way is to argue from general principles:

By assumption, the two vectors v_1 and v_2 are linearly independent. Thus, they form a basis of their span U. Hence, the span U is 2-dimensional (since it has a basis consisting of two vectors). In other words, $\dim U = 2$. But recall that $\dim\left(U^\perp\right) + \dim U = 3$ (since U is a subspace of $\mathbb{R}^3$). Thus, $\dim\left(U^\perp\right) = 3 - \underbrace{\dim U}_{=2} = 3 - 2 = 1$. Hence, any nonzero vector in $U^\perp$ already spans $U^\perp$ (because any nonzero vector in $U^\perp$ spans a 1-dimensional subspace of $U^\perp$, and because of $\dim\left(U^\perp\right) = 1$ this 1-dimensional subspace must already be the whole $U^\perp$). Therefore, in order to prove (1), it suffices to show that the vector

$$(b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1)^T$$

is a nonzero vector in $U^\perp$. Denote this vector by g. We thus must show that g is a nonzero vector in $U^\perp$.

Proving that $g \in U^\perp$ is straightforward: We have

$$\begin{aligned}
\langle g, v_1\rangle &= \left\langle (b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1)^T, (a_1, b_1, c_1)^T \right\rangle \\
&= (b_1c_2 - b_2c_1)\, a_1 + (c_1a_2 - c_2a_1)\, b_1 + (a_1b_2 - a_2b_1)\, c_1 \\
&= b_1c_2a_1 - b_2c_1a_1 + c_1a_2b_1 - c_2a_1b_1 + a_1b_2c_1 - a_2b_1c_1 \\
&= a_1b_1c_2 - a_1b_2c_1 + a_2b_1c_1 - a_1b_1c_2 + a_1b_2c_1 - a_2b_1c_1 = 0
\end{aligned}$$

and thus $g \perp v_1$. Similarly, $g \perp v_2$. Thus, g is orthogonal to both vectors v_1 and v_2, and thus also to each of their linear combinations; in other words, $g \in U^\perp$. It remains to prove that g is nonzero.

Well, assume the contrary. Thus, g is zero, so that each of its three coordinates $b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1$ is zero. In other words, $b_1c_2 = b_2c_1$, $c_1a_2 = c_2a_1$ and $a_1b_2 = a_2b_1$. The vector v_1 is nonzero (since v_1 and v_2 are linearly independent), and thus has at least one nonzero coordinate. We can WLOG assume that $a_1 \neq 0$ (since otherwise, we can cyclically rotate the coordinates of the two vectors v_1 and v_2 until a nonzero coordinate of v_1 hits the first position). Thus, the equalities $c_1a_2 = c_2a_1$ and $a_1b_2 = a_2b_1$ can be solved for c_2 and b_2, respectively, yielding $c_2 = \frac{a_2}{a_1}c_1$ and $b_2 = \frac{a_2}{a_1}b_1$. Combining this with $a_2 = \frac{a_2}{a_1}a_1$, we conclude that each coordinate of v_2 equals $\frac{a_2}{a_1}$ times the corresponding coordinate of v_1. Hence, $v_2 = \frac{a_2}{a_1}v_1$. In other words, $\frac{a_2}{a_1}v_1 - v_2 = \vec{0}$. This contradicts the fact that v_1 and v_2 are linearly indepdendent. Hence, our assumption was wrong, and thus we have shown that g is nonzero. This completes our proof that g is a nonzero vector in $U^\perp$; and as we have said above, this shows that (1) holds (no matter what happens during Gaussian elimination).

[*Remark:* The vector g introduced above is known as the *cross-product* of v_1 and v_2. If you know it from geometry, you should not be surprised that it spans $U^\perp$: In fact, it is orthogonal to both v_1 and v_2, and thus belongs to $U^\perp$.]