Math 4242 Fall 2016 (Darij Grinberg): homework set 5 (corrected)
due: Wed, 9 Nov 2016, in class

Exercise 1. Let U be the subspace $\operatorname{span}\left((1,1,1)^T, (1,2,3)^T\right)$ of $\mathbb{R}^3$. (Recall that $\operatorname{span}(u_1, u_2, \ldots, u_k)$ is our new notation for the span of $u_1, u_2, \ldots, u_k$, formerly known as $\langle u_1, u_2, \ldots, u_k \rangle$.)

(a) Find $U^\perp$. ("Finding" a subspace means writing it as a span throughout this exercise.)

(b) Find $U \cap U^\perp$.

(c) Find $U + U^\perp$.

(d) Find an orthogonal basis of U.

(e) Find an orthogonal basis of $U^\perp$.

(f) Find a subspace W of $\mathbb{R}^3$ such that $U = W^\perp$. [30 points]

Exercise 2. (a) Find the least-squares solution to the equation $Ax = b$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$. (That is, find the vector x for which $||Ax - b||$ is minimum.)

(b) Find the least-squares solution to the equation $Ax = b$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 4 \end{pmatrix}$. [20 points]

(The point of the preceding exercise is to show that "duplicating" a row can change the least-squares solution. This is unlike the case of exact solutions, where a duplicate row adds no information and therefore has no effect on the solution. Visually speaking, the more often a row appears, the closer the least-squares solution comes to satisfying it exactly.)

Exercise 3. Consider the equation $Ax = b$, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. Let's say we want to find the least-squares solution, i.e., the vector x for which $||Ax - b||$ is minimum. However, the algorithm in class breaks down, since $K = A^T A$ is not invertible. And indeed, there is no "**the**" least-squares solution, but there are infinitely many – all having the same (minimum) value $||Ax - b||$. The purpose of this exercise is to find them.

(a) Compute a basis of $\operatorname{Col} A$. (Its size is the rank of A.)

(b) Let A' be the $4 \times r$-matrix whose columns are the elements of this basis (where r is the rank of A). Write down A'.

(c) What is $\operatorname{Col}(A')$?

(d) Find the projection u of b onto $\operatorname{Col} A$.

(e) Now, the least-squares solutions to $Ax = b$ are precisely the vectors $x \in \mathbb{R}^3$ satisfying $Ax = u$. Find them.

[**Note:** This procedure – where we use A' instead of A to compute the projection – can be used as a general method for solving underdetermined least-squares problems like the one in this exercise. However, it is important to keep in mind that the least-square solution to $Ax = b$ is not unique when the problem is underdetermined.] [20 points]

[**Remark** (added in hindsight): The preceding exercise is bad, and I am really not proud of it. I wrote it to give an algorithm for solving underdetermined least-squares problems; but there is much simpler (and less confusing) method to do so: Namely, the least-squares solutions to $Ax = b$ (for any $n \times m$-matrix A and any column vector b of size n) are precisely the (exact!) solutions to $A^T Ax = A^T b$. The latter can be computed using Gaussian elimination.]

Exercise 4. Find QR factorizations of the two matrices A from Exercise 2.
[16 points]

Exercise 5. Fill in the blanks in the following proof. [24 points]

Proposition 0.1. Let $n \in \mathbb{N}$. Let U_1 and U_2 be two subspaces of $\mathbb{R}^n$. Then,

$$(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp.$$

(Recall that $U_1 + U_2$ is defined as the subset $\{u_1 + u_2 \mid u_1 \in U_1 \text{ and } u_2 \in U_2\}$ of $\mathbb{R}^n$.)

Proof of Proposition 0.1. We shall first show that $(U_1 + U_2)^\perp \subseteq U_1^\perp \cap U_2^\perp$, then show that $U_1^\perp \cap U_2^\perp \subseteq (U_1 + U_2)^\perp$.

Proof of $(U_1 + U_2)^\perp \subseteq U_1^\perp \cap U_2^\perp$*:*

Let $v \in (U_1 + U_2)^\perp$. Thus, $\{v\} \perp U_1 + U_2$. In other words,

$$v \perp x \qquad \text{for each } x \in U_1 + U_2. \tag{1}$$

We shall show that $v \in U_1^\perp$ and $v \in U_2^\perp$:

1. Let $y \in U_1$. We have $\overrightarrow{0} \in U_2$ (since U_2 is a subspace of $\mathbb{R}^n$, and thus contains the zero vector). From $y \in U_1$ and $\overrightarrow{0} \in U_2$, we obtain $y + \overrightarrow{0} \in U_1 + U_2$ (by the definition of $U_1 + U_2$). In other words, $y \in U_1 + U_2$ (since $y + \overrightarrow{0} = y$). Hence, (1) (applied to $x = y$) yields $v \perp y$.

   Thus, we have shown that $v \perp y$ for each $y \in U_1$. In other words, $\{v\} \perp U_1$. In other words, $v \in U_1^\perp$.

2. Let $z \in U_2$. We have $\overrightarrow{0} \in U_1$ (since U_1 is a subspace of $\mathbb{R}^n$, and thus contains the zero vector). From ________________, we obtain ________________ (by the definition of $U_1 + U_2$). In other words, ________________ (since ________________). Hence, (1) (applied to ________________) yields ________________.

   Thus, we have shown that $v \perp z$ for each $z \in U_2$. In other words, $\{v\} \perp U_2$. In other words, $v \in U_2^{\perp}$.

Combining $v \in U_1^{\perp}$ with $v \in U_2^{\perp}$, we obtain $v \in U_1^{\perp} \cap U_2^{\perp}$.

Now, we have shown that $v \in U_1^{\perp} \cap U_2^{\perp}$ for each $v \in (U_1 + U_2)^{\perp}$. In other words, $(U_1 + U_2)^{\perp} \subseteq U_1^{\perp} \cap U_2^{\perp}$.

Proof of $U_1^{\perp} \cap U_2^{\perp} \subseteq (U_1 + U_2)^{\perp}$:

Let $v \in U_1^{\perp} \cap U_2^{\perp}$. Thus, $v \in U_1^{\perp} \cap U_2^{\perp} \subseteq U_1^{\perp}$. In other words, $\{v\} \perp U_1$. In other words,

$$v \perp x \qquad \text{for each } x \in U_1. \tag{2}$$

Similarly,

$$v \perp x \qquad \text{for each } x \in U_2. \tag{3}$$

Let now $y \in U_1 + U_2$. Thus,

$$y \in U_1 + U_2 = \{u_1 + u_2 \mid u_1 \in U_1 \text{ and } u_2 \in U_2\}.$$

In other words, $y = u_1 + u_2$ for some $u_1 \in U_1$ and $u_2 \in U_2$. Consider these u_1 and u_2.

We have $v \perp u_1$ (by ________________, applied to ________________). In other words, $\langle v, u_1 \rangle = 0$. Similarly, $\langle v, u_2 \rangle = 0$.

Now, $y = u_1 + u_2$, and thus

$$\begin{aligned}\langle v, y \rangle = \langle v, u_1 + u_2 \rangle &= \underbrace{\langle v, u_1 \rangle}_{=0} + \underbrace{\langle v, u_2 \rangle}_{=0} \\ &\qquad \text{(by the distributive law for inner products)} \\ &= 0 + 0 = 0.\end{aligned}$$

In other words, $v \perp y$.

Thus, we have shown that $v \perp y$ for each $y \in U_1 + U_2$. In other words, $\{v\} \perp U_1 + U_2$. In other words, $v \in (U_1 + U_2)^{\perp}$.

Now, we have shown that $v \in (U_1 + U_2)^{\perp}$ for each $v \in U_1^{\perp} \cap U_2^{\perp}$. In other words, $U_1^{\perp} \cap U_2^{\perp} \subseteq (U_1 + U_2)^{\perp}$.

Combining our two results $(U_1 + U_2)^{\perp} \subseteq U_1^{\perp} \cap U_2^{\perp}$ and $U_1^{\perp} \cap U_2^{\perp} \subseteq (U_1 + U_2)^{\perp}$, we obtain $(U_1 + U_2)^{\perp} = U_1^{\perp} \cap U_2^{\perp}$. Proposition 0.1 is thus proven. □

Exercise 6. Define two subspaces P_1 and P_2 of $\mathbb{R}^3$ as follows:

$$P_1 = \operatorname{span}\left((1,0,1)^T, (1,2,1)^T\right);$$
$$P_2 = \operatorname{span}\left((0,4,5)^T, (-2,1,3)^T\right).$$

(a) Find a vector subspace U_1 of $\mathbb{R}^3$ such that $P_1 = U_1^\perp$.
(b) Find a vector subspace U_2 of $\mathbb{R}^3$ such that $P_2 = U_2^\perp$.
(c) Write the sum $U_1 + U_2$ as a span.
(d) Write the intersection $P_1 \cap P_2$ as a span.
[**Hint:** Use Proposition 0.1 for part **(d)**. This is a general method for writing the intersection of two spans as a span.] [24 points]

Exercise 7. Let $v_1 = (a_1, b_1, c_1)^T$ and $v_2 = (a_2, b_2, c_2)^T$ be two linearly independent vectors in $\mathbb{R}^3$. Let $U = \operatorname{span}(v_1, v_2)$. Write the 1-dimensional subspace $U^\perp$ of $\mathbb{R}^3$ as the span of a single vector. [10 points]