Math 4242 Fall 2016 (Darij Grinberg): homework set 4
due: Wed, 26 Oct 2016, in class
(or **earlier** by moodle)

Exercise 1. (a) A square matrix A is said to be *symmetric* if it satisfies $A^T = A$. For example, symmetric 3×3-matrices have the form $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ with $a, b, c, d, e, f \in \mathbb{R}$.

Find a basis of the vector space of all symmetric 3×3-matrices.

(b) A square matrix A is said to be *skew-symmetric* if it satisfies $A^T = -A$. For example, skew-symmetric 3×3-matrices have the form $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ with $a, b, c \in \mathbb{R}$.

Find a basis of the vector space of all skew-symmetric 3×3-matrices.

(c) Find the dimension of the vector space of all symmetric 6×6-matrices.

(d) Find the dimension of the vector space of all skew-symmetric 6×6-matrices. [16 points]

Solution. Recall that $E_{i,j}$ denotes the $n \times m$-matrix whose (i, j)-th entry is 1 and whose all other entries are 0. (The choice of n and m depends on the context: e.g., when we are working with 3×3-matrices, then $n = 3$ and $m = 3$.)

(a) Let $\mathcal{S}_3$ be the vector space of all symmetric 3×3-matrices. A basis (of course, not the only basis) of $\mathcal{S}_3$ is $(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2} + E_{2,1}, E_{1,3} + E_{3,1}, E_{2,3} + E_{3,2})$. Explicitly, its entries are:

$$E_{1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{2,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{3,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$E_{1,2} + E_{2,1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{1,3} + E_{3,1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$E_{2,3} + E_{3,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

[*Proof:* We want to prove that the list $(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2} + E_{2,1}, E_{1,3} + E_{3,1}, E_{2,3} + E_{3,2})$ is a basis of $\mathcal{S}_3$. In order to do so, we must show that this list spans $\mathcal{S}_3$ and is linearly independent.

Proof of the fact that our list spans $\mathcal{S}_3$*:* All six entries $E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2} + E_{2,1}, E_{1,3} + E_{3,1}, E_{2,3} + E_{3,2}$ of our list are symmetric matrices (this should be clear by a look at these matrices, or from the fact that $\left(E_{i,j}\right)^T = E_{j,i}$ for all i and j). Hence, they belong to $\mathcal{S}_3$. Consequently,

$$\operatorname{span}\left(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2}+E_{2,1}, E_{1,3}+E_{3,1}, E_{2,3}+E_{3,2}\right) \subseteq \mathcal{S}_3. \qquad (1)$$

We shall now prove that

$$\mathcal{S}_3 \subseteq \operatorname{span}\left(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2}+E_{2,1}, E_{1,3}+E_{3,1}, E_{2,3}+E_{3,2}\right). \qquad (2)$$

Indeed, let $A \in \mathcal{S}_3$. Thus, A is a symmetric 3×3-matrix. Hence, $A_{1,1} = A_{1,1}$, $A_{1,2} = A_{2,1}$, $A_{1,3} = A_{3,1}$, $A_{2,1} = A_{1,2}$, $A_{2,2} = A_{2,2}$, $A_{2,3} = A_{3,2}$, $A_{3,1} = A_{1,3}$, $A_{3,2} = A_{2,3}$ and $A_{3,3} = A_{3,3}$. Therefore, A has the form $A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ for some $a, b, c, d, e, f \in \mathbb{R}$. Consider these a, b, c, d, e, f. Now,

$$\begin{aligned}
A &= \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \\
&= a \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=E_{1,1}} + d \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=E_{2,2}} + f \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=E_{3,3}} \\
&\qquad + b \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=E_{1,2}+E_{2,1}} + c \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{=E_{1,3}+E_{3,1}} + e \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{=E_{2,3}+E_{3,2}} \\
&= aE_{1,1} + dE_{2,2} + fE_{3,3} + b\left(E_{1,2}+E_{2,1}\right) + c\left(E_{1,3}+E_{3,1}\right) + e\left(E_{2,3}+E_{3,2}\right) \\
&\qquad \text{(check this equality, if you don't find it obvious)} \\
&\in \operatorname{span}\left(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2}+E_{2,1}, E_{1,3}+E_{3,1}, E_{2,3}+E_{3,2}\right).
\end{aligned}$$

Since we have proven this for every $A \in \mathcal{S}_3$, we thus have shown that

$$\mathcal{S}_3 \subseteq \operatorname{span}\left(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2}+E_{2,1}, E_{1,3}+E_{3,1}, E_{2,3}+E_{3,2}\right).$$

Thus, (2) is proven.

Combining (1) with (2), we conclude that

$$\mathcal{S}_3 = \operatorname{span}\left(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2}+E_{2,1}, E_{1,3}+E_{3,1}, E_{2,3}+E_{3,2}\right).$$

In other words, the list $\left(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2}+E_{2,1}, E_{1,3}+E_{3,1}, E_{2,3}+E_{3,2}\right)$ spans $\mathcal{S}_3$.

Proof of the fact that our list is linearly independent: We must now show that our list $\left(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2}+E_{2,1}, E_{1,3}+E_{3,1}, E_{2,3}+E_{3,2}\right)$ is linearly independent. In other words, we must show that if a, b, c, d, e, f are six reals satisfying

$$aE_{1,1} + bE_{2,2} + cE_{3,3} + d\left(E_{1,2}+E_{2,1}\right) + e\left(E_{1,3}+E_{3,1}\right) + f\left(E_{2,3}+E_{3,2}\right) = \overrightarrow{0}, \qquad (3)$$

then all the six reals a, b, c, d, e, f are 0.

So let a, b, c, d, e, f be six reals satisfying (3). Recall that the zero vector $\overrightarrow{0}$ of the vector space $\mathcal{S}_3$ is the zero matrix $0_{3\times 3}$. Thus,

$$\begin{aligned} 0_{3\times 3} = \overrightarrow{0} &= aE_{1,1} + bE_{2,2} + cE_{3,3} + d\left(E_{1,2} + E_{2,1}\right) + e\left(E_{1,3} + E_{3,1}\right) + f\left(E_{2,3} + E_{3,2}\right) \\ &= \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \end{aligned}$$

(by straightforward computation). Hence,

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = 0_{3\times 3}.$$

Therefore, all of the six reals a, b, c, d, e, f are 0 (being entries of the zero matrix $0_{3\times 3}$). This completes the proof of the fact that our list is linearly independent.

Altogether, we now know that the list $(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2} + E_{2,1}, E_{1,3} + E_{3,1}, E_{2,3} + E_{3,2})$ spans $\mathcal{S}_3$ and is linearly independent. Hence, this list is a basis of $\mathcal{S}_3$.]

(b) Let $\mathcal{A}_3$ be the vector space of all skew-symmetric 3×3-matrices. A basis (of course, not the only basis) of $\mathcal{A}_3$ is $(E_{1,2} - E_{2,1}, E_{1,3} - E_{3,1}, E_{2,3} - E_{3,2})$. Explicitly, its entries are:

$$E_{1,2} - E_{2,1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{1,3} - E_{3,1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$E_{2,3} - E_{3,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

[The *proof* is similar to the proof for part **(a)**, except that we need to make one more observation: Namely, if $A \in \mathcal{A}_3$, then $A_{1,1} = -A_{1,1}$, $A_{1,2} = -A_{2,1}$, $A_{1,3} = -A_{3,1}$, $A_{2,1} = -A_{1,2}$, $A_{2,2} = -A_{2,2}$, $A_{2,3} = -A_{3,2}$, $A_{3,1} = -A_{1,3}$, $A_{3,2} = -A_{2,3}$ and $A_{3,3} = -A_{3,3}$. Thus, $A_{1,1} = 0$ (since $A_{1,1} = -A_{1,1}$) and similarly $A_{2,2} = 0$ and $A_{3,3} = 0$.]

(c) The dimension is $6 + 5 + 4 + 3 + 2 + 1 = 21$.

In fact, this generalizes: For each $n \in \mathbb{N}$, let $\mathcal{S}_n$ denote the vector space of all symmetric $n \times n$-matrices. Then, $\dim\left(\mathcal{S}_n\right) = 1 + 2 + \cdots + n = \dfrac{n\left(n+1\right)}{2}$.

[*Sketch of a proof:* In part **(a)**, we have seen that $(E_{1,1}, E_{2,2}, E_{3,3}, E_{1,2} + E_{2,1}, E_{1,3} + E_{3,1}, E_{2,3} + E_{3,2})$ is a basis of $\mathcal{S}_3$. Similarly, a list that begins with the n diagonal $n \times n$-matrices $E_{1,1}, E_{2,2}, \ldots, E_{n,n}$ and continues with all possible $n \times n$-matrices of the form $E_{i,j} + E_{j,i}$ (with $i < j$) can be shown to be a basis of $\mathcal{S}_n$. Thus, $\dim\left(\mathcal{S}_n\right)$ is the size of this list. Now, what is the size of this list? Its first part consists of the n matrices $E_{1,1}, E_{2,2}, \ldots, E_{n,n}$. Its second part consists of all matrices of the form $E_{i,j} + E_{j,i}$

(with $i < j$). The first part clearly has size n; it thus remains to compute the size of the second part. This size is clearly the number of all pairs (i, j) of elements $i, j \in \{1, 2, \ldots, n\}$ satisfying $i < j$. For each $j \in \{1, 2, \ldots, n\}$, there are exactly $j - 1$ elements $i \in \{1, 2, \ldots, n\}$ satisfying $i < j$. Therefore, the second part has total size $(1-1) + (2-1) + \cdots + (n-1) = 1 + 2 + \cdots + (n-1)$. The size of the whole basis is therefore

$$\underbrace{(\text{size of the first part})}_{=n} + \underbrace{(\text{size of the second part})}_{=1+2+\cdots+(n-1)}$$
$$= n + (1 + 2 + \cdots + (n-1)) = 1 + 2 + \cdots + n.$$

Hence, $\dim(\mathcal{S}_n) = 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ (by the well-known formula for triangular numbers).]

(d) The dimension is $5 + 4 + 3 + 2 + 1 = 15$.

In fact, this generalizes: For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ denote the vector space of all skew-symmetric $n \times n$-matrices. Then, $\dim(\mathcal{A}_n) = 1 + 2 + \cdots + (n-1) = \dfrac{(n-1)n}{2}$.

[*Sketch of a proof:* In part **(b)**, we have seen that $(E_{1,2} - E_{2,1}, E_{1,3} - E_{3,1}, E_{2,3} - E_{3,2})$ is a basis of $\mathcal{A}_3$. Similarly, a list that consists of all possible $n \times n$-matrices of the form $E_{i,j} - E_{j,i}$ (with $i < j$) can be shown to be a basis of $\mathcal{A}_n$. Thus, $\dim(\mathcal{A}_n)$ is the size of this list. This list consists of all matrices of the form $E_{i,j} - E_{j,i}$ (with $i < j$). Hence, its size is the number of all pairs (i, j) of elements $i, j \in \{1, 2, \ldots, n\}$ satisfying $i < j$. For each $j \in \{1, 2, \ldots, n\}$, there are exactly $j - 1$ elements $i \in \{1, 2, \ldots, n\}$ satisfying $i < j$. Therefore, the list has total size $(1-1) + (2-1) + \cdots + (n-1) = 1 + 2 + \cdots + (n-1)$.

Hence, $\dim(\mathcal{A}_n) = 1 + 2 + \cdots + (n-1) = \dfrac{(n-1)n}{2}$ (by the well-known formula for triangular numbers).] $\square$

Exercise 2. **(a)** Find the rank of the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$.

(b) Find the rank of the matrix $B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

(c) Find the rank of the matrix $C = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$.

(d) Find the rank of the matrix $D = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$. [8 points]

Solution. **(a)** The rank of A is 2.

[*Proof:* The two columns of A are linearly independent[1]. Hence, the column space of A has dimension 2. In other words, A has rank 2 (because the rank of a matrix is the dimension of its column space.]

(b) The rank of B is 2.

[*Proof:* The two rows of B are linearly independent[2]. Hence, the row space of B has dimension 2. In other words, B has rank 2 (because the rank of a matrix is the dimension of its row space.]

(c) The rank of C is 2.

[*Proof:* The two columns of C are linearly independent[3]. Hence, the column space of C has dimension 2. In other words, C has rank 2 (because the rank of a matrix is the dimension of its column space.]

(d) The rank of D is 2.

[*Proof:* Let us find a basis of the column space $\operatorname{Col} D$ of D.

Clearly, $\operatorname{Col} D = \operatorname{span}(v_1, v_2, v_3)$, where $v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ (since the column space of a matrix is the span of its columns). Thus, (v_1, v_2, v_3) is a spanning list for $\operatorname{Col} D$.

Now, we are looking for a basis of $\operatorname{Col} D$. As we know, we can find such a basis by removing redundant vectors from our spanning list (v_1, v_2, v_3). To find redundant vectors, we search for a linear dependency between v_1, v_2, v_3. In other words, we search for three reals $\lambda_1, \lambda_2, \lambda_3$, not all zero, satisfying $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \overrightarrow{0}$.

This is an easy exercise in Gaussian elimination: The equation $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \overrightarrow{0}$ is equivalent to the system of linear equations $\begin{cases} 0\lambda_1 + 1\lambda_2 + 2\lambda_3 = 0; \\ 1\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0; \\ 2\lambda_1 + 3\lambda_2 + 4\lambda_3 = 0 \end{cases}$, and solving this system yields $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} r \\ -2r \\ r \end{pmatrix}$ with $r \in \mathbb{R}$. By taking $r = 1$, we find the nonzero solution $\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. This gives us the linear dependency relation $1v_1 + (-2) v_2 + 1v_3 = \overrightarrow{0}$. Solving this for v_3 (the last vector to appear in this relation with a nonzero coefficient), we find $v_3 = 2v_2 - v_1$. Hence, the vector v_3 in the spanning list (v_1, v_2, v_3) is redundant.

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[1]This can be checked rather easily: If λ_1, λ_2 are two reals such that $\lambda_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \overrightarrow{0}$, then $\overrightarrow{0} = \lambda_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 + 2\lambda_2 \end{pmatrix}$, so that both λ_2 and $\lambda_1 + 2\lambda_2$ must be 0, and therefore both λ_1 and λ_2 must be 0.

[2]This is straightforward to check once again.

[3]Again, checking this is left to the reader.

Thus, removing it yields a smaller spanning list (v_1, v_2). Is this smaller list already a basis or does it have further redundant vectors that can be removed? Let us see. We search for a linear dependency between v_1, v_2. In other words, we search for two reals λ_1, λ_2, not all zero, satisfying $\lambda_1 v_1 + \lambda_2 v_2 = \overrightarrow{0}$.

Again, this is an easy exercise in Gaussian elimination: The equation $\lambda_1 v_1 + \lambda_2 v_2 = \overrightarrow{0}$ is equivalent to the system of linear equations $\begin{cases} 0\lambda_1 + 1\lambda_2 = 0; \\ 1\lambda_1 + 2\lambda_2 = 0; \\ 2\lambda_1 + 3\lambda_2 = 0 \end{cases}$, and solving this system yields $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus, the only solution has $(\lambda_1, \lambda_2) = (0, 0)$. In other words, there exist no two reals λ_1, λ_2, not all zero, satisfying $\lambda_1 v_1 + \lambda_2 v_2 = \overrightarrow{0}$. In other words, the spanning list (v_1, v_2) is linearly independent, and thus is a basis of $\operatorname{Col} D$. Therefore, $\dim(\operatorname{Col} D) = 2$ (because this basis has size 2). But the rank of D is $\operatorname{rank} D = \dim(\operatorname{Col} D) = 2$.] □

Exercise 3. Find the four subspaces (kernel, column space, row space, left kernel) of the four matrices from Exercise 2.

[Keep in mind that the column space and the kernel consist of column vectors, whereas the row space and the left kernel consist of row vectors.] [16 points]

Solution. [I have been ambiguous about what "find" means: Does finding a subspace mean writing it as a span, or writing it as a non-redundant span (i.e., finding a basis), or just characterizing it somehow? I shall do all of these further below.

I will only give proofs for the four subspaces of B. The other matrices aren't much different.]

(a) We have

$$\begin{aligned} \operatorname{Ker} A &= \operatorname{span}() = \left\{ \overrightarrow{0} \right\}; \\ \operatorname{Col} A &= \operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = \mathbb{R}^2; \\ \operatorname{Row} A &= \operatorname{span}\left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \end{pmatrix} \right) = \mathbb{R}^{1\times 2}; \\ \left(\operatorname{Ker}\left(A^T \right) \right)^T &= \operatorname{span}() = \left\{ \overrightarrow{0} \right\}. \end{aligned}$$

(b) We have

$$\begin{aligned} \operatorname{Ker} B &= \operatorname{span}\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right); \\ \operatorname{Col} B &= \operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right) = \mathbb{R}^2; \\ \operatorname{Row} B &= \operatorname{span}\left(\begin{pmatrix} 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \right); \\ \left(\operatorname{Ker}\left(B^T \right) \right)^T &= \operatorname{span}() = \left\{ \overrightarrow{0} \right\}. \end{aligned}$$

[*Proof sketches:* To find $\operatorname{Ker} B$, proceed using the algorithm that is illustrated in Examples 4.34, 4.35 and 4.36 in the present version of the lecture notes. This involves solving the equation $Bx = 0_{2\times 1}$, which rewrites as $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if we set $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; this further rewrites as $\begin{cases} 0x_1 + 1x_2 + 2x_3 = 0; \\ 1x_1 + 2x_2 + 3x_3 = 0 \end{cases}$, and this is a system of linear equations whose solutions can be described as the vectors of the form $\begin{pmatrix} r \\ -2r \\ r \end{pmatrix}$ for $r \in \mathbb{R}$. Hence,

$$\begin{aligned} \operatorname{Ker} B &= \left\{ \begin{pmatrix} r \\ -2r \\ r \end{pmatrix} \mid r \in \mathbb{R} \right\} = \left\{ r \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \mid r \in \mathbb{R} \right\} \\ &= \operatorname{span}\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Thus, $\operatorname{Ker} B$ has been computed.

Similarly, $\operatorname{Ker}\left(B^T\right)$ can be computed, and thus $\left(\operatorname{Ker}\left(B^T\right)\right)^T$ as well (since getting from $\operatorname{Ker}\left(B^T\right)$ to $\left(\operatorname{Ker}\left(B^T\right)\right)^T$ just requires transposing every vector).

The row space $\operatorname{Row} B$ is the span of the rows of B (by definition), which are $\begin{pmatrix} 0 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$. Thus,

$$\operatorname{Row} B = \operatorname{span}\left(\begin{pmatrix} 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \right).$$

Since the vectors $\begin{pmatrix} 0 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ are linearly independent (this is easy to check), this list $\left(\begin{pmatrix} 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \right)$ is actually a basis of $\operatorname{Row} B$.

The column space $\operatorname{Col} B$ is the span of the columns of B (by definition), which are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Hence,

$$\operatorname{Col} B = \operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right).$$

However, the spanning list $\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right)$ has a redundant vector (namely, $\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$), and thus can be shrunk to a spanning list $\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$. This latter spanning list has no redundant vectors any more, and thus is a basis of $\operatorname{Col} B$.

Why is $\operatorname{Col} B = \mathbb{R}^2$? Well, we know that $\dim(\operatorname{Col} B) = \operatorname{rank} B = 2$ (from Exercise 2), and thus $\operatorname{Col} B$ is a 2-dimensional subspace of $\mathbb{R}^2$. But $\mathbb{R}^2$ is a 2-dimensional vector space itself, and thus the only 2-dimensional subspace of $\mathbb{R}^2$ is the whole $\mathbb{R}^2$ (this follows from Proposition 0.1 **(c)** below). Hence, $\operatorname{Col} B = \mathbb{R}^2$.]

(c) We have

$$\begin{aligned}
\operatorname{Ker} C &= \operatorname{span}() = \left\{ \overrightarrow{0} \right\}; \\
\operatorname{Col} C &= \operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right); \\
\operatorname{Row} C &= \operatorname{span}\left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \end{pmatrix} \right) = \mathbb{R}^2; \\
\left(\operatorname{Ker}\left(C^T\right)\right)^T &= \operatorname{span}\left(\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \right).
\end{aligned}$$

[*Remark:* We have $C = B^T$. Thus, the four subspaces of C can easily be obtained from the four subspaces of B. Namely,

$$\operatorname{Ker} C = \left(\left(\operatorname{Ker}\left(B^T\right)\right)^T \right)^T; \qquad \operatorname{Col} C = (\operatorname{Row} B)^T;$$
$$\operatorname{Row} C = (\operatorname{Col} B)^T; \qquad \left(\operatorname{Ker}\left(C^T\right)\right)^T = (\operatorname{Ker} B)^T.$$

]

(d) We have

$$\begin{aligned}
\operatorname{Ker} D &= \operatorname{span}\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right); \\
\operatorname{Col} D &= \operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right) \\
&= \operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right); \\
\operatorname{Row} D &= \operatorname{span}\left(\begin{pmatrix} 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \right) \\
&= \operatorname{span}\left(\begin{pmatrix} 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \right); \\
\left(\operatorname{Ker}\left(D^T\right)\right)^T &= \operatorname{span}\left(\begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \right).
\end{aligned}$$

□

Now, for the theory part. Here is a fact which I half-proved in class:

Proposition 0.1. Let U be a subspace of a finite-dimensional vector space V.
(a) The vector space U is finite-dimensional.
(b) We have $\dim U \leq \dim V$.
(c) If $\dim U = \dim V$, then $U = V$.

Proof of Proposition 0.1. Let $d = \dim V$.

Recall that (by one of our many propositions)

$$\text{every linearly independent list of vectors in } V \text{ can be extended to a basis of } V. \tag{4}$$

[4] In particular,

$$\text{every linearly independent list of vectors in } V \text{ has size } \leq d \tag{5}$$

(because (4) shows that this list can be extended to a basis of V; but the latter basis must have size $\dim V = d$, and therefore the former list must have size $\leq d$). In particular, every linearly independent list of vectors in U has size $\leq d$ (because vectors in U are also vectors in V).

Now, fix a linearly independent list $(u_1, u_2, \ldots, u_k)$ of vectors in U **of longest possible size**. (There is indeed a "longest possible size", because every linearly independent list of vectors in U has size $\leq d$.)

We have $\operatorname{span}(u_1, u_2, \ldots, u_k) \subseteq U$ (since $u_1, u_2, \ldots, u_k$ are vectors in U). On the other hand, it is easy to see that $U \subseteq \operatorname{span}(u_1, u_2, \ldots, u_k)$ [5]. Combined with $\operatorname{span}(u_1, u_2, \ldots, u_k) \subseteq U$, this yields $U = \operatorname{span}(u_1, u_2, \ldots, u_k)$. Thus, the list $(u_1, u_2, \ldots, u_k)$ is a basis of U (since we already know that this list is linearly independent).

From $U = \operatorname{span}(u_1, u_2, \ldots, u_k)$, we see that there exists a finite list that spans U (namely, the list $(u_1, u_2, \ldots, u_k)$). Thus, U is finite-dimensional. This proves Proposition 0.1 **(a)**.

[4] As usual: To "extend" means to attach further vectors to it.

[5] *Proof.* Let $u \in U$.

The list $(u_1, u_2, \ldots, u_k, u)$ is a list of vectors in U that is longer than the list $(u_1, u_2, \ldots, u_k)$, and thus must be linearly dependent (because $(u_1, u_2, \ldots, u_k)$ was a linearly independent list of vectors in U **of longest possible size**). In other words, there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda \in \mathbb{R}$, **not all zero**, such that

$$\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k + \lambda u = 0. \tag{6}$$

Consider these scalars.

If we had $\lambda = 0$, then (6) would simplify to $\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k = 0$; this would yield that all of $\lambda_1, \lambda_2, \ldots, \lambda_k$ are zero (since $(u_1, u_2, \ldots, u_k)$ is linearly independent), and therefore all of our scalars $\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda$ would be zero (since $\lambda = 0$ too); but this would contradict the assumption that $\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda$ are not all zero. Hence, we cannot have $\lambda = 0$. Thus, $\lambda \neq 0$. Hence, we can solve (6) for u, obtaining

$$u = \frac{-1}{\lambda} (\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k) \in \operatorname{span}(u_1, u_2, \ldots, u_k).$$

Thus, we have proven that every $u \in U$ satisfies $u \in \operatorname{span}(u_1, u_2, \ldots, u_k)$. In other words, $U \subseteq \operatorname{span}(u_1, u_2, \ldots, u_k)$.

The basis $(u_1, u_2, \ldots, u_k)$ of U has size k. Hence, $\dim U = k$. But $(u_1, u_2, \ldots, u_k)$ is a linearly independent list of vectors in V. Thus, this list has size $\leq d$ (by (5)). In other words, $k \leq d$ (since the size of this list is k). Since $k = \dim U$, we now have $\dim U = k \leq d = \dim V$. This proves Proposition 0.1 **(b)**.

(c) Assume that $\dim U = \dim V$. Recall that $(u_1, u_2, \ldots, u_k)$ is a linearly independent list of vectors in V. Thus, (4) shows that this list can be extended to a basis of V. In other words, there exists a basis of V having the form $(u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m)$ for some additional vectors $v_1, v_2, \ldots, v_m$.

The list $(u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m)$ has size $k + m$. But on the other hand, the same list has size $\dim V$ (since it is a basis of V). Thus, $k + m = \dim V = \dim U$ (since $\dim U = \dim V$). Thus, $k + m = \dim U = k$, and therefore $m = 0$. Hence, there are no additional vectors $v_1, v_2, \ldots, v_m$. In other words, the list $(u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_m)$ is just our old list $(u_1, u_2, \ldots, u_k)$. Thus, the latter list is a basis of V (since the former list is a basis of V). Therefore,

$$V = \operatorname{span}(u_1, u_2, \ldots, u_k) = U.$$

This proves Proposition 0.1 **(c)**. □

Now, let me recall some more properties of matrices:

- If D is a $k \times \ell$-matrix, then

$$\begin{aligned} \operatorname{Col} D = D\mathbb{R}^\ell &= \left\{ Dx \mid x \in \mathbb{R}^\ell \right\} \\ &= \operatorname{span}(\operatorname{col}_1 D, \operatorname{col}_2 D, \ldots, \operatorname{col}_\ell D) \end{aligned} \tag{7}$$

  is the column space of D.

- If D is a $k \times \ell$-matrix, then

$$\begin{aligned} \operatorname{Row} D = \mathbb{R}^{1\times k} D &= \left\{ yD \mid y \in \mathbb{R}^{1\times k} \right\} \\ &= \operatorname{span}(\operatorname{row}_1 D, \operatorname{row}_2 D, \ldots, \operatorname{row}_k D) \end{aligned} \tag{8}$$

  is the row space of D. (It can also be written as $\left(D^T \mathbb{R}^k\right)^T$.)

These two facts are completely analogous (since $\mathbb{R}^\ell$ is just shorthand for $\mathbb{R}^{\ell\times 1}$).

Exercise 4. Prove Proposition 0.2 **(a)** below. [**Hint:** Use row spaces instead of column spaces. Use (8) instead of (7). Do not hesitate to copy my proof of Proposition 0.2 **(a)**, as long as you make the necessary changes.] [10 points]

Proposition 0.2. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $p \in \mathbb{N}$. Let A be an $n \times m$-matrix. Let B be an $m \times p$-matrix.

(a) We have $\operatorname{rank}(AB) \leq \operatorname{rank} B$.

(b) We have $\operatorname{rank}(AB) \leq \operatorname{rank} A$.

Proof of Proposition 0.2. **(b)** Recall that the rank of a matrix is the dimension of its column space. Thus,

$$\operatorname{rank}(AB) = \dim(\operatorname{Col}(AB)) \qquad \text{and} \tag{9}$$
$$\operatorname{rank} A = \dim(\operatorname{Col} A). \tag{10}$$

But (7) (applied to $D = AB$, $k = n$ and $\ell = p$) shows that $\operatorname{Col}(AB) = \{ABx \mid x \in \mathbb{R}^p\}$. On the other hand, (7) (applied to $D = A$, $k = n$ and $\ell = m$) shows that $\operatorname{Col} A = \{Ax \mid x \in \mathbb{R}^m\}$.

Now, let us prove that $\operatorname{Col}(AB) \subseteq \operatorname{Col} A$. Indeed, let $v \in \operatorname{Col}(AB)$ be arbitrary. Thus, $v \in \operatorname{Col}(AB) = \{ABx \mid x \in \mathbb{R}^p\}$. In other words, v has the form $v = ABx$ for some $x \in \mathbb{R}^p$. Denote this x by y (because we will have another use for the letter x later). Thus, $y \in \mathbb{R}^p$ and $v = ABy$. Thus, v has the form $v = Ax$ for some $x \in \mathbb{R}^m$ (namely, for $x = By$). In other words, $v \in \{Ax \mid x \in \mathbb{R}^m\}$. In other words, $v \in \operatorname{Col} A$ (since $\operatorname{Col} A = \{Ax \mid x \in \mathbb{R}^m\}$).

Thus, we have proven that every $v \in \operatorname{Col}(AB)$ lies in $\operatorname{Col} A$. In other words, $\operatorname{Col}(AB) \subseteq \operatorname{Col} A$. Furthermore, $\operatorname{Col}(AB)$ is a subspace of $\mathbb{R}^n$, and thus contains the zero vector, is closed under addition, and is closed under scaling. Hence, $\operatorname{Col}(AB)$ is also a subspace of $\operatorname{Col} A$ (since $\operatorname{Col}(AB) \subseteq \operatorname{Col} A$). Thus, Proposition 0.1 **(b)** (applied to $U = \operatorname{Col}(AB)$ and $V = \operatorname{Col} A$) yields $\dim(\operatorname{Col}(AB)) \leq \dim(\operatorname{Col} A)$. In view of (9) and (10), this rewrites as $\operatorname{rank}(AB) \leq \operatorname{rank} A$. This proves Proposition 0.2 **(b)**.

(a) Recall that the rank of a matrix is the dimension of its row space. Thus,

$$\operatorname{rank}(AB) = \dim(\operatorname{Row}(AB)) \qquad \text{and} \tag{11}$$
$$\operatorname{rank} B = \dim(\operatorname{Row} B). \tag{12}$$

But (8) (applied to $D = AB$, $k = n$ and $\ell = p$) shows that $\operatorname{Row}(AB) = \{yAB \mid y \in \mathbb{R}^{1\times n}\}$. On the other hand, (8) (applied to $D = B$, $k = m$ and $\ell = p$) shows that $\operatorname{Row} B = \{yB \mid y \in \mathbb{R}^{1\times m}\}$.

Now, let us prove that $\operatorname{Row}(AB) \subseteq \operatorname{Row} B$. Indeed, let $v \in \operatorname{Row}(AB)$ be arbitrary. Thus, $v \in \operatorname{Row}(AB) = \{yAB \mid y \in \mathbb{R}^{1\times n}\}$. In other words, v has the form $v = yAB$ for some $y \in \mathbb{R}^{1\times n}$. Denote this y by z (because we will have another use for the letter y later). Thus, $z \in \mathbb{R}^{1\times n}$ and $v = zAB$. Thus, v has the form $v = yB$ for some $y \in \mathbb{R}^{1\times m}$ (namely, for $y = zA$). In other words, $v \in \{yB \mid y \in \mathbb{R}^{1\times m}\}$. In other words, $v \in \operatorname{Row} B$ (since $\operatorname{Row} B = \{yB \mid y \in \mathbb{R}^{1\times m}\}$).

Thus, we have proven that every $v \in \operatorname{Row}(AB)$ lies in $\operatorname{Row} B$. In other words, $\operatorname{Row}(AB) \subseteq \operatorname{Row} B$. Furthermore, $\operatorname{Row}(AB)$ is a subspace of $\mathbb{R}^{1\times p}$, and thus contains the zero vector, is closed under addition, and is closed under scaling. Hence, $\operatorname{Row}(AB)$ is also a subspace of $\operatorname{Row} B$ (since $\operatorname{Row}(AB) \subseteq \operatorname{Row} B$). Thus, Proposition 0.1 **(b)** (applied to $U = \operatorname{Row}(AB)$ and $V = \operatorname{Row} B$) yields $\dim(\operatorname{Row}(AB)) \leq \dim(\operatorname{Row} B)$. In view of (11) and (12), this rewrites as $\operatorname{rank}(AB) \leq \operatorname{rank} B$. This proves Proposition 0.2 **(a)**. □

Recall a few more facts:

- The *rank-nullity theorem* states that any $n \times m$-matrix A satisfies

$$\operatorname{rank} A + \dim(\operatorname{Ker} A) = m. \tag{13}$$

  (I did this in class, though I used $\dim(A\mathbb{R}^m)$ instead of $\operatorname{rank} A$ because I had not defined $\operatorname{rank} A$ yet.)

- We have

$$\operatorname{rank}(I_n) = n \tag{14}$$

  for every $n \in \mathbb{N}$. (To prove this, argue that $I_n\mathbb{R}^n = \left\{ \underbrace{I_n x}_{=x} \mid x \in \mathbb{R}^n \right\} = \{x \mid x \in \mathbb{R}^n\} = \mathbb{R}^n$ and thus $\operatorname{rank}(I_n) = \dim \underbrace{(I_n\mathbb{R}^n)}_{=\mathbb{R}^n} = \dim(\mathbb{R}^n) = n$.)

- If A is an $n \times m$-matrix, then

$$\operatorname{rank} A \leq \min\{n, m\}. \tag{15}$$

  (Recall the reason why this is true: We have $\operatorname{rank} A = \dim(\operatorname{Col} A) \leq m$ because the column space of A is spanned by m vectors, and we have $\operatorname{rank} A = \dim(\operatorname{Row} A) \leq n$ because the row space of D is spanned by n vectors.)

Now, we can prove some statements that were left unproven in class long ago, back before we introduced vector spaces:

Definition 0.3. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let A be an $n \times m$-matrix.
(a) A *right inverse* of A means an $m \times n$-matrix B satisfying $AB = I_n$. If a right inverse of A exists, then A is said to be *right-invertible*.
(b) A *left inverse* of A means an $m \times n$-matrix B satisfying $BA = I_m$. If a left inverse of A exists, then A is said to be *left-invertible*.
(c) An *inverse* of A means an $m \times n$-matrix B satisfying both $AB = I_n$ and $BA = I_m$. If an inverse of A exists, then A is said to be *invertible*.

Exercise 5. Fill in the big blank in the following proof.
[**Hint:** What is the transpose of $A^T B$?] [10 points]

Proposition 0.4. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let A be an $n \times m$-matrix.
(a) The matrix A is left-invertible if and only if the matrix A^T is right-invertible.
(b) We have $\operatorname{rank} A = \operatorname{rank}\left(A^T\right)$.

Proof of Proposition 0.4. **(a)** $\Longrightarrow$: Assume that A is left-invertible. We must show that A^T is right-invertible.

The matrix A is left-invertible. In other words, it has a left inverse. That is, there exists an $m \times n$-matrix B satisfying $BA = I_m$. Consider this B.

Now, $(BA)^T = A^T B^T$ (by Proposition 3.18 **(e)** in the current version of the notes, applied to m, n, m, B and A instead of n, m, p, A and B), so that $A^T B^T = \left(\underbrace{BA}_{=I_m}\right)^T = (I_m)^T = I_m$. Hence, B^T is a right inverse of A^T. Thus, A^T is right-invertible. This proves the $\Longrightarrow$ direction of Proposition 0.4 **(a)**.

$\Longleftarrow$: Assume that A^T is right-invertible. We must show that A is left-invertible.

The matrix A^T is right-invertible. In other words, it has a right inverse. That is, there exists an $n \times m$-matrix B satisfying $A^T B = I_n$. Consider this B.

Set $C = B^T$.

Now, $\left(A^T B\right)^T = B^T \left(A^T\right)^T$ (by Proposition 3.18 **(e)** in the current version of the notes, applied to m, n, m, A^T and B instead of n, m, p, A and B). Comparing this with $\left(\underbrace{A^T B}_{=I_n}\right)^T = (I_n)^T = I_n$, we find $I_n = \underbrace{B^T}_{=C} \underbrace{\left(A^T\right)^T}_{=A} = CA$. Hence, $CA = I_n$.

Hence, C is a left inverse of A. Thus, A is left-invertible. This proves the $\Longleftarrow$ direction of Proposition 0.4 **(a)**.

(b) Recall a notation I introduced in class: If V is a set of column vectors, then V^T denotes the set of their transposes (rigorously speaking, this means that $V^T = \left\{v^T \mid v \in V\right\}$). I shall call V^T the *elementwise transpose* of V. The sets V and V^T are "essentially the same" except that the former consists of column vectors and the latter of row vectors. In particular, if V is a subspace of $\mathbb{R}^n$, then V^T is a subspace of $\mathbb{R}^{1\times n}$, and their dimensions are the same:

$$\dim\left(V^T\right) = \dim V. \tag{16}$$

Now, the rows of A^T are the transposes of the columns of A. Hence, the span of the rows of A^T is the elementwise transpose of the span of the columns of A. This rewrites as follows:

$$\operatorname{Row}\left(A^T\right) = (\operatorname{Col} A)^T$$

(because the span of the rows of A^T is the row space $\operatorname{Row}\left(A^T\right)$, and the span of the columns of A is the column space $\operatorname{Col} A$). Thus,

$$\dim\left(\operatorname{Row}\left(A^T\right)\right) = \dim\left((\operatorname{Col} A)^T\right) = \dim\left(\operatorname{Col} A\right)$$

(by (16), applied to $V = \operatorname{Col} A$). But $\operatorname{rank}\left(A^T\right) = \dim\left(\operatorname{Row}\left(A^T\right)\right)$ (since the rank of a matrix is the dimension of its row space) and $\operatorname{rank} A = \dim\left(\operatorname{Col} A\right)$ (since the rank of a matrix is the dimension of its column space). Hence,

$$\operatorname{rank}\left(A^T\right) = \dim\left(\operatorname{Row}\left(A^T\right)\right) = \dim\left(\operatorname{Col} A\right) = \operatorname{rank} A.$$

Thus, Proposition 0.4 **(b)** is proven. □

Exercise 6. Fill in the blanks in the following proof. [27 points]

Proposition 0.5. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let A be an $n \times m$-matrix.

(a) The matrix A is right-invertible if and only if $\operatorname{rank} A = n$.

(b) The matrix A is left-invertible if and only if $\operatorname{rank} A = m$.

(c) The matrix A is invertible if and only if $\operatorname{rank} A = n = m$. (In particular, only square matrices can be invertible!)

Proof of Proposition 0.5. **(a)** $\Longrightarrow$: Let us first show that if A is right-invertible, then $\operatorname{rank} A = n$.

Indeed, assume that A is right-invertible. In other words, there exists some $m \times n$-matrix B satisfying $AB = I_n$. Consider this B.

Proposition 0.2 **(b)** yields $\operatorname{rank}(AB) \leq \operatorname{rank} A$, so that $\operatorname{rank} A \geq \operatorname{rank}\left(\underbrace{AB}_{=I_n}\right) = \operatorname{rank}(I_n) = n$ (by (14)). But (15) yields $\operatorname{rank} A \leq \min\{n, m\} \leq n$. Combining this with $\operatorname{rank} A \geq n$, we find $\operatorname{rank} A = n$. This completes the proof of the $\Longrightarrow$ direction.

$\Longleftarrow$: Let us now show that if $\operatorname{rank} A = n$, then A is right-invertible.

Indeed, assume that $\operatorname{rank} A = n$. We shall explicitly construct a right inverse B to A.

Recall that the rank of a matrix is the dimension of its column space. Thus, $\operatorname{rank} A = \dim(A\mathbb{R}^m)$ (since the column space of A is $A\mathbb{R}^m$). Hence, $\dim(A\mathbb{R}^m) = \operatorname{rank} A = n = \dim(\mathbb{R}^n)$.

But $A\mathbb{R}^m$ is a subspace of $\mathbb{R}^n$. Hence, from $\dim(A\mathbb{R}^m) = \dim(\mathbb{R}^n)$, we obtain $A\mathbb{R}^m = \mathbb{R}^n$ (by Proposition 0.1 **(c)**, applied to $U = A\mathbb{R}^m$ and $V = \mathbb{R}^n$).

For each $j \in \{1, 2, \ldots, n\}$, we have

$$\operatorname{col}_j(I_n) \in \mathbb{R}^n = A\mathbb{R}^m = \{Ax \mid x \in \mathbb{R}^m\}.$$

[6] In other words, for each $j \in \{1, 2, \ldots, n\}$, we can write $\operatorname{col}_j(I_n)$ in the form Ax for some $x \in \mathbb{R}^m$. Pick such an x, and denote it by x_j. Thus, x_j (for each $j \in \{1, 2, \ldots, n\}$) is a column vector in $\mathbb{R}^m$ satisfying $\operatorname{col}_j(I_n) = Ax_j$.

Now we have chosen n vectors $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^m$ such that

$$\operatorname{col}_j(I_n) = Ax_j \qquad \text{for every } j \in \{1, 2, \ldots, n\}. \tag{17}$$

Let B be the $m \times n$-matrix whose columns are $x_1, x_2, \ldots, x_n$. Thus,

$$\operatorname{col}_j B = x_j \qquad \text{for every } j \in \{1, 2, \ldots, n\}. \tag{18}$$

Now, every $j \in \{1, 2, \ldots, n\}$ satisfies

$$\begin{aligned} \operatorname{col}_j(AB) &= A \cdot \operatorname{col}_j B \qquad \text{(by what is currently Proposition 2.19 **(d)** in the notes)} \\ &= Ax_j \qquad \text{(by (18))} \\ &= \operatorname{col}_j(I_n) \qquad \text{(by (17))}. \end{aligned}$$

[6] By the way, $\operatorname{col}_j(I_n)$ is the standard basis vector $e_j = (0, 0, \ldots, 0, 1, 0, 0, \ldots 0)^T$; but this is unimportant for us here.

In other words, each column of the $n \times n$-matrix AB equals the corresponding column of I_n. Hence, these two matrices are equal. In other words, $AB = I_n$. Thus, B is a right inverse of A. Hence, A is right-invertible. This proves the $\Longleftarrow$ direction.

Altogether, we have thus proven Proposition 0.5 **(a)**.

(b) Applying Proposition 0.5 **(a)** to m, n and A^T instead of n, m and A, we conclude the following: The matrix A^T is right-invertible if and only if $\operatorname{rank}\left(A^T\right) = m$. Combined with Proposition 0.4 **(a)**, this shows that the matrix A is left-invertible if and only if $\operatorname{rank}\left(A^T\right) = m$. Because of Proposition 0.4 **(b)**, we can replace $\operatorname{rank}\left(A^T\right)$ by $\operatorname{rank} A$ here, and we obtain precisely the claim of Proposition 0.5 **(b)**.

(c) $\Longrightarrow$: Let us first show that if A is invertible, then $\operatorname{rank} A = n = m$.

Indeed, assume that A is invertible. In other words, there exists some $m \times n$-matrix B satisfying both $AB = I_n$ and $BA = I_m$. Consider this B.

We have $AB = I_n$; hence, B is a right inverse of A. Thus, A is right-invertible. Therefore, Proposition 0.5 **(a)** shows that $\operatorname{rank} A = n$.

We have $BA = I_m$; hence, B is a left inverse of A. Thus, A is left-invertible. Therefore, Proposition 0.5 **(b)** shows that $\operatorname{rank} A = m$.

Combining $\operatorname{rank} A = n$ with $\operatorname{rank} A = m$, we obtain $\operatorname{rank} A = n = m$. This proves the $\Longrightarrow$ direction of Proposition 0.5 **(c)**.

$\Longleftarrow$: Let us now show that if $\operatorname{rank} A = n = m$, then A is invertible.

Indeed, assume that $\operatorname{rank} A = n = m$.

From $\operatorname{rank} A = n$, we conclude (using Proposition 0.5 **(a)**) that A is right-invertible. In other words, A has a right inverse R. Consider this R.

From $\operatorname{rank} A = m$, we conclude (using Proposition 0.5 **(b)** that A is left-invertible. In other words, A has a left inverse L. Consider this L.

Proposition 3.6 **(d)** from the notes now shows that the matrix $L = R$ is the only inverse of A. In particular, it is an inverse of A; thus, A is invertible. This proves the $\Longleftarrow$ direction of Proposition 0.5 **(c)**. □

Exercise 7. Which of the matrices in Exercise 2 are invertible? [10 points]

Solution. Only the matrix A.

[*Proof:* **(a)** We have $\operatorname{rank} A = 2$ (as we know from solving Exercise 2). The matrix A is a 2×2-matrix. Hence, Proposition 0.5 **(c)** (applied to 2 and 2 instead of n and m) shows that the matrix A is invertible if and only if $\operatorname{rank} A = 2 = 2$. Since we have $\operatorname{rank} A = 2 = 2$, this shows that A is invertible.

(b) The matrix B is a 2×3-matrix. Hence, Proposition 0.5 **(c)** (applied to B, 2 and 3 instead of A, n and m) shows that the matrix B is invertible if and only if $\operatorname{rank} B = 2 = 3$. Since $\operatorname{rank} B = 2 = 3$ is clearly false, this shows that B is not invertible.

(c) The matrix C is a 3×2-matrix. Hence, Proposition 0.5 **(c)** (applied to C, 3 and 2 instead of A, n and m) shows that the matrix C is invertible if and only if $\operatorname{rank} C = 3 = 2$. Since $\operatorname{rank} C = 3 = 2$ is clearly false, this shows that C is not invertible.

(d) We have $\operatorname{rank} D = 2$ (as we know from solving Exercise 2). The matrix D is a 3×3-matrix. Hence, Proposition 0.5 **(c)** (applied to D, 3 and 3 instead of A,

n and m) shows that the matrix D is invertible if and only if $\operatorname{rank} D = 3 = 3$. Since $\operatorname{rank} D = 3 = 3$ is false (because $\operatorname{rank} D = 2 \neq 3$), this shows that D is not invertible.] □

Exercise 8. Let $n \in \mathbb{N}$. Let A be a left-invertible $n \times n$-matrix. Prove that A is invertible. [10 points]

[**Hint:** All ingredients of the proof are on this problem set; you have to combine them.]

Solution. Recall that A is an $n \times n$-matrix. Hence, Proposition 0.5 **(b)** (applied to $m = n$) shows that the matrix A is left-invertible if and only if $\operatorname{rank} A = n$. Thus, $\operatorname{rank} A = n$ (since A is left-invertible).

But Proposition 0.5 **(c)** (applied to $m = n$) shows that the matrix A is invertible if and only if $\operatorname{rank} A = n = n$. Thus, A is invertible (since $\operatorname{rank} A = n = n$). □