

Math 4242 Fall 2016 (Darij Grinberg): homework set 4
due: Wed, 26 Oct 2016, in class
 (or **earlier** by moodle)

Exercise 1. (a) A square matrix A is said to be *symmetric* if it satisfies $A^T = A$. For example, symmetric 3×3 -matrices have the form $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$ with $a, b, c, d, e, f \in \mathbb{R}$.

Find a basis of the vector space of all symmetric 3×3 -matrices.

(b) A square matrix A is said to be *skew-symmetric* if it satisfies $A^T = -A$. For example, skew-symmetric 3×3 -matrices have the form $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ with $a, b, c \in \mathbb{R}$.

Find a basis of the vector space of all skew-symmetric 3×3 -matrices.

(c) Find the dimension of the vector space of all symmetric 6×6 -matrices.

(d) Find the dimension of the vector space of all skew-symmetric 6×6 -matrices. [16 points]

Exercise 2. (a) Find the rank of the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$.

(b) Find the rank of the matrix $B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

(c) Find the rank of the matrix $C = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$.

(d) Find the rank of the matrix $D = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$. [8 points]

Exercise 3. Find the four subspaces (kernel, column space, row space, left kernel) of the four matrices from Exercise 2.

[Keep in mind that the column space and the kernel consist of column vectors, whereas the row space and the left kernel consist of row vectors.] [16 points]

Now, for the theory part. Here is a fact which I half-proved in class:

Proposition 0.1. Let U be a subspace of a finite-dimensional vector space V .

(a) The vector space U is finite-dimensional.

(b) We have $\dim U \leq \dim V$.

(c) If $\dim U = \dim V$, then $U = V$.

Proof of Proposition 0.1. Let $d = \dim V$.

Recall that (by one of our many propositions)

every linearly independent list of vectors in V
can be extended to a basis of V . (1)

¹ In particular,

every linearly independent list of vectors in V has size $\leq d$ (2)

(because (1) shows that this list can be extended to a basis of V ; but the latter basis must have size $\dim V = d$, and therefore the former list must have size $\leq d$). In particular, every linearly independent list of vectors in U has size $\leq d$ (because vectors in U are also vectors in V).

Now, fix a linearly independent list (u_1, u_2, \dots, u_k) of vectors in U **of longest possible size**. (There is indeed a “longest possible size”, because every linearly independent list of vectors in U has size $\leq d$.)

We have $\langle u_1, u_2, \dots, u_k \rangle \subseteq U$ (since u_1, u_2, \dots, u_k are vectors in U). On the other hand, it is easy to see that $U \subseteq \langle u_1, u_2, \dots, u_k \rangle$ ². Combined with $\langle u_1, u_2, \dots, u_k \rangle \subseteq U$, this yields $U = \langle u_1, u_2, \dots, u_k \rangle$. Thus, the list (u_1, u_2, \dots, u_k) is a basis of U (since we already know that this list is linearly independent).

From $U = \langle u_1, u_2, \dots, u_k \rangle$, we see that there exists a finite list that spans U (namely, the list (u_1, u_2, \dots, u_k)). Thus, U is finite-dimensional. This proves Proposition 0.1 (a).

The basis (u_1, u_2, \dots, u_k) of U has size k . Hence, $\dim U = k$. But (u_1, u_2, \dots, u_k) is a linearly independent list of vectors in V . Thus, this list has size $\leq d$ (by (2)). In other words, $k \leq d$ (since the size of this list is k). Since $k = \dim U$, we now have $\dim U = k \leq d = \dim V$. This proves Proposition 0.1 (b).

¹As usual: To “extend” means to attach further vectors to it.

²*Proof.* Let $u \in U$.

The list $(u_1, u_2, \dots, u_k, u)$ is a list of vectors in U that is longer than the list (u_1, u_2, \dots, u_k) , and thus must be linearly dependent (because (u_1, u_2, \dots, u_k) was a linearly independent list of vectors in U **of longest possible size**). In other words, there exist scalars $\lambda_1, \lambda_2, \dots, \lambda_k, \lambda \in \mathbb{R}$, **not all zero**, such that

$$\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_k u_k + \lambda u = 0. \quad (3)$$

Consider these scalars.

If we had $\lambda = 0$, then (3) would simplify to $\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_k u_k = 0$; this would yield that all of $\lambda_1, \lambda_2, \dots, \lambda_k$ are zero (since (u_1, u_2, \dots, u_k) is linearly independent), and therefore all of our scalars $\lambda_1, \lambda_2, \dots, \lambda_k, \lambda$ would be zero (since $\lambda = 0$ too); but this would contradict the assumption that $\lambda_1, \lambda_2, \dots, \lambda_k, \lambda$ are not all zero. Hence, we cannot have $\lambda = 0$. Thus, $\lambda \neq 0$. Hence, we can solve (3) for u , obtaining

$$u = \frac{-1}{\lambda} (\lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_k u_k) \in \langle u_1, u_2, \dots, u_k \rangle.$$

Thus, we have proven that every $u \in U$ satisfies $u \in \langle u_1, u_2, \dots, u_k \rangle$. In other words, $U \subseteq \langle u_1, u_2, \dots, u_k \rangle$.

(c) Assume that $\dim U = \dim V$. Recall that (u_1, u_2, \dots, u_k) is a linearly independent list of vectors in V . Thus, (1) shows that this list can be extended to a basis of V . In other words, there exists a basis of V having the form $(u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_m)$ for some additional vectors v_1, v_2, \dots, v_m .

The list $(u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_m)$ has size $k + m$. But on the other hand, the same list has size $\dim V$ (since it is a basis of V). Thus, $k + m = \dim V = \dim U$ (since $\dim U = \dim V$). Thus, $k + m = \dim U = k$, and therefore $m = 0$. Hence, there are no additional vectors v_1, v_2, \dots, v_m . In other words, the list $(u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_m)$ is just our old list (u_1, u_2, \dots, u_k) . Thus, the latter list is a basis of V (since the former list is a basis of V). Therefore,

$$V = \langle u_1, u_2, \dots, u_k \rangle = U.$$

This proves Proposition 0.1 (c). □

Now, let me recall some more properties of matrices:

- If D is a $k \times \ell$ -matrix, then

$$\begin{aligned} \text{Col } D &= D\mathbb{R}^\ell = \{Dx \mid x \in \mathbb{R}^\ell\} \\ &= \langle \text{col}_1 D, \text{col}_2 D, \dots, \text{col}_\ell D \rangle \end{aligned} \quad (4)$$

is the column space of D .

- If D is a $k \times \ell$ -matrix, then

$$\begin{aligned} \text{Row } D &= \mathbb{R}^{1 \times k} D = \{yD \mid y \in \mathbb{R}^{1 \times k}\} \\ &= \langle \text{row}_1 D, \text{row}_2 D, \dots, \text{row}_k D \rangle \end{aligned} \quad (5)$$

is the row space of D . (It can also be written as $(D^T \mathbb{R}^k)^T$.)

These two facts are completely analogous (since \mathbb{R}^ℓ is just shorthand for $\mathbb{R}^{\ell \times 1}$).

Exercise 4. Prove Proposition 0.2 (a) below. [Hint: Use row spaces instead of column spaces. Use (5) instead of (4). Do not hesitate to copy my proof of Proposition 0.2 (a), as long as you make the necessary changes.] [10 points]

Proposition 0.2. Let $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $p \in \mathbb{N}$. Let A be an $n \times m$ -matrix. Let B be an $m \times p$ -matrix.

(a) We have $\text{rank}(AB) \leq \text{rank } B$.

(b) We have $\text{rank}(AB) \leq \text{rank } A$.

Proof of Proposition 0.2 (b). (b) Recall that the rank of a matrix is the dimension of its column space. Thus,

$$\text{rank}(AB) = \dim(\text{Col}(AB)) \quad \text{and} \quad (6)$$

$$\text{rank } A = \dim(\text{Col } A). \quad (7)$$

But (4) (applied to $D = AB$, $k = n$ and $\ell = p$) shows that $\text{Col}(AB) = \{ABx \mid x \in \mathbb{R}^p\}$. On the other hand, (4) (applied to $D = A$, $k = n$ and $\ell = m$) shows that $\text{Col } A = \{Ax \mid x \in \mathbb{R}^m\}$.

Now, let us prove that $\text{Col}(AB) \subseteq \text{Col } A$. Indeed, let $v \in \text{Col}(AB)$ be arbitrary. Thus, $v \in \text{Col}(AB) = \{ABx \mid x \in \mathbb{R}^p\}$. In other words, v has the form $v = ABx$ for some $x \in \mathbb{R}^p$. Denote this x by y (because we will have another use for the letter x later). Thus, $y \in \mathbb{R}^p$ and $v = AB y$. Thus, v has the form $v = Ax$ for some $x \in \mathbb{R}^m$ (namely, for $x = By$). In other words, $v \in \{Ax \mid x \in \mathbb{R}^m\}$. In other words, $v \in \text{Col } A$ (since $\text{Col } A = \{Ax \mid x \in \mathbb{R}^m\}$).

Thus, we have proven that every $v \in \text{Col}(AB)$ lies in $\text{Col } A$. In other words, $\text{Col}(AB) \subseteq \text{Col } A$. Furthermore, $\text{Col}(AB)$ is a subspace of \mathbb{R}^n , and thus contains the zero vector, is closed under addition, and is closed under scaling. Hence, $\text{Col}(AB)$ is also a subspace of $\text{Col } A$ (since $\text{Col}(AB) \subseteq \text{Col } A$). Thus, Proposition 0.1 (b) (applied to $U = \text{Col}(AB)$ and $V = \text{Col } A$) yields $\dim(\text{Col}(AB)) \leq \dim(\text{Col } A)$. In view of (6) and (7), this rewrites as $\text{rank}(AB) \leq \text{rank } A$. This proves Proposition 0.2 (b). \square

Recall a few more facts:

- The *rank-nullity theorem* states that any $n \times m$ -matrix A satisfies

$$\text{rank } A + \dim(\text{Ker } A) = m. \quad (8)$$

(I did this in class, though I used $\dim(A\mathbb{R}^m)$ instead of $\text{rank } A$ because I had not defined $\text{rank } A$ yet.)

- We have

$$\text{rank}(I_n) = n \quad (9)$$

for every $n \in \mathbb{N}$. (To prove this, argue that $I_n \mathbb{R}^n = \left\{ \underbrace{I_n x}_{=x} \mid x \in \mathbb{R}^n \right\} = \{x \mid x \in \mathbb{R}^n\} = \mathbb{R}^n$ and thus $\text{rank}(I_n) = \dim(\underbrace{I_n \mathbb{R}^n}_{=\mathbb{R}^n}) = \dim(\mathbb{R}^n) = n$.)

- If A is an $n \times m$ -matrix, then

$$\text{rank } A \leq \min\{n, m\}. \quad (10)$$

(Recall the reason why this is true: We have $\text{rank } A = \dim(\text{Col } A) \leq m$ because the column space of A is spanned by m vectors, and we have $\text{rank } A = \dim(\text{Row } A) \leq n$ because the row space of D is spanned by n vectors.)

Now, we can prove some statements that were left unproven in class long ago, back before we introduced vector spaces:

Definition 0.3. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let A be an $n \times m$ -matrix.

(a) A *right inverse* of A means an $m \times n$ -matrix B satisfying $AB = I_n$. If a right inverse of A exists, then A is said to be *right-invertible*.

(b) A *left inverse* of A means an $m \times n$ -matrix B satisfying $BA = I_m$. If a left inverse of A exists, then A is said to be *left-invertible*.

(c) An *inverse* of A means an $m \times n$ -matrix B satisfying both $AB = I_n$ and $BA = I_m$. If an inverse of A exists, then A is said to be *invertible*.

Exercise 5. Fill in the big blank in the following proof.

[Hint: What is the transpose of $A^T B$?]

[10 points]

Proposition 0.4. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let A be an $n \times m$ -matrix.

(a) The matrix A is left-invertible if and only if the matrix A^T is right-invertible.

(b) We have $\text{rank } A = \text{rank } (A^T)$.

Proof of Proposition 0.4. (a) \implies : Assume that A is left-invertible. We must show that A^T is right-invertible.

The matrix A is left-invertible. In other words, it has a left inverse. That is, there exists an $m \times n$ -matrix B satisfying $BA = I_m$. Consider this B .

Now, $(BA)^T = A^T B^T$ (by Proposition 3.18 (e) in the current version of the notes, applied to m, n, m, B and A instead of n, m, p, A and B), so that $A^T B^T =$

$\left(\underbrace{BA}_{=I_m} \right)^T = (I_m)^T = I_m$. Hence, B^T is a right inverse of A^T . Thus, A^T is right-invertible. This proves the \implies direction of Proposition 0.4 (a).

\impliedby : Assume that A^T is right-invertible. We must show that A is left-invertible.

The matrix A^T is right-invertible. In other words, it has a right inverse. That is, there exists an $n \times m$ -matrix B satisfying $A^T B = I_n$. Consider this B .

Set $C = B^T$.

Hence, C is a left inverse of A . Thus, A is left-invertible. This proves the \impliedby direction of Proposition 0.4 (a).

(b) Recall a notation I introduced in class: If V is a set of column vectors, then V^T denotes the set of their transposes (rigorously speaking, this means that $V^T = \{v^T \mid v \in V\}$). I shall call V^T the *elementwise transpose* of V . The sets V and V^T are “essentially the same” except that the former consists of column vectors and the latter of row vectors. In particular, if V is a subspace of \mathbb{R}^n , then V^T is a subspace of $\mathbb{R}^{1 \times n}$, and their dimensions are the same:

$$\dim(V^T) = \dim V. \quad (11)$$

Now, the rows of A^T are the transposes of the columns of A . Hence, the span of the rows of A^T is the elementwise transpose of the span of the columns of A . This rewrites as follows:

$$\text{Row}(A^T) = (\text{Col } A)^T$$

(because the span of the rows of A^T is the row space $\text{Row}(A^T)$, and the span of the columns of A is the column space $\text{Col } A$). Thus,

$$\dim(\text{Row}(A^T)) = \dim((\text{Col } A)^T) = \dim(\text{Col } A)$$

(by (11), applied to $V = \text{Col } A$). But $\text{rank}(A^T) = \dim(\text{Row}(A^T))$ (since the rank of a matrix is the dimension of its row space) and $\text{rank } A = \dim(\text{Col } A)$ (since the rank of a matrix is the dimension of its column space). Hence,

$$\text{rank}(A^T) = \dim(\text{Row}(A^T)) = \dim(\text{Col } A) = \text{rank } A.$$

Thus, Proposition 0.4 (b) is proven. \square

Exercise 6. Fill in the blanks in the following proof.

[27 points]

Proposition 0.5. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let A be an $n \times m$ -matrix.

(a) The matrix A is right-invertible if and only if $\text{rank } A = n$.

(b) The matrix A is left-invertible if and only if $\text{rank } A = m$.

(c) The matrix A is invertible if and only if $\text{rank } A = n = m$. (In particular, only square matrices can be invertible!)

Proof of Proposition 0.5. (a) \implies : Let us first show that if A is right-invertible, then $\text{rank } A = n$.

Indeed, assume that A is right-invertible. In other words, there exists some $m \times n$ -matrix B satisfying $AB = I_n$. Consider this B .

Proposition 0.2 (b) yields $\text{rank}(AB) \leq \text{rank } A$, so that $\text{rank } A \geq \text{rank} \left(\underbrace{AB}_{=I_n} \right) = \text{rank}(I_n) = n$ (by (9)). But (10) yields $\text{rank } A \leq \min\{n, m\} \leq n$. Combining this with $\text{rank } A \geq n$, we find $\text{rank } A = n$. This completes the proof of the \implies direction.

\Leftarrow : Let us now show that if $\text{rank } A = n$, then A is right-invertible.

Indeed, assume that $\text{rank } A = n$. We shall explicitly construct a right inverse B to A .

Recall that the rank of a matrix is the dimension of its _____. Thus, $\text{rank } A = \dim(A\mathbb{R}^m)$ (since the _____ of A is $A\mathbb{R}^m$). Hence, $\dim(A\mathbb{R}^m) = \text{rank } A = n = \dim(\mathbb{R}^n)$.

But $A\mathbb{R}^m$ is a subspace of \mathbb{R}^n . Hence, from $\dim(A\mathbb{R}^m) = \dim(\mathbb{R}^n)$, we obtain $A\mathbb{R}^m = \mathbb{R}^n$ (by Proposition _____, applied to $U = \text{_____}$ and $V = \text{_____}$).

For each $j \in \{1, 2, \dots, n\}$, we have

$$\text{col}_j(I_n) \in \mathbb{R}^n = A\mathbb{R}^m = \{Ax \mid x \in \mathbb{R}^m\}.$$

³ In other words, for each $j \in \{1, 2, \dots, n\}$, we can write $\text{col}_j(I_n)$ in the form Ax for some $x \in \mathbb{R}^m$. Pick such an x , and denote it by x_j . Thus, x_j (for each $j \in \{1, 2, \dots, n\}$) is a column vector in \mathbb{R}^m satisfying $\text{col}_j(I_n) = Ax_j$.

Now we have chosen n vectors x_1, x_2, \dots, x_n in \mathbb{R}^m such that

$$\text{col}_j(I_n) = Ax_j \quad \text{for every } j \in \{1, 2, \dots, n\}. \quad (12)$$

Let B be the $m \times n$ -matrix whose columns are x_1, x_2, \dots, x_n . Thus,

$$\text{col}_j B = x_j \quad \text{for every } j \in \{1, 2, \dots, n\}. \quad (13)$$

Now, every $j \in \{1, 2, \dots, n\}$ satisfies

$$\begin{aligned} \text{col}_j(AB) &= \text{_____} && \text{(by what is currently Proposition 2.19 (d) in the notes)} \\ &= \text{_____} && \text{(by (13))} \\ &= \text{_____} && \text{(by _____).} \end{aligned}$$

In other words, each column of the $n \times n$ -matrix AB equals the corresponding column of I_n . Hence, these two matrices are equal. In other words, $AB = I_n$. Thus, B is a right inverse of A . Hence, A is right-invertible. This proves the \Leftarrow direction.

Altogether, we have thus proven Proposition 0.5 (a).

(b) Applying Proposition 0.5 (a) to m, n and A^T instead of n, m and A , we conclude the following: The matrix A^T is right-invertible if and only if $\text{rank}(A^T) = m$. Combined with Proposition 0.4 (a), this shows that the matrix A is left-invertible if and only if $\text{rank}(A^T) = m$. Because of Proposition 0.4 (b), we can replace $\text{rank}(A^T)$ by $\text{rank } A$ here, and we obtain precisely the claim of Proposition 0.5 (b).

(c) \Rightarrow : Let us first show that if A is invertible, then $\text{rank } A = n = m$.

Indeed, assume that A is invertible. In other words, there exists some $m \times n$ -matrix B satisfying both $AB = I_n$ and $BA = I_m$. Consider this B .

³By the way, $\text{col}_j(I_n)$ is the standard basis vector $e_j = (0, 0, \dots, 0, 1, 0, 0, \dots, 0)^T$; but this is unimportant for us here.

We have $AB = I_n$; hence, B is a right inverse of A . Thus, A is right-invertible. Therefore, Proposition 0.5 (a) shows that $\text{rank } A = n$.

We have $BA = I_m$; hence, B is a left inverse of A . Thus, A is left-invertible. Therefore, Proposition 0.5 (b) shows that $\text{rank } A = m$.

Combining $\text{rank } A = n$ with $\text{rank } A = m$, we obtain $\text{rank } A = n = m$. This proves the \implies direction of Proposition 0.5 (c).

\impliedby : Let us now show that if $\text{rank } A = n = m$, then A is invertible.

Indeed, assume that $\text{rank } A = n = m$.

From $\text{rank } A = n$, we conclude (using Proposition 0.5 (a)) that A is right-invertible. In other words, A has a right inverse R . Consider this R .

From $\text{rank } A = m$, we conclude (using Proposition 0.5 (b)) that A is left-invertible. In other words, A has a left inverse L . Consider this L .

Proposition 3.6 (d) from the notes now shows that the matrix $L = R$ is the only inverse of A . In particular, it is an inverse of A ; thus, A is invertible. This proves the \impliedby direction of Proposition 0.5 (c). \square

Exercise 7. Which of the matrices in Exercise 2 are invertible? [10 points]

Exercise 8. Let $n \in \mathbb{N}$. Let A be a left-invertible $n \times n$ -matrix. Prove that A is invertible. [10 points]

[Hint: All ingredients of the proof are on this problem set; you have to combine them.]
