Math 4242, Section 070
Homework 3

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1) Throughout this exercise I will denote $0_{m\times n} = \mathbf{0}$.

a. $\mathbf{S_1}$: **This is a subspace.** Clearly $\mathbf{0} \in S_1$ since $0+0 = 0+0$. Now assume that we have $\mathbf{a} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \in S_1$. Then $\mathbf{a}+\mathbf{b} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix}$. Then we can see that $(a+w)+(d+z) = (a+d)+(w+z) = (b+c)+(x+y) = (b+x)+(c+y)$. So $\mathbf{a}+\mathbf{b} \in S_1$. It is also clear that $\lambda\mathbf{a} \in S_1$.
$\mathbf{S_2}$: **This is a subspace.** This space is extremely similar to S_1, except one could imagine that $b+c=0$, although this is not actually a requirement. Hence a similar argument follows
$\mathbf{S_3}$: **This is not a subspace.** Consider $\mathbf{a} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix}$ Then we clearly have $\det(\mathbf{a}) = 0$ and $\det(\mathbf{b}) = 0$. However, $\mathbf{a}+\mathbf{b} = \begin{pmatrix} 5 & 9 \\ 1 & 1 \end{pmatrix}$ and so $\det(\mathbf{a}+\mathbf{b}) = -4 \neq 0$.
$\mathbf{S_4}$: **This is not a subspace.** Consider the matrices $\mathbf{a} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is easy to check that $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{0}$. However, $\mathbf{a}+\mathbf{b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $(\mathbf{a}+\mathbf{b})^2 = I_2$.
$\mathbf{S_5}$: **This is a subspace.** We are essentially solving the equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \mathbf{0}$. This gives us the equations $3a+b=0$ and $3c+d=0$. Clearly we have $\mathbf{0} \in S_5$, and the arguments for sums and scales follow similarly from S_1.
$\mathbf{S_6}$: **This is a subspace.** We solve the equation:

$$\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3a+6c+b+2d = 0$$

Once again we have that $\mathbf{0} \in S_6$, and sums and scales follow quickly.

b. We will find spans for all 4 of the subspaces.
For S_1, we can let $b=r$, $c=s$ and $d=t$. Then we have $a = r+s-t$. So we have the following span:

$$S_1 = \langle E_{1,1}+E_{1,2}, E_{1,1}+E_{2,1}, -E_{1,1}+E_{2,2}\rangle$$

For S_2, we require that $a=-d$, while b and c can be anything. So we get the span:

$$\langle E_{1,1}-E_{2,2}, E_{1,2}, E_{2,1}\rangle$$

For S_5, as said in part (a) we require that $b=-3a$ and $d=-3c$. So we get the span:

$$\langle E_{1,2}-3E_{1,1}, E_{2,2}-3E_{2,1}\rangle$$

For S_6 we can have $b = -3a-6c-2d$, with a, c, d being anything. Hence we get:

$$\langle E_{1,1}-3E_{1,2}, E_{2,1}-6E_{1,2}, E_{2,2}-2E_{1,2}\rangle$$

2) a. Yes this does span all of $\mathbb{R}^3$. Note that $(1,1,0)^T - (0,1,1)^T + (1,0,1)^T = (2,0,0)^T$, $(1,1,0)^T + (0,1,1)^T - (1,0,1)^T = (0,2,0)^T$ and $(1,0,1)^T + (0,1,1)^T - (1,1,0)^T = (0,0,2)^T$. So for any

$(a,b,c)^T \in \mathbb{R}^3$, we can write it as $(a/2)(2,0,0)^T + (b/2)(0,2,0)^T + (c/2)(0,0,2)^T$. So $(a,b,c)^T$ is in the span.

b. This does not span all of $\mathbb{R}^4$. Consider the vector $(1,0,1,0)^T \in \mathbb{R}^4$. Then we require that:

$$\lambda_1 \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \lambda_3 \begin{pmatrix}0\\0\\1\\1\end{pmatrix} + \lambda_4 \begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

So we get the following system:

$$\begin{cases} \lambda_1 & + & \lambda_4 & = & 1 \\ \lambda_1 & + & \lambda_2 & = & 0 \\ \lambda_2 & + & \lambda_3 & = & 1 \\ \lambda_3 & + & \lambda_4 & = & 0 \end{cases}$$

Hence we get that $\lambda_1 = -\lambda_2$ and $\lambda_3 = -\lambda_4$. Thus we get the equations $\lambda_4 - \lambda_2 = 1$ and $\lambda_2 - \lambda_4 = 1$. Clearly this is not solvable system, so there are no such λ_i that satisfy the matrix equation, so $(1,0,1,0)^T$ is not in the span, so the given matrices do not span $\mathbb{R}^4$.

3) The blanks are filled with $(u_1 + u_2 + \cdots + u_j) \in U$ (from the induction hypothesis) and $u_{j+1} \in U$. So this is indeed the sum of two vectors of U.

4) (Step 1): The vector $\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)^T$ satisfies $\lambda x_1 + \lambda x_2 + \cdots + \lambda x_n = \lambda(x_1 + x_2 + \cdots + x_n) = \lambda(0) = 0$.
(Step 8): We have X as a subspace of $\mathbb{R}^n$ since it is the span of vectors that are in $\mathbb{R}^n$, and any span of vectors in a space create a subspace of that space.
Applying Proposition 0.1 (c) to $U = X$, $k = n-1$ and $u_i = e_i - e_n$.

5) a. We are looking to solve for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $(2,3,1,-2,-4)^T = \lambda_1(1,0,0,0,-1)^T + \lambda_2(0,1,0,0,-1)^T + \lambda_3(0,0,1,0,-1)^T + (0,0,0,1,-1)^T$. We can read off almost immediately that $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1$ and $\lambda_4 = -2$. Then we see that $-(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = -(2+3+1-2) = -4$ as expected. Hence:

$$(2,3,1,-2,-4)^T = 2(e_1 - e_5) + 3(e_2 - e_5) + 1(e_3 - e_5) - 2(e_4 - e_5)$$

b. We are solving $(2,3,1,-2,-4)^T = \lambda_1(1,-1,0,0,0)^T + \lambda_2(0,1,-1,0,0)^T + \lambda_3(0,0,1,-1,0)^T + \lambda_4(0,0,0,1,-1)^T$ for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. We get the following system:

$$\begin{cases} \lambda_1 & + & 0 & = & 2 \\ \lambda_2 & - & \lambda_1 & = & 3 \\ \lambda_3 & - & \lambda_2 & = & 1 \\ \lambda_4 & - & \lambda_3 & = & -2 \\ 0 & - & \lambda_4 & = & -4 \end{cases}$$

Solving this through substitution is rather simple. We get $\lambda_1 = 2, \lambda_2 = 5, \lambda_3 = 6, \lambda_4 = 4$. Hence:

$$(2,3,1,-2,-4)^T = 2(e_1 - e_2) + 5(e_2 - e_3) + 6(e_3 - e_4) + 4(e_4 - e_5)$$

6) a. We wish to determine if there are $\lambda_1, \lambda_2, \lambda_3$ where not all three $\lambda_i = 0$, such that $\lambda_1(1,1,0)^T + \lambda_2(0,1,1)^T + \lambda_3(1,0,1)^T = \mathbf{0}$. In other words, we want to solve the system:

$$\begin{cases} \lambda_1 & + & 0 & + & \lambda_3 & = & 0 \\ \lambda_1 & + & \lambda_2 & + & 0 & = & 0 \\ 0 & + & \lambda_2 & + & \lambda_3 & = & 0 \end{cases}$$

From a non-rigorous standpoint, we have three equations and three variables so we expect there to be only one solution. Solving through, we get $\lambda_2 = -\lambda_3$, so we have $\lambda_1 - \lambda_3 = 0$ and $\lambda_1 + \lambda_3 = 0$. Hence we must have $\lambda_i = 0$ for $i = 1,2,3$. **Thus these vectors linearly independent.**

b. These are not linearly independent. We can take:

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Hence it is possible to obtain $\mathbf{0}$ without setting all coefficients of the vectors to 0, meaning they are linearly dependent.

7) 1: We can divide the inequality by λ_i because we chose $\lambda_i \neq 0$.
2:$\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\}$
3: $\{v_1, \ldots, v_k\}$

8) **a.** The only such i is $i = 4$, as v_1, v_2 and v_3 are linearly independent. We can write $v_4 = v_1 + 2v_2 + 3v_3$.
b. We can take $i = 3, 4$, since $v_3 = 2v_1 - v_2$ and $v_4 = -v_1 + v_2 + 2v_3$.
c. We can only take $i = 1$, because $v_1 = \mathbf{0}$ is a linear combination of ().