Math 4242 Fall 2016 (Darij Grinberg): homework set 3
due: Wed, 12 Oct 2016, 23:00 (Minneapolis time) by moodle
or in class on 12 Oct 2016

[**Note:** This is an updated version of the homework set, where I have switched to the notation $\operatorname{span}(u_1, u_2, \ldots, u_k)$ for the span of k vectors $u_1, u_2, \ldots, u_k$. The original version used the notation $\langle u_1, u_2, \ldots, u_k \rangle$.]

Exercise 1. Consider the vector space $\mathbb{R}^{2\times 2}$ of all 2×2-matrices.

(a) Which of the following subsets of $\mathbb{R}^{2\times 2}$ are subspaces?

$$S_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \mid a + d = b + c \right\};$$
$$S_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \mid a + d = 0 \right\};$$
$$S_3 = \left\{ A \in \mathbb{R}^{2\times 2} \mid \det A = 0 \right\};$$
$$S_4 = \left\{ A \in \mathbb{R}^{2\times 2} \mid A^2 = 0_{2\times 2} \right\};$$
$$S_5 = \left\{ A \in \mathbb{R}^{2\times 2} \mid A \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 0_{2\times 1} \right\};$$
$$S_6 = \left\{ A \in \mathbb{R}^{2\times 2} \mid \begin{pmatrix} 1 & 2 \end{pmatrix} A \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 0_{1\times 1} \right\}.$$

(b) For **at least three** of the above subsets that are subspaces, find a list of 2×2-matrices that spans it.

[**Example:** The subspace $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \mid a = b \right\}$ is the span $\operatorname{span}(E_{1,1} + E_{1,2}, E_{2,1}, E_{2,2})$, where $E_{i,j}$ are as defined in §3.6 of the notes.]
[20 points]

Exercise 2. (a) Does the span $\operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$ equal the whole vector space $\mathbb{R}^3$?

(b) Does the span $\operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$ equal the whole vector space $\mathbb{R}^4$? [20 points]

The following "fill in the blanks" exercises are an experiment. I would have preferred to have some more interesting proofs in there, but I didn't come to the interesting parts in class... (Each correctly filled blank is worth 3 points.)

Exercise 3. Fill in the blanks in the following proof: [6 points]

Proposition 0.1. Let V be a vector space. Let U be a subspace of V.

(a) If $u_1, u_2, \ldots, u_k$ are elements of U, then $u_1 + u_2 + \cdots + u_k \in U$.

(b) The set U is closed under linear combination. In other words: If $u_1, u_2, \ldots, u_k$ are elements of U, then every linear combination of $u_1, u_2, \ldots, u_k$ also lies in U.

(c) Let $u_1, u_2, \ldots, u_k$ be elements of U. Then,

$$\operatorname{span}(u_1, u_2, \ldots, u_k) \subseteq U.$$

Proof. **(a)** Roughly speaking, this is just a matter of applying the "closed under addition" axiom several times. But there is a subtlety involved (the sum of 0 elements of U is not obtained by addition, but rather defined as $\overrightarrow{0}$), and I want to illustrate the principle of induction, so I am going for a detailed and boring formal proof.

Let $u_1, u_2, \ldots, u_k$ be elements of U. We must show that $u_1 + u_2 + \cdots + u_k \in U$.

We shall prove that

$$u_1 + u_2 + \cdots + u_i \in U \qquad \text{for every } i \in \{0, 1, \ldots, k\}. \tag{1}$$

We will prove (1) by *induction over* i. (If you have never seen a proof by induction: this here is an example.) This means that we shall prove the following two claims:

Claim 1: (1) holds for $i = 0$.

Claim 2: If $j \in \{0, 1, \ldots, k-1\}$ is such that (1) holds for $i = j$, then (1) also holds for $i = j + 1$.

Once these two claims are proven, the *principle of mathematical induction* will yield that (1) holds for all $i \in \{0, 1, \ldots, k\}$. In fact:

- Claim 1 shows that (1) holds for $i = 0$;
- thus, Claim 2 (applied to $j = 0$) shows that (1) holds for $i = 1$;
- thus, Claim 2 (applied to $j = 1$) shows that (1) holds for $i = 2$;
- thus, Claim 2 (applied to $j = 2$) shows that (1) holds for $i = 3$;
- and so on, applying Claim 2 for higher and higher j, until we arrive at $i = k$.

See Chapter 5 in Lehman/Leighton/Meyer for an introduction to proofs by induction.

Of course, we still have to prove the two claims.

1. *Proof of Claim 1:* For $i = 0$, the statement (1) claims that $u_1 + u_2 + \cdots + u_0 \in U$. In order to make sense of this, we must recall that empty sums of vectors are defined to mean $\overrightarrow{0}$. Thus,

$$u_1 + u_2 + \cdots + u_0 = (\text{empty sum of vectors}) = \overrightarrow{0}.$$

But U is a subspace of V, and thus contains $\overrightarrow{0}$ (this is one of the axioms for a subspace). Thus, $\overrightarrow{0} \in U$, so that $u_1 + u_2 + \cdots + u_0 = \overrightarrow{0} \in U$. In other words, (1) holds for $i = 0$. This proves Claim 1.

2. *Proof of Claim 2:* Let $j \in \{0, 1, \ldots, k-1\}$ be such that (1) holds for $i = j$. (The statement that (1) holds for $i = j$ is called the *induction hypothesis*.) We must show that (1) also holds for $i = j + 1$.

Since (1) holds for $i = j$, we have $u_1 + u_2 + \cdots + u_j \in U$. Now,

$$u_1 + u_2 + \cdots + u_{j+1} = \underbrace{(\rule{3cm}{0.4pt})}_{\in U} + \underbrace{\rule{3cm}{0.4pt}}_{\in U}.$$

This is a sum of two vectors in U, and thus belongs to U (since U is closed under addition). In other words, $u_1 + u_2 + \cdots + u_{j+1} \in U$. Thus, (1) also holds for $i = j + 1$. This proves Claim 2.

Now, both Claims 1 and 2 are proven, so that the proof of (1) is complete.

(Usually, the proof of Claim 1 is called the "induction base", and the proof of Claim 2 is called the "induction step".)

Now that (1) is proven, we can simply apply (1) to $i = k$, and conclude that $u_1 + u_2 + \cdots + u_k \in U$. This proves Proposition 0.1 **(a)**.

(b) Let $u_1, u_2, \ldots, u_k$ be elements of U. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be real numbers.

The set U is a subspace of V, and thus is closed under scaling. Hence, the vectors $\lambda_1 u_1, \lambda_2 u_2, \ldots, \lambda_k u_k$ all belong to U (since the vectors $u_1, u_2, \ldots, u_k$ belong to U). Thus, Proposition 0.1 **(a)** (applied to $\lambda_1 u_1, \lambda_2 u_2, \ldots, \lambda_k u_k$ instead of $u_1, u_2, \ldots, u_k$) shows that $\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k \in U$.

Now, we have shown that $\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k \in U$ whenever $\lambda_1, \lambda_2, \ldots, \lambda_k$ are real numbers. In other words: every linear combination of $u_1, u_2, \ldots, u_k$ also lies in U. This proves Proposition 0.1 **(b)**.

(c) Proposition 0.1 **(b)** shows that every linear combination of $u_1, u_2, \ldots, u_k$ lies in U. But $\operatorname{span}(u_1, u_2, \ldots, u_k)$ is precisely the set of these linear combinations. Hence, $\operatorname{span}(u_1, u_2, \ldots, u_k) \subseteq U$. This proves Proposition 0.1 **(c)**. $\square$

A remark: The "empty span" $\operatorname{span}()$ (that is, the span of no vectors) is the subspace $\left\{ \overrightarrow{0} \right\}$ (not the empty set!). This is because $\overrightarrow{0}$ counts as a linear combination of an empty list of vectors (being the empty sum). This will be important later.

Exercise 4. Fill in the blanks in the following proof: [21 points]

Proposition 0.2. Let n be a positive integer. Define a subset W of $\mathbb{R}^n$ by

$$W = \left\{ (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \mid x_1 + x_2 + \cdots + x_n = 0 \right\}.$$

In other words, W is the set of all column vectors of size n whose entries sum to 0.

For each $i \in \{1, 2, \ldots, n\}$, let e_i be the column vector $E_{i,1} = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)^T$ (where the 1 is in the i-th position) in $\mathbb{R}^n$.

(a) We have

$$W = \operatorname{span}(e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n).$$

(b) We have

$$W = \operatorname{span}(e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n).$$

Proof. (The following proof is more detailed and nicer than the one given in class.)

Step 1: First, let us prove that the subset W is a subspace of $\mathbb{R}^n$. Indeed:

- The zero vector $\overrightarrow{0} = 0_{n \times 1} = (0, 0, \ldots, 0)^T$ belongs to W, since it satisfies $0 + 0 + \cdots + 0 = 0$.

- Let $x \in W$ and $y \in W$. We want to show that $x + y \in W$. Write $x \in W$ as $x = (x_1, x_2, \ldots, x_n)^T$, and write $y \in W$ as $y = (y_1, y_2, \ldots, y_n)^T$. Then, $x_1 + x_2 + \cdots + x_n = 0$ (since $(x_1, x_2, \ldots, x_n)^T = x \in W$) and $y_1 + y_2 + \cdots + y_n = 0$ (for similar reasons). Now, the vector $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)^T$ satisfies

$$\begin{aligned} &(x_1 + y_1) + (x_2 + y_2) + \cdots + (x_n + y_n) \\ &= \underbrace{(x_1 + x_2 + \cdots + x_n)}_{=0} + \underbrace{(y_1 + y_2 + \cdots + y_n)}_{=0} = 0 + 0 = 0, \end{aligned}$$

  and therefore also lies in W. We thus have proven that $x + y \in W$ for all $x \in W$ and $y \in W$. In other words, W is closed under addition.

- Let $x \in W$ and $\lambda \in \mathbb{R}$. We want to show that $\lambda x \in W$. Write $x \in W$ as $x = (x_1, x_2, \ldots, x_n)^T$. Then, $x_1 + x_2 + \cdots + x_n = 0$ (since $(x_1, x_2, \ldots, x_n)^T = x \in W$). Now, the vector $\lambda x =$ ________________ satisfies

  ____

  and therefore also lies in W. We thus have proven that $\lambda x \in W$ for all $x \in W$ and $\lambda \in \mathbb{R}$. In other words, W is closed under scaling.

This shows that W is a subspace of $\mathbb{R}^n$.

Step 2: For each $i \in \{1, 2, \ldots, n-1\}$, the vector $e_i - e_n$ has the form

$$(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0, -1)^T$$

(where the 1 stands in position i). Thus, the sum of its entries is $0 + 0 + \cdots + 0 + 1 + 0 + 0 + \cdots + 0 + (-1) = 0$, and this means that it lies in W. Thus, we have shown that $e_i - e_n \in W$ for each $i \in \{1, 2, \ldots, n-1\}$. In other words, the $n-1$ vectors $e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n$ all lie in W.

Step 3: The set W is a subspace of $\mathbb{R}^n$ (by Step 1). Also, the $n-1$ vectors $e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n$ all lie in W (by Step 2). Hence, Proposition 0.1 **(c)** (applied to $U = W$, $k = n-1$ and $u_i = e_i - e_n$) shows that

$$\operatorname{span}(e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n) \subseteq W. \tag{2}$$

Step 4: Now, we are going to prove

$$W \subseteq \operatorname{span}(e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n). \tag{3}$$

Indeed, let $w \in W$. Write w as $w = (w_1, w_2, \ldots, w_n)^T$. Then, $w_1 + w_2 + \cdots + w_n = 0$ (since $(w_1, w_2, \ldots, w_n)^T = w \in W$), so that

$$w_1 + w_2 + \cdots + w_{n-1} = -w_n. \tag{4}$$

But now,

$$\begin{aligned}
&w_1(e_1 - e_n) + w_2(e_2 - e_n) + \cdots + w_{n-1}(e_{n-1} - e_n) \\
&= w_1e_1 + w_2e_2 + \cdots + w_{n-1}e_{n-1} - \underbrace{(w_1 + w_2 + \cdots + w_{n-1})}_{\substack{= -w_n \\ \text{(by (4))}}} e_n \\
&= w_1e_1 + w_2e_2 + \cdots + w_{n-1}e_{n-1} - (-w_n)e_n \\
&= w_1e_1 + w_2e_2 + \cdots + w_{n-1}e_{n-1} + w_ne_n \\
&= (w_1, 0, 0, \ldots, 0, 0)^T + (0, w_2, 0, \ldots, 0, 0)^T \\
&\qquad + \cdots + (0, 0, 0, \ldots, w_{n-1}, 0)^T + (0, 0, 0, \ldots, 0, w_n)^T \\
&= (w_1, w_2, \ldots, w_n)^T = w,
\end{aligned}$$

so that

$$\begin{aligned}
w &= w_1(e_1 - e_n) + w_2(e_2 - e_n) + \cdots + w_{n-1}(e_{n-1} - e_n) \\
&\in \operatorname{span}(e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n).
\end{aligned}$$

Since we have proven this for **every** $w \in W$, we thus have proven (3). Combining (3) with (2), we find $W = \operatorname{span}(e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n)$. Thus, Proposition 0.2 **(a)** is shown.

Step 5: For each $i \in \{1, 2, \ldots, n-1\}$, the vector $e_i - e_{i+1}$ has the form

$$(0, 0, \ldots, 0, 1, -1, 0, 0, \ldots, 0)^T$$

(where the 1 stands in position i, and the -1 stands in position $i+1$). Thus, the sum of its entries is $0 + 0 + \cdots + 0 + 1 + (-1) + 0 + 0 + \cdots + 0 = 0$, and this means that it lies in W. Thus, we have shown that $e_i - e_{i+1} \in W$ for each $i \in \{1, 2, \ldots, n-1\}$. In other words, the $n-1$ vectors $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n$ all lie in W.

Step 6: Set

$$X = \operatorname{span}(e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n).$$

In order to prove Proposition 0.2 **(b)**, we must show that $W = X$.

The $n-1$ vectors $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n$ all lie in W (by Step 5). Hence, Proposition 0.1 **(c)** (applied to $U = W$, $k = n-1$ and $u_i = e_i - e_{i+1}$) shows that

$$\operatorname{span}(e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n) \subseteq W$$

(since W is a subspace of $\mathbb{R}^n$). Since the left hand side of this relation is X, it thus rewrites as

$$X \subseteq W. \tag{5}$$

Step 7: For each $i \in \{1, 2, \ldots, n-1\}$, we have

$$\begin{aligned}
&(e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + (e_{i+2} - e_{i+3}) + \cdots + (e_{n-1} - e_n) \\
&= (e_i + e_{i+1} + e_{i+2} + \cdots + e_{n-1}) - (e_{i+1} + e_{i+2} + e_{i+3} + \cdots + e_n) \\
&= e_i - e_n
\end{aligned}$$

(we have just cancelled the common terms of the two sums), so that

$$\begin{aligned}
e_i - e_n &= (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + (e_{i+2} - e_{i+3}) + \cdots + (e_{n-1} - e_n) \\
&\in \operatorname{span}(e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n) = X.
\end{aligned}$$

In other words, the $n-1$ vectors $e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n$ all lie in X.

Step 8: The set X is a subspace of $\mathbb{R}^n$ (since it is a ________________, and every ________________ is a subspace). But the $n-1$ vectors $e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n$ all lie in X (by Step 7). Hence, Proposition 0.1 **(c)** (applied to $U =$ ________________, $k =$ ________________ and $u_i =$ ________________) shows that

$$\operatorname{span}(e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n) \subseteq X.$$

Now, Proposition 0.2 **(a)** (which we have already proven) yields

$$W = \operatorname{span}(e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n) \subseteq X.$$

Combining this with (5), we find $W = X$. This completes the proof of Proposition 0.2 **(b)**. □

Exercise 5. Let $n = 5$. Define W as in Proposition 0.2. Let w be the vector $(2, 3, 1, -2, -4)^T \in W$.

(a) Write w as a linear combination of $e_1 - e_5, e_2 - e_5, e_3 - e_5, e_4 - e_5$. (This is possible, since Proposition 0.2 **(a)** shows that $W = \operatorname{span}(e_1 - e_5, e_2 - e_5, e_3 - e_5, e_4 - e_5)$.)

(b) Write w as a linear combination of $e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5$. (This is possible, since Proposition 0.2 **(b)** shows that $W = \operatorname{span}(e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5)$.) [10 points]

Proposition 0.3. Let $v_1, v_2, \ldots, v_k$ be k vectors in some vector space V, and let $w_1, w_2, \ldots, w_\ell$ be ℓ vectors in V.

(a) If $\{v_1, v_2, \ldots, v_k\} \subseteq \{w_1, w_2, \ldots, w_\ell\}$, then $\operatorname{span}(v_1, v_2, \ldots, v_k) \subseteq \operatorname{span}(w_1, w_2, \ldots, w_\ell)$.

(b) If $\{v_1, v_2, \ldots, v_k\} = \{w_1, w_2, \ldots, w_\ell\}$, then $\operatorname{span}(v_1, v_2, \ldots, v_k) = \operatorname{span}(w_1, w_2, \ldots, w_\ell)$.

(Proposition 0.3 **(b)** shows that rearranging the vectors in a span and/or duplicating some of them does not change the span. For instance, $\operatorname{span}(\alpha, \beta, \gamma) = \operatorname{span}(\gamma, \alpha, \beta) = \operatorname{span}(\alpha, \beta, \gamma, \beta, \alpha)$ whenever α, β, γ are three vectors. Proposition 0.3 **(a)** shows that, for example, $\operatorname{span}(\alpha, \gamma)$ and $\operatorname{span}(\gamma, \beta, \gamma)$ are subsets of $\operatorname{span}(\alpha, \beta, \gamma)$.)

Proof of Proposition 0.3. Let U denote the span $\operatorname{span}(w_1, w_2, \ldots, w_\ell)$. Then, U is a subspace of V (since every span is a subspace).

(a) Assume that $\{v_1, v_2, \ldots, v_k\} \subseteq \{w_1, w_2, \ldots, w_\ell\}$. Each of the vectors $w_1, w_2, \ldots, w_\ell$ is a linear combination of $w_1, w_2, \ldots, w_\ell$, and thus lies in the span $\operatorname{span}(w_1, w_2, \ldots, w_\ell)$. In other words,

$$\{w_1, w_2, \ldots, w_\ell\} \subseteq \operatorname{span}(w_1, w_2, \ldots, w_\ell).$$

Thus,

$$\{v_1, v_2, \ldots, v_k\} \subseteq \{w_1, w_2, \ldots, w_\ell\} \subseteq \operatorname{span}(w_1, w_2, \ldots, w_\ell) = U.$$

Hence, $v_1, v_2, \ldots, v_k$ are elements of U. Proposition 0.1 **(c)** (applied to $u_i = v_i$) thus shows that

$$\operatorname{span}(v_1, v_2, \ldots, v_k) \subseteq U = \operatorname{span}(w_1, w_2, \ldots, w_\ell).$$

This proves Proposition 0.3 **(a)**.

(b) Assume that $\{v_1, v_2, \ldots, v_k\} = \{w_1, w_2, \ldots, w_\ell\}$. Then, of course, $\{v_1, v_2, \ldots, v_k\} \subseteq \{w_1, w_2, \ldots, w_\ell\}$ (since a set is always a subset of itself). Hence, Proposition 0.3 **(a)** yields

$$\operatorname{span}(v_1, v_2, \ldots, v_k) \subseteq \operatorname{span}(w_1, w_2, \ldots, w_\ell). \tag{6}$$

But we can also apply Proposition 0.3 **(a)** with the roles of $v_1, v_2, \ldots, v_k$ and the roles of $w_1, w_2, \ldots, w_\ell$ interchanged (because $\{v_1, v_2, \ldots, v_k\} = \{w_1, w_2, \ldots, w_\ell\}$ also yields $\{w_1, w_2, \ldots, w_\ell\} \subseteq \{v_1, v_2, \ldots, v_k\}$); thus we obtain

$$\operatorname{span}(w_1, w_2, \ldots, w_\ell) \subseteq \operatorname{span}(v_1, v_2, \ldots, v_k).$$

Combining this with (6), we end up with $\operatorname{span}(v_1, v_2, \ldots, v_k) = \operatorname{span}(w_1, w_2, \ldots, w_\ell)$. This proves Proposition 0.3 **(b)**. $\square$

Exercise 6. **(a)** Is the list $\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$ linearly independent?

(b) Is the list $\left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$ linearly independent? [10 points]

Exercise 7. Fill in the blanks in the following proof. [9 points]

Proposition 0.4. Let V be a vector space. Let $v_1, v_2, \ldots, v_k$ be k vectors in V. Assume that the list $(v_1, v_2, \ldots, v_k)$ is linearly dependent. Then, there exists some $i \in \{1, 2, \ldots, k\}$ such that $v_i \in \operatorname{span}(v_1, v_2, \ldots, v_{i-1})$ and

$$\operatorname{span}(v_1, v_2, \ldots, v_k) = \operatorname{span}(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k). \quad (7)$$

(The right hand side of (7) is the span of the vectors $v_1, v_2, \ldots, v_k$ with the vector v_i skipped over.)

[Notice that when $i = 1$, the span $\operatorname{span}(v_1, v_2, \ldots, v_{i-1})$ has to be interpreted as the "empty span" $\operatorname{span}() = \left\{ \overrightarrow{0} \right\}$.]

Proof. We have assumed that $(v_1, v_2, \ldots, v_k)$ is linearly dependent. In other words, there exists some numbers $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$, **not all zero**, such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k = \overrightarrow{0}. \quad (8)$$

Consider such numbers $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$. [1]

There exists **at least one** $i \in \{1, 2, \ldots, k\}$ such that $\lambda_i \neq 0$ (because $\lambda_1, \lambda_2, \ldots, \lambda_k$ are not all zero). Consider the **largest** such i. Thus, $\lambda_i \neq 0$, but all of the numbers $\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_k$ are 0.

Now, (8) yields

$$\begin{aligned} \overrightarrow{0} &= \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k \\ &= (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_i v_i) + \underbrace{(\lambda_{i+1} v_{i+1} + \lambda_{i+2} v_{i+2} + \cdots + \lambda_k v_k)}_{\substack{= \overrightarrow{0} \\ \text{(since all of } \lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_k \text{ are 0)}}} \\ &= \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_i v_i \\ &= (\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{i-1} v_{i-1}) + \lambda_i v_i. \end{aligned}$$

[1] They are, of course, not uniquely determined – for instance, you could scale them all by 2, and they would still satisfy (8) – but one collection of such numbers will be enough for us.

Solving this for the $\lambda_i v_i$, we obtain

$$\lambda_i v_i = -\left(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{i-1} v_{i-1}\right).$$

We can divide this equality by λ_i (since ________________), and thus obtain

$$\begin{aligned} v_i &= \frac{-\left(\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{i-1} v_{i-1}\right)}{\lambda_i} \\ &= \frac{-\lambda_1}{\lambda_i} v_1 + \frac{-\lambda_2}{\lambda_i} v_2 + \cdots + \frac{-\lambda_{i-1}}{\lambda_i} v_{i-1} \\ &\in \operatorname{span}\left(v_1, v_2, \ldots, v_{i-1}\right). \end{aligned}$$

Thus, we have found some i satisfying $v_i \in \operatorname{span}(v_1, v_2, \ldots, v_{i-1})$. We now need to show that (7) holds as well.

We have $\{v_1, v_2, \ldots, v_{i-1}\} \subseteq \{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\}$ and thus

$$\operatorname{span}(v_1, v_2, \ldots, v_{i-1}) \subseteq \operatorname{span}(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$$

(by Proposition 0.3 **(a)**, with $v_1, v_2, \ldots, v_{i-1}$ and $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$ taking the roles of $v_1, v_2, \ldots, v_k$ and $w_1, w_2, \ldots, w_\ell$).

Let $U = \operatorname{span}(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$. Thus, U is a span, therefore a subspace of V. We have

$$v_i \in \operatorname{span}(v_1, v_2, \ldots, v_{i-1}) \subseteq \operatorname{span}(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) = U. \tag{9}$$

Now, all of the k vectors $v_1, v_2, \ldots, v_k$ lie in U (in fact, for the vector v_i it follows from (9), while for the others it follows from the fact that $\{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\} \subseteq \operatorname{span}(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) = U$). Thus, Proposition 0.1 **(c)** (applied to $u_i = v_i$) shows that

$$\operatorname{span}(v_1, v_2, \ldots, v_k) \subseteq U = \operatorname{span}(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k). \tag{10}$$

But Proposition 0.3 **(a)** (with ________________ and ________________ taking the roles of $v_1, v_2, \ldots, v_k$ and $w_1, w_2, \ldots, w_\ell$) shows that

$$\operatorname{span}(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) \subseteq \operatorname{span}(v_1, v_2, \ldots, v_k).$$

Combined with (10), this yields (7). This proves Proposition 0.4. □

Exercise 8. Let V be the vector space $\mathbb{R}^3$ of column vectors of size 3. Let v_1, v_2, v_3, v_4 be four vectors in V. As shown in class, v_1, v_2, v_3, v_4 must automatically be linearly dependent. Hence, by Proposition 0.4, there exists some $i \in \{1, 2, 3, 4\}$ satisfying $v_i \in \operatorname{span}(v_1, v_2, \ldots, v_{i-1})$.

(a) Find all such i if $v_1 = (1, 0, 0)^T$, $v_2 = (0, 1, 0)^T$, $v_3 = (0, 0, 1)^T$ and $v_4 = (1, 2, 3)^T$.

(b) Find all such i if $v_1 = (1, 1, 2)^T$, $v_2 = (2, 1, 3)^T$, $v_3 = (0, 1, 1)^T$ and $v_4 = (1, 2, 3)^T$. [**Hint:** There are two such i now.]

(c) Find all such i if $v_1 = (0, 0, 0)^T$, $v_2 = (1, 0, 1)^T$, $v_3 = (0, 1, 1)^T$ and $v_4 = (0, 0, 1)^T$. [**Hint:** $\operatorname{span}() = \left\{\overrightarrow{0}\right\}$.] [21 points]