Math 4242 Fall 2016 (Darij Grinberg): homework set 2

Exercise 1. Let $U = \begin{pmatrix} 6 & 3 & -2 & 5 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) Find all column vectors x of size 4 satisfying $Ux = b$, where $b = \begin{pmatrix} 1 \\ 5 \\ 2 \\ 0 \end{pmatrix}$.

(b) Find all column vectors x of size 4 satisfying $Ux = b'$, where $b' = \begin{pmatrix} 1 \\ 5 \\ 2 \\ 1 \end{pmatrix}$.

(c) Find all column vectors x of size 4 satisfying $Ux = x$.

Solution. The matrix U is in row-echelon form. Thus, systems of the form $Ux = a$ for a **constant** vector a can be solved by back-substitution. Parts **(a)** and **(b)** are such systems, so this is how we will solve them. Part **(c)** is slightly different.

(a) Writing x as $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, the equation $Ux = b$ rewrites as the system $\begin{cases} 6x_1 + 3x_2 + (-2)\,x_3 + 5x_4 = 1; \\ (-1)\,x_3 + 2x_4 = 5; \\ 1x_4 = 2; \\ 0 = 0 \end{cases}$. This system can be solved by back-substitution:
The fourth equation ($0 = 0$) is automatically satisfied; the third equation can be solved for x_4 (yielding $x_4 = 2$); the second equation can then be solved for x_3 using our already-obtained value of x_4 (yielding $x_3 = -1$); the lack of an equation with "leading variable" x_2 shows that x_2 will be a free variable (say, $x_2 = s$); finally, the first equation can be solved for x_1 using our already-obtained values for x_2, x_3, x_4 (this yields $x_1 = -\frac{1}{2}s - \frac{11}{6}$). Thus, the solution is

$$x = \begin{pmatrix} -\frac{1}{2}s - \frac{11}{6} \\ s \\ -1 \\ 2 \end{pmatrix}$$

with a free variable $s \in \mathbb{R}$.

(b) Writing x as $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, the equation $Ux = b'$ rewrites as the system

$\begin{cases} 6x_1 + 3x_2 + (-2)\, x_3 + 5x_4 = 1; \\ (-1)\, x_3 + 2x_4 = 5; \\ 1x_4 = 2; \\ 0 = 1 \end{cases}$. This system can be solved by back-substitution:
The fourth equation ($0 = 1$) is unsatisfiable, so **there are no solutions**.

(c) Writing x as $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, the equation $Ux = x$ rewrites as the system $\begin{cases} 6x_1 + 3x_2 + (-2)\, x_3 + 5x_4 = x_1; \\ (-1)\, x_3 + 2x_4 = x_2; \\ 1x_4 = x_3; \\ 0 = x_4 \end{cases}$. Bringing the x_1, x_2, x_3, x_4 onto the left hand sides transforms this into $\begin{cases} 5x_1 + 3x_2 + (-2)\, x_3 + 5x_4 = 0; \\ (-1)\, x_2 + (-1)\, x_3 + 2x_4 = 0; \\ (-1)\, x_3 + 1x_4 = 0; \\ (-1)\, x_4 = 0 \end{cases}$. This system can again be solved by back-substitution, leading to the only solution

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

[*Alternative solution for **(c)**:* Rewrite the equation $Ux = x$ as $Ux - x = 0_{4\times 1}$. Since $Ux - x = Ux - I_4 x = (U - I_4)\, x$, this can be furthermore rewritten as $(U - I_4)\, x = 0_{4\times 1}$. Set $A = U - I_4$. Then, our equation $(U - I_4)\, x = 0_{4\times 1}$ can be rewritten as $Ax = 0_{4\times 1}$. So we need to find all column vectors x satisfying $Ax = 0_{4\times 1}$.

But in my notes, there is the following theorem ([lina, Theorem 3.99]):

Theorem 0.1. Let $n \in \mathbb{N}$. Let A be an invertibly upper-triangular $n \times n$-matrix. Then, A is invertible, and its inverse A^{-1} is again invertibly upper-triangular.

The matrix

$$A = U - I_4 = \begin{pmatrix} 6 & 3 & -2 & 5 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 & -2 & 5 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is invertibly upper-triangular. Hence, Theorem 0.1 shows that A is invertible. Now, the equation $Ax = 0_{4\times 1}$ rewrites as $x = A^{-1} 0_{4\times 1}$, thus as $x = 0_{4\times 1}$ (since $A^{-1} 0_{4\times 1} = 0_{4\times 1}$). So the only solution is $x = 0_{4\times 1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.] □

Exercise 2. Let $A = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 2 & 4 & 1 \end{pmatrix}$.

(a) Find all column vectors x of size 4 satisfying $Ax = b$, where $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(b) Find all column vectors x of size 4 satisfying $Ax = b$, where $b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Solution. Any form of Gaussian elimination will do here. Let me do it the way we did in class:

Apply row operations to bring A into row echelon form:

$$\begin{pmatrix} 1 & 4 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 2 & 4 & 1 \end{pmatrix} \underset{A_{2,1}^{1}}{\overset{A_{2,1}^{-1}}{\rightleftarrows}} \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 2 & 4 & 1 \end{pmatrix} \underset{A_{3,1}^{1}}{\overset{A_{3,1}^{-1}}{\rightleftarrows}} \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \end{pmatrix}$$
$$\underset{A_{3,2}^{2}}{\overset{A_{3,2}^{-2}}{\rightleftarrows}} \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $A = EU$, where $E = A_{2,1}^{1} A_{3,1}^{1} A_{3,2}^{2}$ is a product of elementary matrices (thus invertible) and $U = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is in row-echelon form.

Now, $A = EU$, so that each 3×1-matrix b satisfies

$$Ax = b \iff EUx = b \iff Ux = E^{-1}b.$$

So what is E^{-1} ? Since $E = A_{2,1}^{1} A_{3,1}^{1} A_{3,2}^{2}$, we have $E^{-1} = A_{3,2}^{-2} A_{3,1}^{-1} A_{2,1}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$.

(More explicitly, E^{-1} is the matrix obtained from I_3 by doing precisely the same row operations that we applied to A to obtain U.)

(a) We must solve $Ax = b$, thus $Ux = E^{-1}b$. Since $U = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $E^{-1}b = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, this means that we must solve $\begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The solution is $x = \begin{pmatrix} 1 - r - 6s \\ s \\ s \\ r \end{pmatrix}$ with two free variables $r, s \in \mathbb{R}$.

(b) We must solve $Ax = b$, thus $Ux = E^{-1}b$. Since $U = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $E^{-1}b = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, this means that we must solve $\begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. There are no solutions. □

Exercise 3. Recall that the determinant of a 2×2-matrix is computed by the formula

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Use this to prove (by direct computation) that $\det(AB) = \det A \cdot \det B$ holds for all 2×2-matrices A and B.

Solution. Setting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, we have $AB = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}$ and thus

$$\begin{aligned} \det(AB) &= (ax+bz)(cy+dw) - (ay+bw)(cx+dz) \\ &= adwx - bcwx - adyz + bcyz. \end{aligned}$$

Comparing this with

$$\underbrace{\det A}_{=ad-bc} \cdot \underbrace{\det B}_{=xw-yz} = (ad-bc)(xw-yz) = adwx - bcwx - adyz + bcyz,$$

we confirm $\det(AB) = \det A \cdot \det B$. □

Exercise 4. I have mentioned in class that determinants of square matrices behave predictably under the standard row operations:

- The operation $A_{u,v}^{\lambda}$ preserves the determinant (that is, $\det\left(A_{u,v}^{\lambda}C\right) = \det C$ for any C).
- The operation S_u^{λ} multiplies the determinant by λ (that is, $\det\left(S_u^{\lambda}C\right) = \lambda \det C$ for any C).
- The operation $T_{u,v}$ negates the determinant (that is, $\det(T_{u,v}C) = -\det C$ for any C).

Also, I have mentioned that the determinant of a triangular matrix is the product of its diagonal entries.

> Compute
> $$\det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 7 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 \end{pmatrix}.$$
> (Mind the 7 in the upper-right corner!)

Solution. We perform row operations to our matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 7 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 \end{pmatrix} \xrightarrow{A_{2,1}^{-2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & -2\cdot 7 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 \end{pmatrix}$$
$$\xrightarrow{A_{3,2}^{-3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & -2\cdot 7 \\ 0 & 0 & 1 & 0 & 0 & 2\cdot 3\cdot 7 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 \end{pmatrix} \xrightarrow{A_{4,3}^{-4}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & -2\cdot 7 \\ 0 & 0 & 1 & 0 & 0 & 2\cdot 3\cdot 7 \\ 0 & 0 & 0 & 1 & 0 & -2\cdot 3\cdot 4\cdot 7 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 \end{pmatrix}$$
$$\xrightarrow{A_{5,4}^{-5}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & -2\cdot 7 \\ 0 & 0 & 1 & 0 & 0 & 2\cdot 3\cdot 7 \\ 0 & 0 & 0 & 1 & 0 & -2\cdot 3\cdot 4\cdot 7 \\ 0 & 0 & 0 & 0 & 1 & 2\cdot 3\cdot 4\cdot 5\cdot 7 \\ 0 & 0 & 0 & 0 & 6 & 1 \end{pmatrix} \xrightarrow{A_{6,5}^{-6}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & -2\cdot 7 \\ 0 & 0 & 1 & 0 & 0 & 2\cdot 3\cdot 7 \\ 0 & 0 & 0 & 1 & 0 & -2\cdot 3\cdot 4\cdot 7 \\ 0 & 0 & 0 & 0 & 1 & 2\cdot 3\cdot 4\cdot 5\cdot 7 \\ 0 & 0 & 0 & 0 & 0 & 1-2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 \end{pmatrix}.$$

All of these operations have preserved the determinant (since any operation $A_{u,v}^{\lambda}$ preserves the determinant). But the result is an upper-triangular matrix, whose determinant is therefore the product of its diagonal entries:

$$1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1\cdot (1-2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7) = -5039.$$

Hence, the determinant of our initial matrix must also be -5039. □

References

[lina] Darij Grinberg, *Notes on linear algebra*, version of 13 December 2016.
`https://github.com/darijgr/lina`