Math 4242, Section 070
Homework 1

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1) a. Matrix B is a 2×2 matrix.

b. Because A has the same number of columns as B has rows, the product AB is defined.

$$AB = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 1-1 & 2-6 \\ 2+0 & 4+0 \\ 3+5 & 6+30 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ 2 & 4 \\ 8 & 36 \end{pmatrix}$$

c. B does not have as many columns as A has rows, so the product BA is not defined.

2) a.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b. Let $S = a + b + c$, $S' = a' + b' + c'$ and $S'' = a'' + b'' + c''$. Then we have:

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} S & S & S \\ S' & S' & S' \\ S'' & S'' & S'' \end{pmatrix}$$

c.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+b+c+d \\ b+c+d \\ c+d \\ d \end{pmatrix}$$

3) a. Given that $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ we wish to find all 2×2 matrices B such that $AB = BA$. Let $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then we set up the matrix equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \implies \begin{pmatrix} x & y \\ 2z & 2w \end{pmatrix} = \begin{pmatrix} x & 2y \\ z & 2w \end{pmatrix}$$

This gives us a system of linear equations:

$$\begin{cases} x &= x \\ y &= 2y \\ 2z &= z \\ 2w &= 2w \end{cases}$$

Thus we can conclude that x and w can be arbitrary parameters a and d, and y and z must be equal to 0. Hence for the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, the family of 2×2 matrices B that satisfy $AB = BA$ are of the form

$B = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$.

b. Similar to part (a), we now have $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and we wish to find all 2×2 matrices B such that $AB = BA$. Let $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then we can set up the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \implies \begin{pmatrix} x+z & y+w \\ z & w \end{pmatrix} = \begin{pmatrix} x & x+y \\ z & z+w \end{pmatrix}$$

This gives us a system of linear equation:

$$\begin{cases} x+z &= x \\ y+w &= x+y \\ z &= z \\ w &= z+w \end{cases}$$

While the third equation implies that z can be a free parameter, the first and last equations show that $z = 0$. The second equation shows us that $x = w$, so they can both be a single parameter r. The second equation also tells us y is a free parameter, call it t. Thus all such B that satisfy $AB = BA$ are $B = \begin{pmatrix} r & t \\ 0 & r \end{pmatrix}$.

4) In order for a matrix B to be a left inverse of A, it must be a 1×2 matrix. Let $B = \begin{pmatrix} x & y \end{pmatrix}$. Then we need $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = (1)$. Thus we have $x + 4y = 1$. Letting y be a free parameter t, we have $x = 1 - 4t$. Thus whenever $B = \begin{pmatrix} 1-4t & t \end{pmatrix}$ it will be a left inverse of A.
In order for B to be a right inverse, it also has to be a 1×2 matrix. So we require $\begin{pmatrix} 1 \\ 4 \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x & y \\ 4x & 4y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Thus we have an inconsistent system, where we need $x = 0$ and $x = 1$. Hence we can conclude that there is no such B that is a right inverse of A.

5) Let A be an invertible $n \times n$ matrix with inverse A^{-1}. Let λ be a non-zero number. Then consider the matrix λA. We know that $\lambda A = \lambda I_n A$. We know that $(\lambda I_n)(\frac{1}{\lambda} I_n) = (\frac{1}{\lambda} I_n)(\lambda I_n) = I_n$, so λI_n is invertible with inverse $\frac{1}{\lambda} I_n$. Hence we have the product of two invertible matrices, λI_n and A. Hence $(\lambda I_n A)^{-1} = A^{-1}(\lambda I_n)^{-1} = A^{-1}\frac{1}{\lambda} I_n = \frac{1}{\lambda} A^{-1} I_n = \frac{1}{\lambda} A^{-1}$. $\square$

6) Let A be an invertible $n \times n$ matrix, $n \in \mathbb{N}$. We aim to show that A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$. Because A is invertible, we have $I_n = I_n^T = (AA^{-1})^T = (A^{-1})^T A^T$, and also $I_n = I_n^T = (A^{-1}A)^T = A^T (A^{-1})^T$. Because we have found a matrix that is both a left and right inverse of A^T, we have found that A^T is invertible with an inverse of $(A^{-1})^T$. Hence we also have $(A^T)^{-1} = (A^{-1})^T$. $\square$

7) Let $n \in \mathbb{N}$ and let A and B be two $n \times n$ lower-triangular matrices. We aim to show that the product AB is also a lower triangular matrix. By definition of a lower-triangular matrix, we know that $(A)_{i,j} = (B)_{i,j} = 0$ whenever $i < j$. Pick i and j such that $i, j \in \{1, 2, \ldots, n\}$ and $i < j$. We will show that $(A)_{i,k}(B)_{k,j} = 0$ for all $k \in \{1, 2, \ldots, n\}$. First we can consider the case where $k \geq j$. Then we have $i < j \leq k$ and hence $(A)_{i,k} = 0$. Otherwise, we have $k < j$. In this case we have $(B)_{k,j} = 0$. Thus each of these products is 0. However, we know that for any i, j such that $i < j$,

$$(AB)_{i,j} = \sum_{k=1}^{n} (A)_{i,k}(B)_{k,j} = 0$$

Thus whenever $i < j$ we have $(AB)_{i,j} = 0$, so AB must be lower-triangular. $\square$

8) Let $A = \begin{pmatrix} a & b & c \\ 0 & b' & c' \\ 0 & 0 & c'' \end{pmatrix}$ such that $a, b', c'' \neq 0$. We aim to compute A^{-1} in order to show that A is invertible. We will begin by computing a right inverse. We set up the equation

$$AX = \begin{pmatrix} a & b & c \\ 0 & b' & c' \\ 0 & 0 & c'' \end{pmatrix} \begin{pmatrix} x & x' & x'' \\ y & y' & y'' \\ z & z' & z'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This can be reduced to three systems of equations using the three columns of matrix X as follows:

$$\begin{cases} ax + by + cz = 1 \\ 0x + b'y + c'z = 0 \\ 0x + 0y + c''z = 0 \end{cases}$$

$$\begin{cases} ax' + by' + cz' = 0 \\ 0x' + b'y' + c'z' = 1 \\ 0x' + 0y' + c''z' = 0 \end{cases}$$

$$\begin{cases} ax'' + by'' + cz'' = 0 \\ 0x'' + b'y'' + c'z'' = 0 \\ 0x'' + 0y'' + c''z'' = 1 \end{cases}$$

We will begin by solving the first system. The third equation implies that $z = 0$. Thus we must also have that $y = 0$, and hence $x = \frac{1}{a}$.
Next we solve the second system. The third equation implies that $z' = 0$. Thus from the second equation $y' = \frac{1}{b'}$. And then we get that $x' = -\frac{b}{ab'}$.
Finally we solve the third system. The first equation implies that $z'' = \frac{1}{c''}$. Thus we get that $y'' = -\frac{c'}{b'c''}$. Substituting these into the first equation we obtain, upon some simplification, that $x'' = \frac{bc'-b'c}{ab'c''}$. Hence the right inverse of A is

$$X = \begin{pmatrix} \frac{1}{a} & -\frac{b}{ab'} & \frac{bc'-b'c}{ab'c''} \\ 0 & \frac{1}{b'} & -\frac{c'}{b'c''} \\ 0 & 0 & \frac{1}{c''} \end{pmatrix}$$

Now we wish to check that this is a left inverse of A.

$$XA = \begin{pmatrix} \frac{1}{a} & -\frac{b}{ab'} & \frac{bc'-b'c}{ab'c''} \\ 0 & \frac{1}{b'} & -\frac{c'}{b'c''} \\ 0 & 0 & \frac{1}{c''} \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & b' & c' \\ 0 & 0 & c'' \end{pmatrix} = \begin{pmatrix} \frac{a}{a} & \frac{b}{a} - \frac{bb'}{b'a} & \frac{c}{a} - \frac{bc'}{ab'} + \frac{c''(bc'-b'c)}{ab'c''} \\ 0 & \frac{b'}{b'} & \frac{c'}{b'} - \frac{c'c''}{b'c''} \\ 0 & 0 & \frac{c''}{c''} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that every term in the denominator was defined to be non-zero, so this matrix is defined. Thus we have shown that A is invertible and that its inverse is

$$X = A^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{ab'} & \frac{bc'-b'c}{ab'c''} \\ 0 & \frac{1}{b'} & -\frac{c'}{b'c''} \\ 0 & 0 & \frac{1}{c''} \end{pmatrix}$$