**Math 4242 Fall 2016 (Darij Grinberg): homework set 1**
**due: Thu, 22 Sep 2016, 23:00** (Minneapolis time) by email (`dgrinber@umn.edu`), or in class on 21 Sep 2016

**Exercise 1.** Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 1 & 6 \end{pmatrix}$.

**(a)** The matrix $A$ is of size $3 \times 2$. What is the size of $B$ ?
**(b)** Is $AB$ defined? If it is, compute it.
**(c)** Is $BA$ defined? If it is, compute it.

**Exercise 2.** **(a)** Compute

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**(b)** Compute $\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ for an arbitrary $3 \times 3$-matrix $\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix}$.

**(c)** Compute $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ for an arbitrary $4 \times 1$-matrix $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

**Exercise 3.** **(a)** Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Show that the $2 \times 2$-matrices $B$ satisfying $AB = BA$ are precisely the matrices of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ (where $a$ and $d$ are any numbers). [**Hint:** Set $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, and rewrite $AB = BA$ as a system of linear equations in $x, y, z, w$. Solve this system.]

**(b)** Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Characterize the $2 \times 2$-matrices $B$ satisfying $AB = BA$.

**Exercise 4.** Let $A = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. When is a matrix $B$ a left inverse of $A$ ? When is a matrix $B$ a right inverse of $A$ ?

**Proposition 0.1.** Let $n \in \mathbb{N}$. Let $\lambda$ be a nonzero number. Let $A$ be an invertible $n \times n$-matrix. Then, the matrix $\lambda A$ is also invertible, and its inverse is $(\lambda A)^{-1} = \lambda^{-1} A^{-1} = \frac{1}{\lambda} A^{-1}$.

**Exercise 5.** Prove Proposition 0.1.

**Proposition 0.2.** **(a)** Let $n \in \mathbb{N}$. Then, $(I_n)^T = I_n$.

**(b)** Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Then, $(0_{n \times m})^T = 0_{m \times n}$.

**(c)** Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $A$ be an $n \times m$-matrix. Let $\lambda$ be a number. Then, $(\lambda A)^T = \lambda A^T$.

**(d)** Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $A$ and $B$ be two $n \times m$-matrices. Then, $(A + B)^T = A^T + B^T$.

**(e)** Let $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $p \in \mathbb{N}$. Let $A$ be an $n \times m$-matrix. Let $B$ be an $m \times p$-matrix. Then, $(AB)^T = B^T A^T$.

**(f)** Let $n \in \mathbb{N}$. Let $A$ be an invertible $n \times n$-matrix. Then, $A^T$ is invertible, and its inverse is $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

**Exercise 6.** Prove Proposition 0.2 **(f)**. (You are free to use all the other parts of Proposition 0.2.)

**Theorem 0.3.** Let $n \in \mathbb{N}$. Let $A$ and $B$ be two lower-triangular $n \times n$-matrices.

**(a)** Then, $AB$ is a lower-triangular $n \times n$-matrix.

**(b)** The diagonal entries of $AB$ are

$$(AB)_{i,i} = A_{i,i} B_{i,i} \qquad \text{for all } i \in \{1, 2, \ldots, n\}.$$

**(c)** Also, $A + B$ is a lower-triangular $n \times n$-matrix. Furthermore, $\lambda A$ is a lower-triangular matrix whenever $\lambda$ is a number.

**Exercise 7.** Prove Theorem 0.3 **(a)**. (Feel free to repeat my proof of the analogous theorem for upper-triangular matrices (in the lecture notes), changing only what little needs to be changed. This is not plagiarism for the purpose of this exercise!)

(Similarly, you can prove Theorem 0.3 **(b)** and **(c)**, but you don't need to write it up.)

**Exercise 8.** Let $A = \begin{pmatrix} a & b & c \\ 0 & b' & c' \\ 0 & 0 & c'' \end{pmatrix}$ be an invertibly upper-triangular $3 \times 3$-matrix. ("Invertibly" means that $a, b', c''$ are nonzero.) Show that $A$ is invertible by explicitly computing the inverse of $A$ (in terms of $a, b, c, b', c', c''$).

[**Hint:** In order to find a right inverse of $A$, it is enough to find three column vectors $u, v, w$ (each of size 3) satisfying the equations

$$Au = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad Av = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad Aw = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In fact, once these vectors are found, assembling them into a matrix yields a right inverse of $A$ (why?). Find these $u, v, w$. Then, check that the resulting right-inverse of $A$ is also a left-inverse.]