A modulus and square root inequality
A problem by Darij Grinberg

For any four reals a, b, c, d, prove the inequality

$$|a+b-c-d| + |a+c-b-d| + |a+d-b-c|$$
$$\geq \left|\sqrt{a^2+b^2} - \sqrt{c^2+d^2}\right| + \left|\sqrt{a^2+c^2} - \sqrt{b^2+d^2}\right| + \left|\sqrt{a^2+d^2} - \sqrt{b^2+c^2}\right|.$$

Solution by Darij Grinberg.
We will show the following generalization of the problem:

> **Theorem 1.** Let V be an Euclidean space. Then, for any four vectors a, b, c, d in V, the inequality
>
> $$|a+b-c-d| + |a+c-b-d| + |a+d-b-c|$$
> $$\geq \left|\sqrt{a^2+b^2} - \sqrt{c^2+d^2}\right| + \left|\sqrt{a^2+c^2} - \sqrt{b^2+d^2}\right| + \left|\sqrt{a^2+d^2} - \sqrt{b^2+c^2}\right|$$
>
> holds.

Here, we are using the following notations:

- For any vector $v \in V$, we denote by $|v|$ the length (i. e., the Euclidean norm) of v.
- For any two vectors v and w in V, we denote by vw the scalar product of v and w. (Note that this is not really a multiplication, since vw is a scalar while v and w are vectors. In general, for three vectors u, v, w in an Euclidean space, we do *not* have $uv \cdot w = vw \cdot u$.)
- For any vector $v \in V$, we abbreviate vv as v^2.

Before we begin proving Theorem 1, we recapitulate two important facts about Euclidean spaces:

Cauchy-Schwarz inequality in vector form: If u and v are two vectors in an Euclidean space, then $|uv| \leq |u| \cdot |v|$.

Triangle inequality: If v and w are two vectors in an Euclidean space, then $|u| + |v| \geq |u+v|$.

Proof of Theorem 1. Denote

$$t = a+b+c+d;$$
$$x = a+b-c-d;$$
$$y = a+c-b-d;$$
$$z = a+d-b-c.$$

Then,

$$\begin{aligned}
tx+yz &= (a+b+c+d)(a+b-c-d)+(a+c-b-d)(a+d-b-c) \\
&= ((a+b)+(c+d))((a+b)-(c+d))+((a-b)+(c-d))((a-b)-(c-d)) \\
&= \left((a+b)^2-(c+d)^2\right)+\left((a-b)^2-(c-d)^2\right) \\
&= \left(\underbrace{(a+b)^2+(a-b)^2}_{\substack{=\left(a^2+2ab+b^2\right)+\left(a^2-2ab+b^2\right)\\ =2a^2+2b^2}}\right)-\left(\underbrace{(c+d)^2+(c-d)^2}_{\substack{=\left(c^2+2cd+d^2\right)+\left(c^2-2cd+d^2\right)\\ =2c^2+2d^2}}\right)=2\left(\left(a^2+b^2\right)-\left(c^2+d^2\right)\right),
\end{aligned}$$

so that

$$\left(a^2+b^2\right)-\left(c^2+d^2\right)=\frac{1}{2}(tx+yz). \qquad (1)$$

The triangle inequality, applied to the two vectors $(a+b,a-b)$ and $(c+d,c-d)$ in the Euclidean space $V\oplus V$ [1], yields

$$|(a+b,a-b)|+|(c+d,c-d)|\geq|(a+b,a-b)+(c+d,c-d)|.$$

Since

$$\begin{aligned}
|(a+b,a-b)| &= \sqrt{(a+b,a-b)^2}=\sqrt{(a+b)^2+(a-b)^2}; \\
|(c+d,c-d)| &= \sqrt{(c+d,c-d)^2}=\sqrt{(c+d)^2+(c-d)^2}; \\
|(a+b,a-b)+(c+d,c-d)| &= |((a+b)+(c+d),(a-b)+(c-d))| \\
&= \sqrt{((a+b)+(c+d),(a-b)+(c-d))^2} \\
&= \sqrt{((a+b)+(c+d))^2+((a-b)+(c-d))^2},
\end{aligned}$$

this rewrites as

$$\sqrt{(a+b)^2+(a-b)^2}+\sqrt{(c+d)^2+(c-d)^2}\geq\sqrt{((a+b)+(c+d))^2+((a-b)+(c-d))^2}.$$

But

$$\begin{aligned}
(a+b)^2+(a-b)^2 &= \left(a^2+2ab+b^2\right)+\left(a^2-2ab+b^2\right)=2a^2+2b^2=2\left(a^2+b^2\right); \\
(c+d)^2+(c-d)^2 &= \left(c^2+2cd+d^2\right)+\left(c^2-2cd+d^2\right)=2c^2+2d^2=2\left(c^2+d^2\right); \\
(a+b)+(c+d) &= a+b+c+d=t; \\
(a-b)+(c-d) &= a+c-b-d=y.
\end{aligned}$$

[1] The canonical scalar product on this space $V\oplus V$ is defined by $(e,f)(g,h)=eg+fh$ for any $(e,f)\in V\oplus V$ and $(g,h)\in V\oplus V$. In particular, we thus have $(e,f)^2=e^2+f^2$ for any $(e,f)\in V\oplus V$.

Hence, this becomes

$$\sqrt{2\left(a^2+b^2\right)}+\sqrt{2\left(c^2+d^2\right)} \geq \sqrt{t^2+y^2}.$$

But $\sqrt{t^2+y^2} \geq \frac{1}{\sqrt{2}}\left(|t|+|y|\right)$ [2]. Hence,

$$\sqrt{2\left(a^2+b^2\right)}+\sqrt{2\left(c^2+d^2\right)} \geq \frac{1}{\sqrt{2}}\left(|t|+|y|\right).$$

Dividing this by $\sqrt{2}$, we obtain

$$\sqrt{a^2+b^2}+\sqrt{c^2+d^2} \geq \underbrace{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}}_{=1/2}\left(|t|+|y|\right); \qquad \text{in other words,}$$

$$\sqrt{a^2+b^2}+\sqrt{c^2+d^2} \geq \frac{1}{2}\left(|t|+|y|\right). \tag{2}$$

By switching c with d (and, consequently, switching y with z) in the above argument, we can similarly prove

$$\sqrt{a^2+b^2}+\sqrt{d^2+c^2} \geq \frac{1}{2}\left(|t|+|z|\right), \qquad \text{what rewrites as}$$

$$\sqrt{a^2+b^2}+\sqrt{c^2+d^2} \geq \frac{1}{2}\left(|t|+|z|\right). \tag{3}$$

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[2] In fact, the AM-QM inequality yields $\sqrt{\frac{|t|^2+|y|^2}{2}} \geq \frac{|t|+|y|}{2}$. Thus, $\sqrt{t^2+y^2} = \sqrt{|t|^2+|y|^2} = \sqrt{2} \cdot \sqrt{\frac{|t|^2+|y|^2}{2}} \geq \sqrt{2} \cdot \frac{|t|+|y|}{2} = \frac{\sqrt{2}}{2}\left(|t|+|y|\right) = \frac{1}{\sqrt{2}}\left(|t|+|y|\right).$

Now,

$$\left|\sqrt{a^2+b^2}-\sqrt{c^2+d^2}\right| = \left|\frac{\left(\sqrt{a^2+b^2}-\sqrt{c^2+d^2}\right)\left(\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\right)}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}}\right|$$

$$= \left|\frac{(a^2+b^2)-(c^2+d^2)}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}}\right| = \left|\frac{\frac{1}{2}(tx+yz)}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}}\right| \qquad \text{(by (1))}$$

$$= \frac{1}{2}\cdot\frac{|tx+yz|}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}} \qquad \left(\text{since } \frac{1}{2} \text{ and } \sqrt{a^2+b^2}+\sqrt{c^2+d^2} \text{ are positive}\right)$$

$$\leq \frac{1}{2}\cdot\frac{|tx|+|yz|}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}} \qquad \left(\text{since } |tx+yz| \leq |tx|+|yz| \text{ by the triangle inequality}\right)$$

$$= \frac{1}{2}\cdot\left(\underbrace{\frac{|tx|}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}}}_{\leq\frac{|tx|}{\frac{1}{2}(|t|+|y|)}\text{ by (2)}} + \underbrace{\frac{|yz|}{\sqrt{a^2+b^2}+\sqrt{c^2+d^2}}}_{\leq\frac{|yz|}{\frac{1}{2}(|t|+|z|)}\text{ by (3)}}\right)$$

$$\leq \frac{1}{2}\cdot\left(\frac{|tx|}{\frac{1}{2}(|t|+|y|)}+\frac{|yz|}{\frac{1}{2}(|t|+|z|)}\right) = \frac{|tx|}{|t|+|y|}+\frac{|yz|}{|t|+|z|} \leq \frac{|t|\cdot|x|}{|t|+|y|}+\frac{|y|\cdot|z|}{|t|+|z|}$$

(since $|tx| \leq |t|\cdot|x|$ and $|yz| \leq |y|\cdot|z|$ by the Cauchy-Schwarz inequality in vector form)

$$= \frac{|t|}{|t|+|y|}\cdot|x| + \frac{|z|}{|t|+|z|}\cdot|y|. \qquad (4)$$

By cyclically permuting the variables b, c, and d (and, consequently, cyclically permuting x, y, and z) in the above argument, we can similarly prove the two inequalities

$$\left|\sqrt{a^2+c^2}-\sqrt{b^2+d^2}\right| \leq \frac{|t|}{|t|+|z|}\cdot|y| + \frac{|x|}{|t|+|x|}\cdot|z|; \qquad (5)$$

$$\left|\sqrt{a^2+d^2}-\sqrt{b^2+c^2}\right| \leq \frac{|t|}{|t|+|x|}\cdot|z| + \frac{|y|}{|t|+|y|}\cdot|x|. \qquad (6)$$

Now, the three inequalities (4), (5), (6) yield

$$\begin{aligned}
&\left|\sqrt{a^2+b^2}-\sqrt{c^2+d^2}\right|+\left|\sqrt{a^2+c^2}-\sqrt{b^2+d^2}\right|+\left|\sqrt{a^2+d^2}-\sqrt{b^2+c^2}\right| \\
&\leq \left(\frac{|t|}{|t|+|y|}\cdot|x|+\frac{|z|}{|t|+|z|}\cdot|y|\right)+\left(\frac{|t|}{|t|+|z|}\cdot|y|+\frac{|x|}{|t|+|x|}\cdot|z|\right)+\left(\frac{|t|}{|t|+|x|}\cdot|z|+\frac{|y|}{|t|+|y|}\cdot|x|\right) \\
&= \left(\frac{|t|}{|t|+|y|}\cdot|x|+\frac{|y|}{|t|+|y|}\cdot|x|\right)+\left(\frac{|t|}{|t|+|z|}\cdot|y|+\frac{|z|}{|t|+|z|}\cdot|y|\right)+\left(\frac{|t|}{|t|+|x|}\cdot|z|+\frac{|x|}{|t|+|x|}\cdot|z|\right) \\
&= \underbrace{\left(\frac{|t|}{|t|+|y|}+\frac{|y|}{|t|+|y|}\right)}_{=1}\cdot|x|+\underbrace{\left(\frac{|t|}{|t|+|z|}+\frac{|z|}{|t|+|z|}\right)}_{=1}\cdot|y|+\underbrace{\left(\frac{|t|}{|t|+|x|}+\frac{|x|}{|t|+|x|}\right)}_{=1}\cdot|z| \\
&= |x|+|y|+|z| = |a+b-c-d|+|a+c-b-d|+|a+d-b-c|\,,
\end{aligned}$$

and Theorem 1 is proven.