A problem on maxima and rearrangements

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Problem

Let $n \in \mathbb{N}$.

Let $a_1, a_2, ..., a_n$ be n nonnegative reals.

Let $b_1, b_2, ..., b_n$ be n nonnegative reals.

Let σ be a permutation of $\{1, 2, ..., n\}$.

For every $k \in \{1, 2, ..., n\}$, let $c_k = \max(\{a_1 b_k, a_2 b_k, ..., a_k b_k\} \cup \{a_k b_1, a_k b_2, ..., a_k b_k\})$.

Prove that

$$a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + ... + a_n b_{\sigma(n)} \leq c_1 + c_2 + ... + c_n.$$

Remark

1) By the rearrangement inequality, it is enough to prove this inequality when σ is the permutation (or, more precisely, one of the permutations) which makes the sequences $(a_1, a_2, ..., a_n)$ and $\left(b_{\sigma(1)}, b_{\sigma(2)}, ..., b_{\sigma(n)}\right)$ equally sorted (because if we treat $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ are constants, then this permutation σ maximizes the left hand side $a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + ... + a_n b_{\sigma(n)}$ of our inequality, whereas the right hand side is constant). But I don't think this helps in solving the problem. It is actually getting the cart before the horse: The rearrangement inequality can be derived from our problem (see the remark after the solution for details).

2) The problem was conceived by me as a lemma to prove the "combinatorialist's Chebyshev inequality", which states that $(a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n) \leq n(c_1 + c_2 + ... + c_n)$ (under the conditions of the problem) and is due to Ahlswede and Blinovsky ([AhlBli08, Lecture 15, Consequences of Theorem 33, no. 2]). This inequality can be derived from our problem by summing the left hand side over $\sigma \in C_n$ (where C_n denotes the subgroup of the symmetric group S_n formed by all *cyclic* permutations). We leave the details to the reader. The proof in [AhlBli08] is completely different.

Solution of the problem

We begin with a simple lemma:

Lemma 1. Let a, a', b and b' be four nonnegative reals. Then,

$$ab' + a'b \leq ab + \max\left\{ab', a'b', a'b\right\}. \qquad (1)$$

(Note that this lemma 1 is the particular case of our problem when $n = 2$ and $\sigma = (1,2)$.)

Proof of Lemma 1. We distinguish between three cases:
Case 1. We have $a \geq a'$.
Case 2. We have $b \geq b'$.
Case 3. We have neither $a \geq a'$ nor $b \geq b'$.
In Case 1, we have $a \geq a'$ and thus

$$\underbrace{ab'}_{\leq \max\{ab', a'b', a'b\}} + \underbrace{a'}_{\leq a} b \leq \max\left\{ab', a'b', a'b\right\} + ab = ab + \max\left\{ab', a'b', a'b\right\}.$$

Thus, (1) is proven in Case 1.
In Case 2, we have $b \geq b'$ and thus

$$a \underbrace{b'}_{\leq b} + \underbrace{a'b}_{\leq \max\{ab', a'b', a'b\}} \leq ab + \max\left\{ab', a'b', a'b\right\}.$$

Thus, (1) is proven in Case 2.
Now let us consider Case 3. In this Case, we have neither $a \geq a'$ nor $b \geq b'$. Thus, $a < a'$ and $b < b'$. Hence, $\underbrace{a}_{<a'} - a' < a' - a' = 0$ and $\underbrace{b}_{<b'} - b' < b' - b' = 0$. Therefore, $(a - a')(b - b') > 0$ (since the product of two negative reals must always be positive), so that

$$0 < \left(a - a'\right)\left(b - b'\right) = ab + \underbrace{a'b'}_{\leq \max\{ab', a'b', a'b\}} - ab' - a'b$$
$$\leq ab + \max\left\{ab', a'b', a'b\right\} - ab' - a'b.$$

This rewrites as

$$ab' + a'b \leq ab + \max\left\{ab', a'b', a'b\right\}.$$

Thus, (1) is proven in Case 3.
We thus have proven (1) in all Cases 1, 2 and 3. Since these cases are clearly the only possible cases to occur, this shows that (1) always holds. This proves Lemma 1. $\square$

Now, we need a trivial lemma from combinatorics:

Lemma 2. Let N be a positive integer, and let σ be a permutation of $\{1,2,...,N\}$ such that $\sigma(N) = N$. Then, there exists a permutation τ of $\{1,2,...,N-1\}$ such that

$$(\sigma(i) = \tau(i) \qquad \text{for every } i \in \{1,2,...,N-1\}). \tag{2}$$

Proof of Lemma 2. For every $i \in \{1,2,...,N-1\}$, we have $\sigma(i) \in \{1,2,...,N-1\}$ [1]. Hence, we can define a map $\tau : \{1,2,...,N-1\} \to \{1,2,...,N-1\}$ by

$$(\tau(u) = \sigma(u) \text{ for every } u \in \{1,2,...,N-1\})).$$

This map τ is injective[2] and surjective[3], thus bijective. Hence, τ is a permutation of $\{1,2,...,N-1\}$. Every $i \in \{1,2,...,N-1\}$ satisfies $\tau(i) = \sigma(i)$ (by the definition of τ). This proves Lemma 2. □

Now, we come to the actual *solution of the problem*:

Solution of the problem. We will solve the problem by induction over n:

Induction base: In the case $n = 0$, the problem is evidently true.[4] This completes the induction base.

Induction step: Fix some integer $N \geq 1$. Assume that the problem has already been solved for $n = N-1$. Now we need to solve the problem for $n = N$.

Let $a_1, a_2, ..., a_N$ be N nonnegative reals.

Let $b_1, b_2, ..., b_N$ be N nonnegative reals.

Let σ be a permutation of $\{1,2,...,N\}$.

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[1] *Proof.* Let $i \in \{1,2,...,N-1\}$. Then, $i \leq N-1 < N$, so that $i \neq N$, so that $\sigma(i) \neq \sigma(N)$ (because σ is a permutation and thus injective). Combining $\sigma(i) \in \{1,2,\ldots,N\}$ with $\sigma(i) \neq \sigma(N) = N$, we obtain $\sigma(i) \in \{1,2,\ldots,N\} \setminus \{N\} = \{1,2,...,N-1\}$, qed.

[2] *Proof.* Let u and v be two elements of $\{1,2,...,N-1\}$ such that $\tau(u) = \tau(v)$. Then, by the definition of τ, we have $\tau(u) = \sigma(u)$ and $\tau(v) = \sigma(v)$, so that $\sigma(u) = \tau(u) = \tau(v) = \sigma(v)$, so that $u = v$ (since σ is a permutation and thus injective). Thus, we have shown that any two elements u and v of $\{1,2,...,N-1\}$ such that $\tau(u) = \tau(v)$ must satisfy $u = v$. In other words, τ is injective, qed.

[3] *Proof.* Let $p \in \{1,2,...,N-1\}$. Then, $p \leq N-1 < N$. Moreover, $p \in \{1,2,...,N-1\} \subseteq \{1,2,...,N\}$, so that $\sigma^{-1}(p)$ is well-defined. We have $\sigma^{-1}(p) \neq N$ (because if $\sigma^{-1}(p)$ were $= N$, then p would be $= \sigma(N) = N$, contradicting $p < N$). Combining $\sigma^{-1}(p) \in \{1,2,...,N\}$ with $\sigma^{-1}(p) \neq N$, we obtain $\sigma^{-1}(p) \in \{1,2,...,N\} \setminus \{N\} = \{1,2,...,N-1\}$. Thus, $\tau(\sigma^{-1}(p))$ is well-defined. By the definition of τ, we have $\tau(\sigma^{-1}(p)) = \sigma(\sigma^{-1}(p)) = p$, and thus $p = \tau(\sigma^{-1}(p)) \in \tau(\{1,2,...,N-1\})$.

So we have proven that every $p \in \{1,2,...,N-1\}$ satisfies $p \in \tau(\{1,2,...,N-1\})$. In other words, τ is surjective.

[4] *Proof.* In the case $n = 0$, the inequality

$$a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + ... + a_n b_{\sigma(n)} \leq c_1 + c_2 + ... + c_n$$

takes the form $0 \leq 0$, which is obviously true. Thus, in the case $n = 0$, the problem is true.

For every $k \in \{1, 2, ..., N\}$, let $c_k = \max\left(\{a_1 b_k, a_2 b_k, ..., a_k b_k\} \cup \{a_k b_1, a_k b_2, ..., a_k b_k\}\right)$.
We must then prove that

$$a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + ... + a_N b_{\sigma(N)} \leq c_1 + c_2 + ... + c_N. \tag{3}$$

We distinguish between two cases:
Case 1: We have $\sigma(N) = N$.
Case 2: We have $\sigma(N) \neq N$.
First, let us consider Case 1. In this case, $\sigma(N) = N$, so that Lemma 2 yields that there exists a permutation τ of $\{1, 2, ..., N-1\}$ such that

$$(\sigma(i) = \tau(i) \qquad \text{for every } i \in \{1, 2, ..., N-1\}). \tag{4}$$

Consider this τ.
We assumed that the problem has already been solved for $n = N - 1$. Hence, we can apply the problem to τ and $N - 1$ instead of σ and n, and obtain

$$a_1 b_{\tau(1)} + a_2 b_{\tau(2)} + ... + a_{N-1} b_{\tau(N-1)} \leq c_1 + c_2 + ... + c_{N-1}. \tag{5}$$

The definition of c_N yields

$$\begin{aligned} c_N &= \max\left(\{a_1 b_N, a_2 b_N, ..., a_N b_N\} \cup \{a_N b_1, a_N b_2, ..., a_N b_N\}\right) \\ &\geq a_N b_{\sigma(N)} \end{aligned} \tag{6}$$

(since $a_N b_{\sigma(N)} \in \{a_N b_1, a_N b_2, ..., a_N b_N\} \subseteq \{a_1 b_N, a_2 b_N, ..., a_N b_N\} \cup \{a_N b_1, a_N b_2, ..., a_N b_N\}$).
We can rewrite (4) as follows: $\tau(1) = \sigma(1)$, $\tau(2) = \sigma(2)$, ..., $\tau(N-1) = \sigma(N-1)$. Thus, (5) becomes

$$a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + ... + a_{N-1} b_{\sigma(N-1)} \leq c_1 + c_2 + ... + c_{N-1}.$$

Adding this inequality to the inequality $a_N b_{\sigma(N)} \leq c_N$ (which follows from (6)), we obtain

$$a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + ... + a_{N-1} b_{\sigma(N-1)} + a_N b_{\sigma(N)} \leq c_1 + c_2 + ... + c_{N-1} + c_N.$$

In other words, (3) holds. We have thus proven (3) in Case 1.
Next, let us consider Case 2. In this case, $\sigma(N) \neq N$.
Let $j = \sigma^{-1}(N)$. Thus, $j = \sigma^{-1}(N) \neq N$, so that $j \in \{1, 2, ..., N-1\}$.
Let η be the permutation $\sigma \circ (N, j)$ of $\{1, 2, ..., N\}$ (where (N, j) is the transposition which transposes N with j). Then,

$$\eta(N) = (\sigma \circ (N, j))(N) = \sigma \underbrace{((N, j)(N))}_{\substack{=j \\ \text{(by the definition of } (N,j))}} = \sigma(j) = N$$

(since $j = \sigma^{-1}(N)$). Hence, Lemma 2 (applied to η instead of σ) yields that there exists a permutation τ of $\{1, 2, ..., N-1\}$ such that

$$(\eta(i) = \tau(i) \qquad \text{for every } i \in \{1, 2, ..., N-1\}). \tag{7}$$

Consider this τ.

We assumed that the problem has already been solved for $n = N-1$. Hence, we can apply the problem to τ and $N-1$ instead of σ and n, and obtain

$$a_1 b_{\tau(1)} + a_2 b_{\tau(2)} + ... + a_{N-1} b_{\tau(N-1)} \leq c_1 + c_2 + ... + c_{N-1}. \tag{8}$$

But for every $i \in \{1,2,...,N-1\} \setminus \{j\}$, we have

$$\begin{aligned}
&\tau(i) \\
&= \eta(i) \qquad \left(\text{by (7), since } i \in \{1,2,...,N-1\} \setminus \{j\} \subseteq \{1,2,...,N-1\}\right) \\
&= (\sigma \circ (N,j))(i) \qquad \left(\text{since } \eta = \sigma \circ (N,j)\right) \\
&= \sigma((N,j)(i)) = \sigma(i) \qquad (9) \\
&\left(\begin{array}{c}\text{since} \\ i \in \underbrace{\{1,2,...,N-1\}}_{=\{1,2,...,N\}\setminus\{N\}} \setminus \{j\} = (\{1,2,...,N\} \setminus \{N\}) \setminus \{j\} = \{1,2,...,N\} \setminus \{N,j\}, \\ \text{and thus } i \notin \{N,j\}, \text{ so that } (N,j)(i) = i \text{ (by the definition of } (N,j)\text{)}\end{array}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\tau(j) &= \eta(j) \qquad \left(\text{by (7), applied to } i = j \text{ (since } j \in \{1,2,...,N-1\}\text{)}\right) \\
&= (\sigma \circ (N,j))(j) \qquad \left(\text{since } \eta = \sigma \circ (N,j)\right) \\
&= \sigma\left(\underbrace{(N,j)(j)}_{\substack{=N \\ \text{(by the definition of } (N,j))}}\right) = \sigma(N).
\end{aligned}$$

Now,

$$\begin{aligned}
&a_1 b_{\tau(1)} + a_2 b_{\tau(2)} + ... + a_{N-1} b_{\tau(N-1)} \\
&= \sum_{i \in \{1,2,...,N-1\}} a_i b_{\tau(i)} = a_j \underbrace{b_{\tau(j)}}_{\substack{=b_{\sigma(N)} \\ \text{(since } \tau(j) = \sigma(N))}} + \sum_{i \in \{1,2,...,N-1\} \setminus \{j\}} a_i \underbrace{b_{\tau(i)}}_{\substack{=b_{\sigma(i)} \\ \text{(by (9))}}} \\
&\qquad \left(\text{since } j \in \{1,2,...,N-1\}\right) \\
&= a_j b_{\sigma(N)} + \sum_{i \in \{1,2,...,N-1\} \setminus \{j\}} a_i b_{\sigma(i)}.
\end{aligned}$$

Hence, (8) rewrites as

$$a_j b_{\sigma(N)} + \sum_{i \in \{1,2,...,N-1\} \setminus \{j\}} a_i b_{\sigma(i)} \leq c_1 + c_2 + ... + c_{N-1}. \tag{10}$$

But

$$\begin{aligned}
& a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + \dots + a_N b_{\sigma(N)} \\
&= \underbrace{\left(a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + \dots + a_{N-1} b_{\sigma(N-1)}\right)}_{\substack{= \sum_{i\in\{1,2,\dots,N-1\}} a_i b_{\sigma(i)} = a_j b_{\sigma(j)} + \sum_{i\in\{1,2,\dots,N-1\}\setminus\{j\}} a_i b_{\sigma(i)} \\ \text{(since } j\in\{1,2,\dots,N-1\})}} + a_N b_{\sigma(N)} \\
&= a_j \underbrace{b_{\sigma(j)}}_{\substack{= b_N \text{ (since } \sigma(j)=N)}} + \sum_{i\in\{1,2,\dots,N-1\}\setminus\{j\}} a_i b_{\sigma(i)} + a_N b_{\sigma(N)} \\
&= a_j b_N + \sum_{i\in\{1,2,\dots,N-1\}\setminus\{j\}} a_i b_{\sigma(i)} + a_N b_{\sigma(N)} \\
&= \underbrace{a_j b_N + a_N b_{\sigma(N)}}_{\substack{\leq a_j b_{\sigma(N)} + \max\left\{a_j b_N, a_N b_N, a_N b_{\sigma(N)}\right\} \\ \text{(by Lemma 1, applied to } a=a_j,\ a'=a_N, \\ b=b_{\sigma(N)} \text{ and } b'=b_N)}} + \sum_{i\in\{1,2,\dots,N-1\}\setminus\{j\}} a_i b_{\sigma(i)} \\
&\leq a_j b_{\sigma(N)} + \max\left\{a_j b_N, a_N b_N, a_N b_{\sigma(N)}\right\} + \sum_{i\in\{1,2,\dots,N-1\}\setminus\{j\}} a_i b_{\sigma(i)} \\
&= \underbrace{a_j b_{\sigma(N)} + \sum_{i\in\{1,2,\dots,N-1\}\setminus\{j\}} a_i b_{\sigma(i)}}_{\substack{\leq c_1 + c_2 + \dots + c_{N-1} \\ \text{(by (10))}}} + \max\left\{a_j b_N, a_N b_N, a_N b_{\sigma(N)}\right\} \\
&\leq c_1 + c_2 + \dots + c_{N-1} + \max\left\{a_j b_N, a_N b_N, a_N b_{\sigma(N)}\right\}. \qquad (11)
\end{aligned}$$

But $\left\{a_j b_N, a_N b_N, a_N b_{\sigma(N)}\right\} \subseteq \{a_1 b_N, a_2 b_N, \dots, a_N b_N\} \cup \{a_N b_1, a_N b_2, \dots, a_N b_N\}$ [5], so that

$$\begin{aligned}
\max\left\{a_j b_N, a_N b_N, a_N b_{\sigma(N)}\right\} &\leq \max\left(\{a_1 b_N, a_2 b_N, \dots, a_N b_N\} \cup \{a_N b_1, a_N b_2, \dots, a_N b_N\}\right) \\
&= c_N
\end{aligned}$$

(since c_N is defined as $\max\left(\{a_1 b_N, a_2 b_N, \dots, a_N b_N\} \cup \{a_N b_1, a_N b_2, \dots, a_N b_N\}\right)$). Hence,

[5] This is because

$$\begin{aligned}
a_j b_N &\in \{a_1 b_N, a_2 b_N, \dots, a_N b_N\} \subseteq \{a_1 b_N, a_2 b_N, \dots, a_N b_N\} \cup \{a_N b_1, a_N b_2, \dots, a_N b_N\}; \\
a_N b_N &\in \{a_1 b_N, a_2 b_N, \dots, a_N b_N\} \subseteq \{a_1 b_N, a_2 b_N, \dots, a_N b_N\} \cup \{a_N b_1, a_N b_2, \dots, a_N b_N\}; \\
a_N b_{\sigma(N)} &\in \{a_N b_1, a_N b_2, \dots, a_N b_N\} \subseteq \{a_1 b_N, a_2 b_N, \dots, a_N b_N\} \cup \{a_N b_1, a_N b_2, \dots, a_N b_N\}.
\end{aligned}$$

(11) becomes

$$a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + ... + a_N b_{\sigma(N)} \leq c_1 + c_2 + ... + c_{N-1} + \underbrace{\max \left\{ a_j b_N, a_N b_N, a_N b_{\sigma(N)} \right\}}_{\leq c_N}$$
$$\leq c_1 + c_2 + ... + c_{N-1} + c_N = c_1 + c_2 + ... + c_N.$$

In other words, (3) holds. We have thus proven (3) in Case 2.

Since the cases 1 and 2 are the only possible cases, and since we have proven (3) in both of these cases, we conclude that (3) is true. We thus have shown that $a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + ... + a_N b_{\sigma(N)} \leq c_1 + c_2 + ... + c_N$. But this means that we have solved the problem for $n = N$. This completes the induction step.

Thus, the induction proof of our problem is complete. □

Remark

The problem that we just solved generalizes the rearrangement inequality. To see why, here is one possible form of the rearrangement inequality:

> **Corollary 3.** Let $n \in \mathbb{N}$.
>
> Let $a_1, a_2, ..., a_n$ be n reals such that $a_1 \leq a_2 \leq ... \leq a_n$.
>
> Let $b_1, b_2, ..., b_n$ be n reals such that $b_1 \leq b_2 \leq ... \leq b_n$.
>
> Let σ be a permutation of $\{1, 2, ..., n\}$.
>
> Then,
>
> $$a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + ... + a_n b_{\sigma(n)} \leq a_1 b_1 + a_2 b_2 + ... + a_n b_n.$$

Proof of Corollary 3. Let $\alpha = \min \{a_1, a_2, ..., a_n\}$ and $\beta = \min \{b_1, b_2, ..., b_n\}$.

For every $k \in \{1, 2, ..., n\}$, let $a'_k = a_k - \alpha$ and $b'_k = b_k - \beta$. Then, every $k \in \{1, 2, ..., n\}$ satisfies $a'_k = a_k - \underbrace{\alpha}_{=\min\{a_1, a_2, ..., a_n\} \leq a_k} \geq a_k - a_k = 0$. In other words, a'_1, a'_2, ..., a'_n are n nonnegative reals. Similarly, b'_1, b'_2, ..., b'_n are n nonnegative reals, i.e., every $k \in \{1, 2, ..., n\}$ satisfies $b'_k \geq 0$.

For every $k \in \{1, 2, ..., n\}$, let

$$c'_k = \max \left(\left\{ a'_1 b'_k, a'_2 b'_k, ..., a'_k b'_k \right\} \cup \left\{ a'_k b'_1, a'_k b'_2, ..., a'_k b'_k \right\} \right).$$

Then, the problem that we have solved (applied to $(a'_1, a'_2, ..., a'_n)$, $(b'_1, b'_2, ..., b'_n)$ and $(c'_1, c'_2, ..., c'_n)$ instead of $(a_1, a_2, ..., a_n)$, $(b_1, b_2, ..., b_n)$ and $(c_1, c_2, ..., c_n)$) yields

$$a'_1 b'_{\sigma(1)} + a'_2 b'_{\sigma(2)} + ... + a'_n b'_{\sigma(n)} \leq c'_1 + c'_2 + ... + c'_n. \qquad (12)$$

But now, it is easy to see that every $k \in \{1, 2, ..., n\}$ satisfies

$$c'_k \leq a'_k b'_k \qquad (13)$$

[6]. Hence, it is almost trivial that every $k \in \{1, 2, ..., n\}$ satisfies $c'_k = a'_k b'_k$ [7]. Thus, $\sum\limits_{k=1}^{n} c'_k = \sum\limits_{k=1}^{n} a'_k b'_k$.

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[6] *Proof of (13).* Let $k \in \{1, 2, ..., n\}$. By definition of c'_k, we have

$$\begin{aligned} c'_k &= \max\left(\left\{a'_1 b'_k, a'_2 b'_k, ..., a'_k b'_k\right\} \cup \left\{a'_k b'_1, a'_k b'_2, ..., a'_k b'_k\right\}\right) \\ &\in \left\{a'_1 b'_k, a'_2 b'_k, ..., a'_k b'_k\right\} \cup \left\{a'_k b'_1, a'_k b'_2, ..., a'_k b'_k\right\} \end{aligned}$$

(since the maximum of a set always belongs to that set). Thus, either $c'_k \in \left\{a'_1 b'_k, a'_2 b'_k, ..., a'_k b'_k\right\}$ or $c'_k \in \left\{a'_k b'_1, a'_k b'_2, ..., a'_k b'_k\right\}$ (or both). In other words, we must be in one of the following two cases:

Case 1. We have $c'_k \in \left\{a'_1 b'_k, a'_2 b'_k, ..., a'_k b'_k\right\}$.

Case 2. We have $c'_k \in \left\{a'_k b'_1, a'_k b'_2, ..., a'_k b'_k\right\}$.

Let us first consider Case 1. In this case, we have $c'_k \in \left\{a'_1 b'_k, a'_2 b'_k, ..., a'_k b'_k\right\}$, so that there exists an $i \in \{1, 2, ..., k\}$ such that $c'_k = a'_i b'_k$. Consider such an i. Then, $i \in \{1, 2, ..., k\}$, so that $i \leq k$ and thus $a_i \leq a_k$ (since $a_1 \leq a_2 \leq ... \leq a_n$) and

$$\begin{aligned} a'_i &= \underbrace{a_i}_{\leq a_k} - \alpha && \left(\text{by the definition of } a'_i\right) \\ &\leq a_k - \alpha = a'_k && \left(\text{since } a'_k = a_k - \alpha \text{ by the definition of } a'_k\right). \end{aligned}$$

Since $b'_k \geq 0$, this yields $a'_i b'_k \leq a'_k b'_k$. Thus, $c'_k = a'_i b'_k \leq a'_k b'_k$. So we have proven $c'_k \leq a'_k b'_k$ in Case 1.

Now let us consider Case 2. In this case, we have $c'_k \in \left\{a'_k b'_1, a'_k b'_2, ..., a'_k b'_k\right\}$, so that there exists an $i \in \{1, 2, ..., k\}$ such that $c'_k = a'_k b'_i$. Consider such an i. Then, $i \in \{1, 2, ..., k\}$, so that $i \leq k$ and thus $b_i \leq b_k$ (since $b_1 \leq b_2 \leq ... \leq b_n$) and

$$\begin{aligned} b'_i &= \underbrace{b_i}_{\leq b_k} - \beta && \left(\text{by the definition of } b'_i\right) \\ &\leq b_k - \beta = b'_k && \left(\text{since } b'_k = b_k - \beta \text{ by the definition of } b'_k\right). \end{aligned}$$

Since $a'_k \geq 0$, this yields $a'_k b'_i \leq a'_k b'_k$. Thus, $c'_k = a'_k b'_i \leq a'_k b'_k$. So we have proven $c'_k \leq a'_k b'_k$ in Case 2.

Thus, we have proven $c'_k \leq a'_k b'_k$ in each of the two cases 1 and 2. Since these two cases are the only possible cases, this yields that $c'_k \leq a'_k b'_k$ always holds. This proves (13).

[7] *Proof.* Let $k \in \{1, 2, ..., n\}$. Then,

$$a'_k b'_k \in \left\{a'_1 b'_k, a'_2 b'_k, ..., a'_k b'_k\right\} \subseteq \left\{a'_1 b'_k, a'_2 b'_k, ..., a'_k b'_k\right\} \cup \left\{a'_k b'_1, a'_k b'_2, ..., a'_k b'_k\right\}.$$

Since every element of a set is always $\leq$ to the maximum of that set, this yields

$$a'_k b'_k \leq \max\left(\left\{a'_1 b'_k, a'_2 b'_k, ..., a'_k b'_k\right\} \cup \left\{a'_k b'_1, a'_k b'_2, ..., a'_k b'_k\right\}\right) = c'_k.$$

Combined with (13), this yields $c'_k = a'_k b'_k$, qed.

Now, (12) can be rewritten as

$$\begin{aligned}
&0 \\
&\geq \underbrace{\left(a'_1 b'_{\sigma(1)} + a'_2 b'_{\sigma(2)} + \dots + a'_n b'_{\sigma(n)}\right)}_{=\sum_{k=1}^{n} a'_k b'_{\sigma(k)}} - \underbrace{\left(c'_1 + c'_2 + \dots + c'_n\right)}_{=\sum_{k=1}^{n} c'_k = \sum_{k=1}^{n} a'_k b'_k} \\
&= \sum_{k=1}^{n} a'_k b'_{\sigma(k)} - \sum_{k=1}^{n} a'_k b'_k \\
&= \sum_{k=1}^{n} \underbrace{a'_k}_{\substack{=a_k - \alpha \\ \text{(by the definition of } a'_k)}} \left(\underbrace{b'_{\sigma(k)}}_{\substack{=b_{\sigma(k)} - \beta \\ \text{(by the definition of } b'_{\sigma(k)})}} - \underbrace{b'_k}_{\substack{=b_k - \beta \\ \text{(by the definition of } b'_k)}} \right) \\
&= \sum_{k=1}^{n} (a_k - \alpha) \underbrace{\left(\left(b_{\sigma(k)} - \beta\right) - (b_k - \beta)\right)}_{=b_{\sigma(k)} - b_k} = \sum_{k=1}^{n} \underbrace{(a_k - \alpha)\left(b_{\sigma(k)} - b_k\right)}_{=a_k\left(b_{\sigma(k)} - b_k\right) - \alpha\left(b_{\sigma(k)} - b_k\right)} \\
&= \sum_{k=1}^{n} \left(a_k \left(b_{\sigma(k)} - b_k\right) - \alpha \left(b_{\sigma(k)} - b_k\right)\right) = \sum_{k=1}^{n} a_k \left(b_{\sigma(k)} - b_k\right) - \alpha \sum_{k=1}^{n} \left(b_{\sigma(k)} - b_k\right).
\end{aligned} \tag{14}$$

But since σ is bijective (since σ is a permutation), we have $\sum_{k \in \{1,2,\dots,n\}} b_{\sigma(k)} = \sum_{k \in \{1,2,\dots,n\}} b_k$ (here, we substituted k for $\sigma(k)$ in the sum, since σ is bijective), so that

$$\begin{aligned}
\sum_{k=1}^{n} \left(b_{\sigma(k)} - b_k\right) &= \underbrace{\sum_{k=1}^{n} b_{\sigma(k)}}_{= \sum_{k \in \{1,2,\dots,n\}} b_{\sigma(k)} = \sum_{k \in \{1,2,\dots,n\}} b_k} - \underbrace{\sum_{k=1}^{n} b_k}_{= \sum_{k \in \{1,2,\dots,n\}} b_k} \\
&= \sum_{k \in \{1,2,\dots,n\}} b_k - \sum_{k \in \{1,2,\dots,n\}} b_k = 0.
\end{aligned}$$

Thus, (14) becomes

$$\begin{aligned}
&0\\
&\geq \sum_{k=1}^{n} a_k \left(b_{\sigma(k)} - b_k\right) - \alpha \underbrace{\sum_{k=1}^{n} \left(b_{\sigma(k)} - b_k\right)}_{=0} = \sum_{k=1}^{n} a_k \left(b_{\sigma(k)} - b_k\right) - \alpha 0\\
&= \sum_{k=1}^{n} a_k \left(b_{\sigma(k)} - b_k\right) = \underbrace{\sum_{k=1}^{n} a_k b_{\sigma(k)}}_{=a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + \ldots + a_n b_{\sigma(n)}} - \underbrace{\sum_{k=1}^{n} a_k b_k}_{=a_1 b_1 + a_2 b_2 + \ldots + a_n b_n}\\
&= \left(a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + \ldots + a_n b_{\sigma(n)}\right) - \left(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n\right).
\end{aligned}$$

In other words,

$$a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + \ldots + a_n b_{\sigma(n)} \leq a_1 b_1 + a_2 b_2 + \ldots + a_n b_n.$$

This proves Corollary 3. □

References

[AhlBli08] Rudolf Ahlswede, Vladimir Blinovsky, *Lectures on Advances in Combinatorics*, Springer 2008.