Witt vectors. Part 1

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Witt#4a: Equigraded power series

[not completed, not proofread]

This little note is there to prove an easy lemma used in [1]. This lemma is about *equigraded power series*. First, let us define this notion:

Definition 1. Let A be a graded commutative ring with unity[1]. A power series $\alpha \in A[[T]]$ is said to be *equigraded* if and only if

$$(\text{for every } n \in \mathbb{N}, \text{ the coefficient of } \alpha \text{ before } T^n \text{ lies in the } n\text{-th graded component of } A).$$

Here and in the following, the symbol $\mathbb{N}$ stands for the set $\{0, 1, 2, ...\}$ (and not for the set $\{1, 2, 3, ...\}$, as it does in various other literature).

We now claim that:

Theorem 1. Let A be a graded commutative ring with unity.

(a) The set

$$\{\alpha \in A[[T]] \mid \text{ the power series } \alpha \text{ is equigraded}\}$$

is a sub-A_0-algebra of $A[[T]]$. In particular, the sum, the difference and the product of finitely many equigraded power series are equigraded as well, and the two power series 0 and 1 are both equigraded.

(b) This set $\{\alpha \in A[[T]] \mid \text{ the power series } \alpha \text{ is equigraded}\}$ is closed with respect to the (T)-adic topology on the ring $A[[T]]$.

(c) If an equigraded power series $\alpha \in A[[T]]$ has a multiplicative inverse $\alpha^{-1} \in A[[T]]$, then α^{-1} is equigraded as well.

(d) If an equigraded power series $\alpha \in A[[T]]$ has a multiplicative inverse $\alpha^{-1} \in A[[T]]$, then α^k is equigraded for every $k \in \mathbb{Z}$.

Proof of Theorem 1. For every $n \in \mathbb{N}$, we denote by A_n the n-th graded component of A. Thus, a power series α is equigraded if and only if

$$(\text{for every } n \in \mathbb{N}, \text{ the coefficient of } \alpha \text{ before } T^n \text{ lies in } A_n).$$

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[1] *Remark.* Different authors sometimes use different (and non-equivalent!) notions of a "graded ring with unity". The one that we are using here is defined as follows:

Definition. A "*graded ring with unity*" means a ring A with unity equipped with a family $(A_n)_{n\in\mathbb{N}}$ of subgroups of the additive group A satisfying $A = \bigoplus_{n\in\mathbb{N}} A_n$ (as abelian groups), $1 \in A_0$ and

$$(A_n A_m \subseteq A_{n+m} \text{ for every } n \in \mathbb{N} \text{ and } m \in \mathbb{N}).$$

Also, we use the following notation:

Definition. If a ring A, equipped with a family $(A_n)_{n\in\mathbb{N}}$, is a graded ring, then the family $(A_n)_{n\in\mathbb{N}}$ is said to be the *grading* of this graded ring A.

Definition. If a ring A, equipped with a family $(A_n)_{n\in\mathbb{N}}$, is a graded ring, then, for each $n \in \mathbb{N}$, the group A_n is called the *n-th graded component* of the graded ring A.

We denote the coefficient of α before T^n by $\operatorname{coeff}_n \alpha$. Thus, a power series α is equigraded if and only if

$$(\text{for every } n \in \mathbb{N}, \text{ we have } \operatorname{coeff}_n \alpha \in A_n). \tag{1}$$

We denote the set

$$\{\alpha \in A[[T]] \mid \text{the power series } \alpha \text{ is equigraded}\}$$

by E.

(a) In order to prove that the set

$$\{\alpha \in A[[T]] \mid \text{the power series } \alpha \text{ is equigraded}\}$$

is a sub-A_0-algebra of $A[[T]]$, we must show the following assertions:

Assertion 1: The power series 0 and 1 are both equigraded.

Assertion 2: If $\alpha \in A[[T]]$ is an equigraded power series, then $-\alpha$ is equigraded as well.

Assertion 3: If $\alpha \in A[[T]]$ and $\beta \in A[[T]]$ are two equigraded power series, then $\alpha + \beta$ and $\alpha\beta$ are equigraded as well.

Assertion 4: If $\alpha \in A[[T]]$ is an equigraded power series, and $u \in A_0$, then $u\alpha$ is equigraded as well.

However, Assertions 1 and 2 are completely obvious, so it will suffice to prove Assertions 3 and 4 only.

Proof of Assertion 3. Let $\alpha \in A[[T]]$ and $\beta \in A[[T]]$ be two equigraded power series.

For every $n \in \mathbb{N}$, we have

$$\operatorname{coeff}_n(\alpha+\beta) = \underbrace{\operatorname{coeff}_n \alpha}_{\substack{\in A_n \text{ (since } \alpha \\ \text{is equigraded)}}} + \underbrace{\operatorname{coeff}_n \beta}_{\substack{\in A_n \text{ (since } \beta \\ \text{is equigraded)}}} \in A_n + A_n \subseteq A_n.$$

Thus, the power series $\alpha + \beta$ is equigraded.

Besides, for every $n \in \mathbb{N}$, we have

$$\operatorname{coeff}_n(\alpha\beta) = \sum_{k=0}^{n} \underbrace{\operatorname{coeff}_k \alpha}_{\substack{\in A_k \text{ (since } \alpha \\ \text{is equigraded)}}} \cdot \underbrace{\operatorname{coeff}_{n-k} \beta}_{\substack{\in A_{n-k} \text{ (since } \beta \\ \text{is equigraded)}}} \qquad \text{(this is how the product of two power series is defined)}$$

$$\in \sum_{k=0}^{n} \underbrace{A_k \cdot A_{n-k}}_{\substack{\subseteq A_n, \text{ since } (A_n)_{n\in\mathbb{N}} \text{ is} \\ \text{a grading of the ring } A}} \subseteq \sum_{k=0}^{n} A_n \subseteq A_n.$$

Thus, the power series $\alpha\beta$ is equigraded. Thus, Assertion 3 is proven.

Proof of Assertion 4. Let $\alpha \in A[[T]]$ be an equigraded power series. Let $u \in A_0$. For every $n \in \mathbb{N}$, we have

$$\operatorname{coeff}_n(u\alpha) = \underbrace{u}_{\in A_0} \underbrace{\operatorname{coeff}_n(\alpha)}_{\substack{\in A_n \text{ (since } \alpha \\ \text{is equigraded)}}} \subseteq A_0 A_n \subseteq A_n$$

$$\left(\text{since } (A_n)_{n\in\mathbb{N}} \text{ is a grading of the ring } A\right).$$

In other words, the power series $u\alpha$ is equigraded. Thus, Assertion 4 is proven.

Now, all four Assertions 1, 2, 3 and 4 are proven. Therefore, the set

$$\{\alpha \in A[[T]] \mid \text{the power series } \alpha \text{ is equigraded}\}$$

is a sub-A_0-algebra of $A[[T]]$. This completes the proof of Theorem 1 **(a)**.

(b) In order to prove Theorem 1 **(b)**, we have to prove that the set E is closed with respect to the (T)-adic topology on the ring $A[[T]]$. This is equivalent to showing that every limit point of the set E lies in E. So let us prove that every limit point of the set E lies in E.

Let α be a limit point of the set E; then, for every neighbourhood U of α, there exists some $\alpha_U \in U \cap E$. We want to show that $\alpha \in E$.

Let $n \in \mathbb{N}$. Let U be the neighbourhood

$$\{\beta \in A[[T]] \mid \operatorname{coeff}_n \beta = \operatorname{coeff}_n \alpha\}$$

of α. Then, $\alpha_U \in U$ (since $\alpha_U \in U \cap E$) yields that $\operatorname{coeff}_n(\alpha_U) = \operatorname{coeff}_n \alpha$, while $\alpha_U \in E$ (since $\alpha_U \in U \cap E$) yields that α_U is equigraded and thus $\operatorname{coeff}_n(\alpha_U) \in A_n$. Hence, $\operatorname{coeff}_n \alpha = \operatorname{coeff}_n(\alpha_U) \in A_n$. Since this holds for every $n \in \mathbb{N}$, we can conclude that α is equigraded. In other words, $\alpha \in E$. So we have shown that every limit point α of the set E lies in E. This completes the proof of Theorem 1 **(b)**.

(c) Let $\alpha \in A[[T]]$ be an equigraded power series that has a multiplicative inverse $\alpha^{-1} \in A[[T]]$. Thus, $\alpha \cdot \alpha^{-1} = 1$. Hence,

$$\begin{aligned} 1 = \operatorname{coeff}_0 1 &= \operatorname{coeff}_0\left(\alpha \cdot \alpha^{-1}\right) \qquad \left(\text{since } 1 = \alpha \cdot \alpha^{-1}\right) \\ &= \operatorname{coeff}_0 \alpha \cdot \operatorname{coeff}_0\left(\alpha^{-1}\right) \end{aligned}$$

(because $\operatorname{coeff}_0(\alpha \cdot \beta) = \operatorname{coeff}_0 \alpha \cdot \operatorname{coeff}_0 \beta$ for any two power series α and β). Hence, the element $\operatorname{coeff}_0 \alpha \in A$ is invertible, and $\operatorname{coeff}_0(\alpha^{-1})$ is its inverse.

Now, for every element $u \in A$ and for every $n \in \mathbb{N}$, let us denote by u_n the n-th graded component of u. Of course, $u_n \in A_n$ for every $u \in A$ and every $n \in \mathbb{N}$.

Note that $(uv)_0 = u_0 v_0$ for any $u \in A$ and any $v \in A$ (because the map $A \to A_0$, $x \mapsto x_0$ is a ring homomorphism). Applied to $u = \operatorname{coeff}_0 \alpha$ and $v = \operatorname{coeff}_0(\alpha^{-1})$, this yields $(\operatorname{coeff}_0 \alpha \cdot \operatorname{coeff}_0(\alpha^{-1}))_0 = (\operatorname{coeff}_0 \alpha)_0 \cdot (\operatorname{coeff}_0(\alpha^{-1}))_0$. But since $\left(\underbrace{\operatorname{coeff}_0 \alpha \cdot \operatorname{coeff}_0\left(\alpha^{-1}\right)}_{=1}\right)_0 = 1_0 = 1$ and $(\operatorname{coeff}_0 \alpha)_0 = \operatorname{coeff}_0 \alpha$ (since $\operatorname{coeff}_0 \alpha \in A_0$, because α is equigraded), this becomes $1 = \operatorname{coeff}_0 \alpha \cdot (\operatorname{coeff}_0(\alpha^{-1}))_0$. Thus, $(\operatorname{coeff}_0(\alpha^{-1}))_0$ is the inverse of the element $\operatorname{coeff}_0 \alpha$ of A. But on the other hand, we know that $\operatorname{coeff}_0(\alpha^{-1})$ is the inverse of the element $\operatorname{coeff}_0 \alpha$ of A. Thus, $\operatorname{coeff}_0(\alpha^{-1}) = (\operatorname{coeff}_0(\alpha^{-1}))_0$ (since the inverse of an element of a ring is unique). Since $(\operatorname{coeff}_0(\alpha^{-1}))_0 \in A_0$, this yields $\operatorname{coeff}_0(\alpha^{-1}) \in A_0$.

Now, we are going to prove that $\operatorname{coeff}_n(\alpha^{-1}) \in A_n$ for every $n \in \mathbb{N}$. In fact, we are going to prove this by strong induction over $n \in \mathbb{N}$: We fix some $n \in \mathbb{N}$, and assume that

$$\operatorname{coeff}_k\left(\alpha^{-1}\right) \in A_k \text{ for every } k \in \mathbb{N} \text{ satisfying } k < n. \tag{2}$$

Our goal is to prove that $\operatorname{coeff}_n(\alpha^{-1}) \in A_n$ for our fixed value of n.

If $n = 0$, then this means proving that $\operatorname{coeff}_0(\alpha^{-1}) \in A_0$, which we have already shown. Hence, if $n = 0$, then we are done. So we can now WLOG assume that $n > 0$.

By the definition of the product of two power series, we have

$$\operatorname{coeff}_n \left(\widetilde{\alpha} \cdot \widetilde{\beta}\right) = \sum_{k=0}^{n} \operatorname{coeff}_k \widetilde{\alpha} \cdot \operatorname{coeff}_{n-k} \widetilde{\beta}$$

for any two power series $\widetilde{\alpha}$ and $\widetilde{\beta}$. Applying this to $\widetilde{\alpha} = \alpha^{-1}$ and $\widetilde{\beta} = \alpha$, we obtain

$$\operatorname{coeff}_n \left(\alpha^{-1} \cdot \alpha\right) = \sum_{k=0}^{n} \operatorname{coeff}_k \left(\alpha^{-1}\right) \cdot \operatorname{coeff}_{n-k} \alpha.$$

But $\operatorname{coeff}_n (\alpha^{-1} \cdot \alpha) = \operatorname{coeff}_n 1 = 0$ (since $n > 0$). Thus,

$$\begin{aligned} 0 = \operatorname{coeff}_n \left(\alpha^{-1} \cdot \alpha\right) &= \sum_{k=0}^{n} \operatorname{coeff}_k \left(\alpha^{-1}\right) \cdot \operatorname{coeff}_{n-k} \alpha \\ &= \sum_{k=0}^{n-1} \operatorname{coeff}_k \left(\alpha^{-1}\right) \cdot \operatorname{coeff}_{n-k} \alpha + \operatorname{coeff}_n \left(\alpha^{-1}\right) \cdot \operatorname{coeff}_0 \alpha, \end{aligned}$$

so that

$$\begin{aligned} \operatorname{coeff}_n \left(\alpha^{-1}\right) \cdot \operatorname{coeff}_0 \alpha &= - \sum_{k=0}^{n-1} \underbrace{\operatorname{coeff}_k \left(\alpha^{-1}\right)}_{\in A_k \text{ (by (2))}} \cdot \underbrace{\operatorname{coeff}_{n-k} \alpha}_{\substack{\in A_{n-k}, \text{ since } \alpha \\ \text{is equigraded}}} \in - \sum_{k=0}^{n-1} \underbrace{A_k \cdot A_{n-k}}_{\substack{\subseteq A_n, \text{ since } (A_n)_{n\in\mathbb{N}} \text{ is} \\ \text{a grading of the ring } A}} \\ &\subseteq - \sum_{k=0}^{n-1} A_n \subseteq A_n. \end{aligned} \tag{3}$$

But

$$\begin{aligned} \operatorname{coeff}_n \left(\alpha^{-1}\right) &= \operatorname{coeff}_n \left(\alpha^{-1}\right) \cdot \operatorname{coeff}_0 \alpha \cdot \operatorname{coeff}_0 \left(\alpha^{-1}\right) && \left(\text{since } 1 = \operatorname{coeff}_0 \alpha \cdot \operatorname{coeff}_0 \left(\alpha^{-1}\right)\right) \\ &\in A_n \cdot \underbrace{\operatorname{coeff}_0 \left(\alpha^{-1}\right)}_{\in A_0} && \text{(by (3))} \\ &\subseteq A_n \cdot A_0 \subseteq A_n && \left(\text{since } (A_n)_{n\in\mathbb{N}} \text{ is a grading of the ring } A\right). \end{aligned}$$

This completes our induction step. Thus, we have proven that $\operatorname{coeff}_n (\alpha^{-1}) \in A_n$ for every $n \in \mathbb{N}$. Consequently, the power series α^{-1} is equigraded. This proves Theorem 1 **(c)**.

(d) Let $\alpha \in A[[T]]$ be an equigraded power series that has a multiplicative inverse $\alpha^{-1} \in A[[T]]$. Let $k \in \mathbb{Z}$. Then, three cases are possible: Either $k > 0$ or $k = 0$ or $k < 0$. We will now show that in each of these cases, α^k is equigraded.

- If $k > 0$, then $\alpha^k = \underbrace{\alpha \cdot \alpha \cdot ... \cdot \alpha}_{k \text{ times}}$ is equigraded (since α is equigraded, and since the product of finitely many equigraded power series is equigraded).
- If $k = 0$, then $\alpha^k = 1$ is equigraded (as we know from Assertion 1).
- If $k < 0$, then $-k > 0$, and thus $\alpha^k = (\alpha^{-1})^{-k} = \underbrace{\alpha^{-1} \cdot \alpha^{-1} \cdot ... \cdot \alpha^{-1}}_{-k \text{ times}}$ is equigraded (since α^{-1} is equigraded (by Theorem 1 **(c)**), and since the product of finitely many equigraded power series is equigraded).

Hence, in each of the three possible cases, α^k is equigraded. This proves Theorem 1 **(d)**.

References

[1] Darij Grinberg: *Witt#4: Some computations in* **Symm**.