An Introduction to Hyperplane Arrangements
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List of additional errata and questions - I
(version 2)
Darij Grinberg, 11 June 2015

Page numbers refer to the page numbers at the top of the pages, not to the page count of the PDF file.

- **page 2:** You speak of the "usual dot product", but there is no "usual dot product" on a general finite-dimensional K-vector space. If you want to work in this generality, you should use the dual space. (Or else just identify V with K^n for this definition, or require the choice of a nondegenerate symmetric bilinear form $V \times V \to K$ which is to serve as a dot product. I personally find it easier to invoke the dual space, because as soon as one introduces additional structures like a basis or a bilinear form, it starts clouding further definitions. Besides, you use linear forms in the next paragraph, even though in the next paragraph you actually use a basis!)

  Similarly, in the next paragraph, "$x = (x_1, \ldots, x_n)$" assumes an isomorphism $V \to K^n$ to be given. And after that, the notion of "normals" assumes a dot product.

- **pages 2-3:** You write: "Let Y be a complementary space in K^n to the subspace X spanned by the normals to hyperplanes in $\mathcal{A}$. Define

  $$W = \{v \in V \ : \ v \cdot y = 0 \ \forall y \in Y\}.$$

  If char $(K) = 0$ then we can simply take $W = X$."

  I understand what you mean here, but it is not correctly explained. First, even if char $(K) = 0$, then we cannot take $W = X$ unless Y is *the orthogonal complement* to X (instead of just a random complementary space in K^n to the subspace X); I think you should say "we can simply take $Y = X^\perp$ and $W = X$" instead of "we can simply take $W = X$" because otherwise your wording suggests that Y can be taken arbitrary here. Second, char $(K) = 0$ does not guarantee that the orthogonal complement to X is indeed a complementary subspace to X (for example, the orthogonal complement to the subspace $\langle (1, i) \rangle$ of $\mathbb{C}^2$ is *not* a complementary subspace to $\langle (1, i) \rangle$, even though char $\mathbb{C} = 0$). So to make sure that we can actually take $W = X$, we

need to require that K is a formally real field (which is a stronger assertion than char $(K) = 0$).

Once again, the whole situation simplifies if you use the dual space. Now, the "normals to hyperplanes in $\mathcal{A}$" become the "linear forms defining the hyperplanes in $\mathcal{A}$", and they form a subspace $\widetilde{X}$ of the dual space V^*. The orthogonal space of this $\widetilde{X}$ (= the joint kernel of the linear forms defining the hyperplanes in $\mathcal{A}$) is a subspace of V; we call it $\widetilde{Y}$. Then, the images of the hyperplanes in $\mathcal{A}$ under the projection map $V \to V/\widetilde{Y}$ are hyperplanes in $V/\widetilde{Y}$ (this is easy to prove[1] – a lot easier than checking your equality (1)). The arrangement they form in $V/\widetilde{Y}$ is isomorphic to your $\mathcal{A}_W$, but defined canonically (thus eliminating the necessity of checking that your $\mathcal{A}_W$ is independent on the choice of W up to isomorphism).

- **page 3:** When you write "$H' \in \mathcal{A}_W$ if and only if $H' \oplus W^\perp \in \mathcal{A}$", it would be good to point out that H' is supposed to be a subspace of W. I would also replace the $\oplus$ sign by a $+$ sign, since $\oplus$ has not been defined for *affine* subspaces (and its standard meaning that involves the intersection being 0 is not correct for affine subspaces).

- **page 3:** You write: "in characteristic p this type of reasoning fails". Yes, but not only in characteristic p. Also for $K = \mathbb{C}$, as I explained above.

- **page 3:** You write: "then $R \in \mathcal{R}(\mathcal{A})$ if and only if $R \cap W \in \mathcal{R}(\mathcal{A}_W)$". This makes little sense (R cannot be any subset of $\mathbb{R}^n$, but what should it be?). I assume you mean that the map

$$\mathcal{R}(\mathcal{A}) \to \mathcal{R}(\mathcal{A}_W),$$
$$R \mapsto R \cap W$$

is well-defined and bijective, with inverse

$$\mathcal{R}(\mathcal{A}_W) \to \mathcal{R}(\mathcal{A}),$$
$$R \mapsto R + W^\perp.$$

- **page 4:** Trivial nitpick: In the definition of "general position", the $H_1, \ldots, H_p$ should be assumed distinct in the formulas.

- **page 4, second line of Example 1.2:** "L line" should be "line L".

- **page 4, second line of Example 1.2:** "$\mathcal{A}_K$" should be "$\mathcal{A}_k$".

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[1] All that needs to be checked is that every hyperplane $H \in \mathcal{A}$ satisfies $\widetilde{Y} \subseteq H'$, where H' is the translate of H that passes through the origin. The proof is easy: From $H^\perp \subseteq \widetilde{X}$, we obtain $\widetilde{X}^\perp \subseteq (H^\perp)^\perp = H$, thus $Y = \widetilde{X}^\perp \subseteq H$. Here, the $\perp$ sign is defined with reference to the canonical pairing $V^* \times V \to K$, not to any non-canonical bilinear form on V.

- **page 8:** You define saturated chains, but you do not define maximal chains. The fact that you use the word "maximal" in the next sentence ("if every maximal chain of P has length n") creates the incorrect impression that "maximal" is a synonym for "saturated".

- **page 8:** You write: "If $x < y$ in a graded poset P then we write $\mathrm{rk}\,(x,y) = \mathrm{rk}\,(y) - \mathrm{rk}\,(x)$". It would be better to replace "$x < y$" by "$x \leq y$" here, since you later use the notation in this mildly greater generality.

- **page 10:** You never seem to explicitly say that Theorem 1.1 is the "Möbius inversion formula". When that name appears in the proof of Theorem 1.1, the reader has to guess.

- **page 12, Exercise (7):** The word "face" has not been defined.

- **page 14, Lemma 2.2:** Do you really want the word "real" here? I haven't read the proof in much detail, but I don't see how you are using this condition (actually, you even use the letter K for the ground field in the proof).

- **page 16, Theorem 2.3:** I find it interesting to observe that this theorem holds whenever L is a finite join-semilattice, not necessarily a lattice. (Of course, the same proof applies, although the algebras now become nonunital algebras.) In this generality, L as a semigroup needs not have a neutral element, but the algebra $A\,(L)$ has a unity nevertheless (because $A'\,(L)$ has a unity, namely $\sum\limits_{x \in L} \sigma'_x$).

- **page 24:** In the last paragraph of this page, replace "by first choosing the size $i = \sharp\kappa\,([n])$ of its image in $\binom{q}{i}$ ways" by "by first choosing the size $i = \sharp\kappa\,([n])$ of its image, then choosing its image $\kappa\,([n])$ itself in $\binom{q}{i}$ ways".

- **page 28:** In the "NOTE", I think "linear extension of $\mathfrak{o}$" should be "linear extension of $\overline{\mathfrak{o}}$". (At least I usually have seen the notion of a "linear extension" defined for posets, sometimes for preorders, but not for acyclic orientations themselves.)

- **page 32:** "diagam" → "diagram".

- **page 33:** After "M is isomorphic to the set of nonzero vectors in the vector space $\mathbb{F}_2^3$", add "(with independence defined as linear independence)". (This is to say, not affine independence.)

- **page 35, proof of Proposition 3.6:** Replace "$\overline{B} = \overline{B}'$" by "$\overline{B} = \overline{B'}$" (notice the $'$ being under the overline).

- **page 35, proof of Proposition 3.6:** You show that $X = X'$ if and only if $\overline{\mathcal{B}} = \overline{\mathcal{B}'}$. But in order to check that $L(M) \cong L(\mathcal{A})$ as lattices, it is also important to prove that $X \subseteq X'$ if and only if $\overline{\mathcal{B}} \supseteq \overline{\mathcal{B}'}$. (Not difficult but worth making explicit.)

- **page 36, Definition 3.9:** In the first sentence of this definition, replace "A finite lattice" by "A finite graded lattice". Similarly, in the last sentence, replace "a finite lattice" by "a finite graded lattice".

- **page 36, proof of Theorem 3.8:** In the first displayed equation of the proof, replace "rk $(\vee I)$" by "rk $(\bigvee I)$".

- **page 36, proof of Theorem 3.8:** You write: "Since L is atomic, there exists $y \in S$ such that $y \nleq \bigvee T$". (Here I have applied the correction that already appears in your errata.) In my opinion, the existence of such a y is not a consequence of L being atomic. Instead, it is the consequence of the argument that if no such y existed, then every $z \in S$ would satisfy $z \leq \bigvee T$, and thus we would have $\bigvee S \leq \bigvee T < \bigvee T' \leq \bigvee S$ (since $T' \subseteq S$), which is absurd.

- **page 36, proof of Theorem 3.8:** You claim that "$L \cong L(M)$". I think this claim is roughly the difficulty of your [2+] exercises (certainly harder than the "[why?]" in the same paragraph) and should get at least a hint to its proof.

  Proof sketch. The idea is to show that the maps

  $$\Phi : L \to L(M),$$
  $$x \mapsto \{a \in A \mid a \leq x\}$$

  and

  $$\Psi : L(M) \to L, \tag{1}$$
  $$F \mapsto \bigvee F$$

  are well-defined, mutually inverse and lattice homomorphisms. The well-definedness of Ψ is obvious. The well-definedness of Φ uses the neat but not completely trivial observation that every $B \subseteq A$ satisfies

  $$\operatorname{rk} B = \operatorname{rk}\left(\bigvee B\right) \tag{2}$$

  (where the first rk means rank in the matroid M, while the second rk means rank in the lattice L). The proof of $\Psi \circ \Phi = \operatorname{id}$ uses the fact that every $x \in L$ satisfies $x = \bigvee(\Phi(x))$, which has a neat indirect proof (the lattice L is atomic, so x is the join $\bigvee C$ of *some* subset C of A, and clearly this C satisfies $C \subseteq \Phi(x)$, so that $x = \bigvee \underbrace{C}_{\subseteq \Phi(x)} \leq \bigvee(\Phi(x))$, but on the

other hand $\bigvee\left(\Phi\left(x\right)\right) \leq x$ for obvious reasons, and thus $x = \bigvee\left(\Phi\left(x\right)\right) = \Psi\left(\Phi\left(x\right)\right) = \left(\Psi \circ \Phi\right)\left(x\right))$. The proof of $\Phi \circ \Psi = \mathrm{id}$ proceeds by observing that if $a \in A$ and $F \in L\left(M\right)$ satisfy $a \leq \bigvee F$, then $\bigvee\left(F \cup \{a\}\right) = \bigvee F$, and thus (2) yields $\mathrm{rk}\left(F \cup \{a\}\right) = \mathrm{rk}\left(\underbrace{\bigvee\left(F \cup \{a\}\right)}_{=\bigvee F}\right) = \mathrm{rk}\left(\bigvee F\right) = \mathrm{rk}\, F$ (by (2)

again), which shows that $a \in F$ (since F is a flat). So the maps Φ and Ψ are mutually inverse. In order to prove that they are lattice homomorphisms, I proceed as follows: The maps Ψ and Φ are poset homomorphisms and form a monotone Galois connection between the posets $L\left(M\right)$ and L; thus, Ψ preserves joins and Φ preserves meets. Since Ψ and Φ are mutually inverse, this yields that both of these maps preserve both joins and meets, and we are done. This would be more noticeably painful to show without knowing the Galois-connection trick.

Am I missing something obvious?

- **page 36, proof of Theorem 3.8:** The assumption that M is simple (in the proof of (2) $\implies$ (1)) seems like overkill to me. It can be lifted easily (you just have to replace "every flat is the join of its elements" by "every flat F satisfies $F = \bigvee_{x \in F} \overline{\{x\}}$"), whereas justifying this assumption requires the use of the $L\left(M\right) \cong L\left(\widehat{M}\right)$ statement on page 34, which does not look easy to prove at all.

- **page 37:** In the paragraph directly after the proof of Theorem 3.8, replace "the intersection lattice $L_{\mathcal{A}}$" by "the intersection lattice $L\left(\mathcal{A}\right)$".

- **page 37:** I am surprised that you don't prove Theorem 3.9 here. In my opinion, the proof is very short (possibly even shorter than justifying why it really is dual to [31, Cor. 3.9.3] – there is an asymmetry in the definition of the Möbius function which "favors" the lower end of the interval[2]), and fits perfectly with what you did in §2:

Proof of Theorem 3.9. We have $a \neq \widehat{0}$ and thus $a > \widehat{0}$. In the Möbius algebra $A\left(L\right)$ (defined in §2), we have $\sigma_{\widehat{0}} = \underbrace{\sum_{y \geq \widehat{0}}}_{=\sum_{y \in L}} \underbrace{\mu\left(\widehat{0}, y\right)}_{=\mu(y)} y = \sum_{y \in L} \mu\left(y\right) y =$

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[2]The formula (2) says $\mu\left(x, y\right) = -\sum_{x \leq z < y} \mu\left(x, z\right)$, rather than $\mu\left(x, y\right) = -\sum_{x < z \leq y} \mu\left(z, y\right)$. Maybe it is worth explaining that the latter equality also holds (because $\mu = \zeta^{-1}$ is a two-sided inverse) and therefore the Möbius function is invariant under "turning the poset upside down".

(While at that, it's also worth pointing out that $\mu\left(x, y\right)$ depends only on the poset $[x, y]$, not on the whole poset P. You use this a lot, but you leave it tacit, defining μ as a function on the whole $\mathrm{Int}\, P$ instead.)

$\sum\limits_{x \in L} \mu(x) x$ and thus

$$\sum_{x \in L} \mu(x) \underbrace{x \vee a}_{=xa} = \sum_{x \in L} \mu(x) xa = \underbrace{\left(\sum_{x \in L} \mu(x) x\right)}_{=\sigma_{\widehat{0}}} \underbrace{a}_{\substack{= \sum\limits_{y \geq a} \sigma_y \\ \text{(by (9))}}} = \sigma_{\widehat{0}} \left(\sum_{y \geq a} \sigma_y\right)$$

$$= \sum_{y \geq a} \underbrace{\sigma_{\widehat{0}} \sigma_y}_{\substack{= \delta_{\widehat{0},y} \sigma_{\widehat{0}} \\ \text{(by the second sentence} \\ \text{of Theorem 2.3)}}} = \sum_{y \geq a} \underbrace{\delta_{\widehat{0},y}}_{\substack{=0 \\ \text{(since } y \geq a > \widehat{0} \\ \text{and thus } y \neq \widehat{0})}} \sigma_{\widehat{0}} = 0.$$

Comparing coefficients before $\widehat{1}$ in this equality yields $\sum\limits_{\substack{x \in L; \\ x \vee a = \widehat{1}}} \mu(x) = 0$.

Theorem 3.9 is proven.

- **page 38, proof of Theorem 3.10:** It took me a while to understand why "The sum on the right is nonempty". The simplest proof of this that I can find is the following: Let A be the set of all atoms of M. The map Ψ defined by (1) is injective (since we have found an inverse to it). Thus, $\bigvee A \neq \bigvee(A \setminus \{a\})$. Hence, $\bigvee(A \setminus \{a\}) < \bigvee A$. Semimodularity of L easily shows that $\bigvee(A \setminus \{a\}) \lessdot \bigvee A$. But $\bigvee A = \widehat{1}$ (which is easy to prove using atomicity of L: the element $\widehat{1}$ must be a join of *some* set of atoms, and thus also of the set A of all atoms), so this becomes $\bigvee(A \setminus \{a\}) \lessdot 1$. But $a \not\leq \bigvee(A \setminus \{a\})$ (since otherwise, we would have $\bigvee(A \setminus \{a\}) = \bigvee A$, contradicting $\bigvee A \neq \bigvee(A \setminus \{a\})$). Thus, there exists an $x \in L$ satisfying $a \not\leq x \lessdot 1$ (namely, $x = \bigvee(A \setminus \{a\})$).

- **page 38, (26):** Replace "M" by "$M_{\mathcal{A}}$".

- **page 41, §4.1:** Throughout this section, whenever you work with $BC(M)$, you need to require M to have no loops. Otherwise, $BC(M)$ is the empty set (since the empty set is a broken circuit), and thus not a simplicial complex.

- **page 42, Lemma 4.4:** Replace "$-c_1 + c_2 - c_3 + \cdots$" by "$c_0 - c_1 + c_2 - c_3 + \cdots$", in order for the lemma to still be valid when P is the one-element poset.

- **page 42, NOTE after Lemma 4.4:** The equality "$\mu\left(\widehat{0}, \widehat{1}\right) = \widetilde{\chi}(\Delta(P'))$" requires that P contain more than one element.

- **page 43:** This is absolutely not an erratum, and not even a suggestion, but I just felt like sharing another proof of Theorem 4.11 (though the probability that it is not new to you is high).

Proof sketch for Theorem 4.11. Let $K = \mathbb{Q}$. For every $i \in \mathbb{P}$, define a function $\zeta_i : \operatorname{Int} P \to K$ by

$$\zeta_i \left[x, y \right] = \left[x \lessdot y \text{ and } \lambda \left(x, y \right) = i \right].$$

Here, we are using the Iverson bracket notation (that is, $[\mathcal{A}] = \begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$ for any logical statement $\mathcal{A}$).

Recall that the functions $\operatorname{Int} P \to K$ form a K-algebra $\mathcal{I}(P)$ (defined in §1.3, and called the incidence algebra of P). So all of the ζ_i are elements of this K-algebra $\mathcal{I}(P)$. These elements ζ_i are locally nilpotent (since they send one-element intervals $[x, x]$ to 0), and the infinite products $\cdots (1 - \zeta_3)(1 - \zeta_2)(1 - \zeta_1)$ and $(1 - \zeta_1)^{-1}(1 - \zeta_2)^{-1}(1 - \zeta_3)^{-1} \cdots$ is well-defined. Here, 1 stands for the unity of the K-algebra $\mathcal{I}(P)$; this is its element δ.

We have

$$\underbrace{(1 - \zeta_1)^{-1}}_{\substack{= \sum\limits_{m \in \mathbb{N}} \zeta_1^m}} \underbrace{(1 - \zeta_2)^{-1}}_{\substack{= \sum\limits_{m \in \mathbb{N}} \zeta_2^m}} \underbrace{(1 - \zeta_3)^{-1}}_{\substack{= \sum\limits_{m \in \mathbb{N}} \zeta_3^m}} \cdots$$

$$= \left(\sum_{m \in \mathbb{N}} \zeta_1^m \right) \left(\sum_{m \in \mathbb{N}} \zeta_2^m \right) \left(\sum_{m \in \mathbb{N}} \zeta_3^m \right) \cdots$$

$$= \sum_{\substack{(m_1, m_2, m_3, \dots) \text{ is a} \\ \text{weak composition}}} \zeta_1^{m_1} \zeta_2^{m_2} \zeta_3^{m_3} \cdots = \sum_{1 \leq a_1 \leq a_2 \leq \cdots \leq a_k} \zeta_{a_1} \zeta_{a_2} \cdots \zeta_{a_k}.$$

Hence, every $x \leq y$ in P satisfy

$$
\left((1-\zeta_1)^{-1} (1-\zeta_2)^{-1} (1-\zeta_3)^{-1} \cdots \right) [x,y]
$$

$$
= \left(\sum_{1 \leq a_1 \leq a_2 \leq \cdots \leq a_k} \zeta_{a_1} \zeta_{a_2} \cdots \zeta_{a_k} \right) [x,y] \tag{3}
$$

$$
= \sum_{1 \leq a_1 \leq a_2 \leq \cdots \leq a_k} \underbrace{(\zeta_{a_1} \zeta_{a_2} \cdots \zeta_{a_k}) [x,y]}_{= \sum\limits_{x=x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq y} \prod\limits_{i=1}^{k} \zeta_{a_i}[x_{i-1},x_i]}
$$

$$
= \sum_{1 \leq a_1 \leq a_2 \leq \cdots \leq a_k} \sum_{x=x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq y} \prod_{i=1}^{k} \underbrace{\zeta_{a_i} [x_{i-1}, x_i]}_{\substack{=[x_{i-1} \lessdot x_i \text{ and } \lambda(x_{i-1},x_i)=a_i] \\ \text{(by the definition of } \zeta_{a_i})}}
$$

$$
= \sum_{1 \leq a_1 \leq a_2 \leq \cdots \leq a_k} \sum_{x=x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq y} \underbrace{\prod_{i=1}^{k} [x_{i-1} \lessdot x_i \text{ and } \lambda(x_{i-1}, x_i) = a_i]}_{=[x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k \text{ and each } i \text{ satisfies } \lambda(x_{i-1},x_i)=i]}
$$

$$
= \sum_{1 \leq a_1 \leq a_2 \leq \cdots \leq a_k} \underbrace{\sum_{x=x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq y} [x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k \text{ and each } i \text{ satisfies } \lambda(x_{i-1}, x_i) = i]}_{=\sharp\{x=x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k=y \,:\, \text{each } i \text{ satisfies } \lambda(x_{i-1},x_i)=i\}}
$$

$$
= \sum_{1 \leq a_1 \leq a_2 \leq \cdots \leq a_k} \sharp \{x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = y \,:\, \text{each } i \text{ satisfies } \lambda(x_{i-1}, x_i) = i\}
$$

$$
= \sharp \{x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = y \,:\, \lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k)\} \tag{4}
$$

$$
= 1 \qquad \text{(by Definition 4.11)}
$$

$$
= \zeta[x,y].
$$

Hence, $(1-\zeta_1)^{-1} (1-\zeta_2)^{-1} (1-\zeta_3)^{-1} \cdots = \zeta$. Inverting both sides of this equality, we obtain $\cdots (1-\zeta_3)(1-\zeta_2)(1-\zeta_1) = \zeta^{-1} = \mu$. Thus,

$$
\mu = \cdots (1-\zeta_3)(1-\zeta_2)(1-\zeta_1) = \sum_{a_1 > a_2 > \cdots > a_k \geq 1} (-1)^k \zeta_{a_1} \zeta_{a_2} \cdots \zeta_{a_k},
$$

Hence, every $x \leq y$ satisfy

$$
\mu[x,y]
$$

$$
= \left(\sum_{a_1 > a_2 > \cdots > a_k \geq 1} (-1)^k \zeta_{a_1} \zeta_{a_2} \cdots \zeta_{a_k} \right) [x,y]
$$

$$
= \sum_{k \in \mathbb{N}} (-1)^k \sharp \{x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = y \,:\, \lambda(x_0, x_1) > \lambda(x_1, x_2) > \cdots > \lambda(x_{k-1}, x_k)\}
$$

(similarly to how we got from (3) to (4)).

The sum on the right hand side has only one (or, rather, at most one) nonzero term, namely the one for $k = \operatorname{rk}(x,y)$ (since P is graded). Hence, this equality rewrites as

$$\mu[x,y]$$
$$= (-1)^{\operatorname{rk}(x,y)} \, \sharp \left\{ x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = y \; : \; \lambda(x_0,x_1) > \lambda(x_1,x_2) > \cdots > \lambda(x_{k-1},x_k) \right\},$$

and we are done.

Now that I have written up this proof, I guess I understand why you didn't want to do it...

- **page 44, proof of Theorem 4.11:** In (27), replace "n" by "$n-1$".

- **page 44, proof of Theorem 4.11:** Replace "$\lambda(x_i) > \lambda(x_{i+1})$" by "$\lambda(x_{i-1},x_i) > \lambda(x_i,x_{i+1})$".

- **page 44, proof of Theorem 4.11:** The case of $n = 0$ should be ruled out somewhere near the beginning of the proof, as there are arguments that tacitly use $n > 0$ throughout the proof. (Compare what I wrote about Lemma 4.4.)

- **page 45, proof of Theorem 4.12:** Again, I think that assuming that M is simple is not worth the hassle. We already have the hypothesis that M has no loops (else, $\operatorname{BC}(M)$ is not a simplicial complex). Without the assumption that M be simple, we can no longer identify the atoms of $L(M)$ with the points of M. But we still have a surjective map

$$M \to \{\text{atoms of } L(M)\},$$
$$x_i \mapsto \overline{\{x_i\}},$$

and your proof goes through if some of the x_i's appearing in it are replaced by the corresponding $\overline{\{x_i\}}$'s.

- **page 45, proof of Theorem 4.12:** In "Figure 3 shows the lattice of flats of the matroid M of Figure 1 with the edge labeling (30)", add "the ordering $\mathcal{O}$ and" after the "with".

- **page 45, proof of Theorem 4.12:** In "Moreover, there is a unique y_1 satisfying $x = x_0 \lessdot y_1 \leq y$ and $\widetilde{\lambda}(x_0,y_1) = j$, viz., $y_1 = x_0 \vee x_j$. (Note that $y_1 > x_0$ by semimodularity.)", replace every of the four occurrences of "x_0" by "x". Then, define y_0 (not x_0) to mean x (this notation is used in the next sentence).

- **page 46, proof of Theorem 4.12:** In "$\widetilde{\lambda}(y_0,y_1) = j > \widetilde{\lambda}(y_1,y_2)$", replace the ">" sign by a "$\geq$" sign.

- **page 46, proof of Theorem 4.12:** In Claim 2, replace both appearances of "$\lambda(C)$" by "$\widetilde{\lambda}(C)$".

  Also, it would be good to define what $\widetilde{\lambda}(C)$ means, and explain the abuse of notation. As far as I understand, you define $\widetilde{\lambda}(C)$ as follows: If C is a chain $0 = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_k$, then you define $\widetilde{\lambda}(C)$ to be the sequence $\left(\widetilde{\lambda}(y_0, y_1), \widetilde{\lambda}(y_1, y_2), \ldots, \widetilde{\lambda}(y_{k-1}, y_k)\right)$. Sometimes you denote the set of the entries of this sequence (rather than this sequence itself) as $\widetilde{\lambda}(C)$. Moreover, you identify this set with the set $\left\{x_{\widetilde{\lambda}(y_0, y_1)}, x_{\widetilde{\lambda}(y_1, y_2)}, \ldots, x_{\widetilde{\lambda}(y_{k-1}, y_k)}\right\} \subseteq S$.

- **page 46, proof of Theorem 4.12:** In Claim 2, replace "increasing chain" by "strictly increasing chain".

- **page 46, proof of Theorem 4.12:** In the proof of Claim 2, replace "$\lambda(C)$" by "$\widetilde{\lambda}(C)$" (in "To prove the distinctness of the labels $\lambda(C)$").

- **page 46, proof of Theorem 4.12:** In the proof of Claim 2, replace "$\widehat{0} := y_0 \lessdot y_1 \lessdot \cdots \lessdot y_k$" by "$\widehat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_k$" (no colon, since you are not defining anything).

- **page 46, proof of Theorem 4.12:** In the proof of Claim 2, you write: "Note that C is saturated by semimodularity". This is only half of the story, because it has to be checked that no y_i equals y_{i-1}. This latter statement follows from

$$\mathrm{rk}(y_i) = \mathrm{rk}\left(\overline{\{x_{a_1}\}} \vee \overline{\{x_{a_2}\}} \vee \cdots \vee \overline{\{x_{a_i}\}}\right) = \mathrm{rk}\,\overline{\{x_{a_1}, x_{a_2}, \ldots, x_{a_i}\}}$$
$$= \mathrm{rk}\{x_{a_1}, x_{a_2}, \ldots, x_{a_i}\} = i$$
$$\left(\begin{array}{c} \text{since } \{x_{a_1}, x_{a_2}, \ldots, x_{a_i}\} \text{ is a subset of the independent set } T, \\ \text{and thus itself independent} \end{array}\right),$$

  where we are tacitly using that the lattice $L(M)$ is graded by the rank of flats in the matroid M.

- **page 46, proof of Theorem 4.12:** In the proof of Claim 2, you write: "Thus

$$\mathrm{rk}(T) = \mathrm{rk}(T \cup \{x_j\}) = i.$$

  Since T is independent, $T \cup \{x_j\}$ contains a circuit Q satisfying $x_j \in Q$, so T contains a broken circuit." This is wrong in two places: first, $\mathrm{rk}(T)$ is not i, and second, x_j might not be larger than $\max T$. Let me suggest the following corrected argument:

  "Let T_i be the subset $\{x_{a_1}, \ldots, x_{a_i}\}$ of T; then, T_i is independent (since T is independent). Moreover, $y_i = \bigvee_{t \in T_i} \overline{\{t\}} = \overline{T_i}$. Hence, from $y_{i-1} \vee \overline{\{x_j\}} =$

y_i, we obtain $\overline{\{x_j\}} \leq y_i$, so $\overline{\{x_j\}} \subseteq y_i = \overline{T_i}$. Thus, the set $T_i \cup \{x_j\}$ is dependent, and thus contains a circuit Q satisfying $x_j \in Q$ (since T_i is independent). Therefore, T_i contains a broken circuit (namely, $Q \setminus \{x_j\}$, since $j > a_i > a_{i-1} > \cdots > a_1$). Thus, T contains a broken circuit (since $T_i \subseteq T$), which is absurd."

- **page 47, Example 4.9 (c):** Replace "and rk $(y) = 2$" by "with rk $(y) = 2$".

- **page 47, Example 4.9 (e):** Replace "$\mathbb{F}_n(q)$" by "$\mathbb{F}_q^n$".

- **page 48, Example 4.9 (e):** Replace "L is a modular geometric lattice" by "$B_n(q)$ is a modular geometric lattice".

- **page 48, Example 4.9 (e):** Replace "every $x \in L$ is modular" by "every $x \in B_n(q)$ is modular".

- **page 48, Example 4.9 (e),** NOTE: Replace "every two points" by "every two distinct points". Similarly, replace "every two lines" by "every two distinct lines".

- **page 49, Example 4.9 (f):** In "$\{a, b, B_1 - a, B_2 - b, \ldots, B_3, \ldots, B_k\}$", remove the first "$\ldots$".

- **page 49, Theorem 4.13:** The "of rank n" is slightly ambiguous: does it refer to the lattice or to z ? (It is meant to refer to L, of course, rather unsurprisingly, but I'd still split such a sentence into two if I were to write it.)

- **page 49, Theorem 4.13:** If I am not mistaken, χ_L and $\chi_{\left[\widehat{0}, z\right]}$ have never been defined: You defined χ_M for matroids, but not χ_L for lattices. I guess it wouldn't be wrong to address this on a more general level and define χ_P for every finite graded poset P which has a $\widehat{0}$ and a $\widehat{1}$, by setting

$$\chi_P(t) = \sum_{x \in P} \mu\left(\widehat{0}, x\right) t^{\operatorname{rk}\widehat{1} - \operatorname{rk} x}.$$

- **page 50:** In the first paragraph of this page, "begins $x^n - ax^{n-1} + \cdots$" should be "begins $t^n - at^{n-1} + \cdots$".

- **page 52, proof of Theorem 4.13:** It took me a while to understand what you mean by "the product will be preserved". Your argument, set up more algebraically, seems to be this: We define a K-module homomorphism $\alpha : K\left[\widehat{0}, z\right] \to K[t]$ by $\alpha(v) = t^{\operatorname{rk} z - \operatorname{rk} v}$ for every $v \in \left[\widehat{0}, z\right]$. We define a K-module homomorphism $\beta : K\left\{w \in L \mid w \wedge z = \widehat{0}\right\} \to K[t]$ by $\beta(y) = t^{n - \operatorname{rk} y - \operatorname{rk} z}$ for every $y \in L$ satisfying $y \wedge z = \widehat{0}$. We define a K-module

homomorphism $\gamma : KL \to K[t]$ by $\gamma(x) = t^{n-\mathrm{rk}\, x}$ for every $x \in L$. Then, you show (using Claim 2) the equality

$$\alpha(v)\,\beta(y) = \gamma(v \vee y) \tag{5}$$

for every $v \in L$ and $y \in L$ satisfying $v \leq z$ and $y \wedge z = \widehat{0}$. By linearity, the same equality thus holds for every $v \in K\left[\widehat{0}, z\right]$ and $y \in K\left\{y \in L \mid y \wedge z = \widehat{0}\right\}$. Now, you apply the map γ to both sides of (33), and simplify the right hand side using (5).

- **page 53, Definition 4.13:** Replace "$L_{\mathcal{A}}$" by "$L(\mathcal{A})$".

- **page 54, Example 4.11 (c):** In "$B_1 \subset B_2 \cdots \subset B_{n-1}$", you forgot a "$\subset$" sign.

- **page 54, Example 4.11 (c):** "The atoms covered by π_i" should be "The atoms $\leq \pi_i$".

- **page 54, Example 4.11 (c):** On the last line of the page, replace "$\mathcal{B}_n(t)$" by "$\mathcal{B}_n$".

- **page 55:** Again, "$L_{\mathcal{A}}$" should be "$L(\mathcal{A})$" (two lines above Theorem 4.14).

- **page 61, proof of Proposition 5.13:** On line 2 of the proof, replace "$v_i, a_i \in \mathbb{Z}^n$" by "$v_i \in \mathbb{Z}^n$ and $a_i \in \mathbb{Z}$".

- **page 62, proof of Proposition 5.13:** On the second line of the page, you write "if and only if at least one". I understand the "only if". The "if" might be true, but is probably not easy to prove (the point is to rule out accidental isomorphisms $L(\mathcal{A}) \cong L(\mathcal{A}_p)$ that could happen if hyperplanes becoming parallel "undo" the damage done by hyperplanes becoming concurrent); either way it is a distraction from the proof.

- **page 62, proof of Theorem 5.15:** Replace "F_q" by "$\mathbb{F}_q$".