On the q-Poincaré sum of a Coxeter group

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Abstract. Given a finite Coxeter group W and an element q of a commutative ring $\mathbf{k}$, we define the element $\mathbf{L}_q := \sum\limits_{w \in W} q^{\ell(w)} w$ of its group algebra $\mathbf{k}[W]$. We say that an element $w \in W$ is *W-length-symmetric* if its coefficient in the commutator $[\mathbf{L}_a, \mathbf{L}_b]$ is 0 for all $\mathbf{k}$ and all $a, b \in \mathbf{k}$. We prove several sufficient criteria for elements to be W-length-symmetric. In particular, we show that any separable permutation (i.e., permutation avoiding the patterns 2413 and 3142) in the symmetric group S_n is S_n-length-symmetric. This is only a sufficient condition; other W-length-symmetric elements include (twisted) involutions and elements of dihedral parabolic subgroups.

Fix a commutative ring $\mathbf{k}$. Consider the n-th symmetric group S_n for a given nonnegative integer n. Any permutation $w \in S_n$ has a *length* $\ell(w)$, which is defined as the number of inversions of w (that is, of pairs (i, j) of integers with $1 \leq i < j \leq n$ and $w(i) > w(j)$). Using these lengths, we can define an element

$$\mathbf{L}_q := \sum_{w \in S_n} q^{\ell(w)} w \in \mathbf{k}[S_n]$$

in the group algebra $\mathbf{k}[S_n]$ of S_n, for any fixed scalar $q \in \mathbf{k}$. For instance, $\mathbf{L}_1$ and $\mathbf{L}_{-1}$ are two central elements of $\mathbf{k}[S_n]$, equalling $n!$ times the central idempotents corresponding to the trivial and sign representations of S_n. For general q, however, the element $\mathbf{L}_q$ is not central, and does not even necessarily commute with another such element $\mathbf{L}_r$ for $r \neq q$.

However, it turns out that if we fix two scalars $q, r \in \mathbf{k}$, then many permutations $w \in S_n$ do not appear in the commutator $[\mathbf{L}_q, \mathbf{L}_r] := \mathbf{L}_q \mathbf{L}_r - \mathbf{L}_r \mathbf{L}_q$. Namely, if $w \in S_n$ is a *separable* permutation, then the coefficient of w in $[\mathbf{L}_q, \mathbf{L}_r]$ is 0. The notion of a separable permutation can be defined in many ways, e.g., algorithmically or by the avoidance of two patterns (2413 and 3142); we refer to Ghys's book [Ghys16, "Separable permutations"] for four equivalent definitions and a survey of their rich theory.

In this paper, we will prove the fact just mentioned, and essentially generalize it to arbitrary finite Coxeter groups.

1. The theorems

We assume the basic properties of Coxeter groups (see [BjoBre05], [Davis25], [Luszti14], [Bourba02], [Heckma18]). A Coxeter system (W, S) consists of a Coxeter group W and its set S of simple reflections. If (W, S) is a Coxeter system, and $w \in W$ is any element, then $\ell(w)$ denotes the Coxeter length of w (that is, the smallest length of a list of simple reflections having product w). Note that $\ell\left(w^{-1}\right) = \ell(w)$ for each $w \in W$.

If W is a finite Coxeter group, then there exists a unique element $w \in W$ whose length is maximum (see [Luszti14, §11.1]); we shall call it w_0. This element w_0 is always an involution (i.e., satisfies $w_0^2 = \mathrm{id}$), since its inverse w_0^{-1} must also have maximum length. It is known as the *longest element* or the *longest word* of W.

We now define our main object of study:

Definition 1.1. Let (W, S) be a Coxeter system with finite Coxeter group W. Let $\mathbf{k}$ be a commutative ring. For any $q \in \mathbf{k}$, we define the element

$$\mathbf{L}_q(W) := \sum_{w \in W} q^{\ell(w)} w$$

in the group algebra $\mathbf{k}[W]$. We denote this element simply by $\mathbf{L}_q$ if the group W is clear from the context. We call it the *q-Poincaré sum* of W.

Example 1.2. Let n be a positive integer. The symmetric group S_n is the Coxeter group of type A_{n-1}; the corresponding Coxeter system is (S_n, S), where $S = \{s_1, s_2, \ldots, s_{n-1}\}$ is the set of all simple reflections in S_n (so that s_i is the transposition swapping i with $i+1$). The length $\ell(w)$ of a permutation $w \in S_n$ is well-known to equal the number of inversions of w (that is, of pairs (i, j) of elements of $\{1, 2, \ldots, n\}$ satisfying $i < j$ and $w(i) > w(j)$). The q-Poincaré sum $\mathbf{L}_q(S_n) = \sum_{w \in S_n} q^{\ell(w)} w \in \mathbf{k}[S_n]$ is then a linear combination of all permutations of $\{1, 2, \ldots, n\}$, where each w appears with coefficient $q^{\ell(w)}$. For instance,

$$\mathbf{L}_q(S_3) = \mathrm{id} + qs_1 + qs_2 + q^2 s_1 s_2 + q^2 s_2 s_1 + q^3 w_0,$$

where $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ is the permutation sending $1, 2, 3$ to $3, 2, 1$.

If q is a nonnegative real, then the element $\mathbf{L}_q(S_n)$ can be viewed as a measure on S_n (by treating each coefficient $q^{\ell(w)}$, divided by the total sum of these coefficients, as the measure of $w \in S_n$). This measure is known as the *Mallows measure* Mallows (n, q) (see, e.g., [Korotk16]).

Example 1.3. Let (W, S) be any Coxeter system with finite Coxeter group W. Then,

$$\mathbf{L}_1(W) = \sum_{w \in W} w \qquad \text{and} \qquad \mathbf{L}_{-1}(W) = \sum_{w \in W} \underbrace{(-1)^{\ell(w)}}_{=\operatorname{sign} w} w$$

are two elements in the center of $\mathbf{k}[W]$, and in fact (up to a factor of $|W|$) are the central idempotents corresponding to the trivial and sign representations of W, respectively. Of course, $\mathbf{L}_0(W) = 1$ lies in the center of $\mathbf{k}[W]$, too.

However, in general, $\mathbf{L}_q = \mathbf{L}_q(W)$ does not lie in the center of $\mathbf{k}[W]$ when $q \notin \{1, -1, 0\}$; and in fact, the elements $\mathbf{L}_q$ for two different values of q usually do not commute. That is, most finite Coxeter groups W do **not** satisfy the equality $\mathbf{L}_a\mathbf{L}_b = \mathbf{L}_b\mathbf{L}_a$ when a and b are generic.

This note is devoted to salvaging this equality to some extent. First, we shall show that it holds when W is dihedral (or smaller):

Theorem 1.4. Let (W, S) be a Coxeter system, where W is finite and $|S| \leq 2$. Then,

$$\mathbf{L}_a\mathbf{L}_b = \mathbf{L}_b\mathbf{L}_a$$

for any $a, b \in \mathbf{k}$.

Example 1.5. Consider the symmetric group S_3 as the Coxeter group A_2. Then one can observe that

$$\begin{aligned}
&\mathbf{L}_a\mathbf{L}_b \\
&= (1 + 2ab + 2a^2b^2 + a^3b^3)\operatorname{id} + (a + b + ab^2 + a^2b + a^3b^2 + a^2b^3)(s_1 + s_2) \\
&\quad + (ab + a^2 + b^2 + a^2b^2 + ab^3 + a^3b)(s_1s_2 + s_2s_1) + (2ab^2 + 2a^2b + a^3 + b^3)w_0
\end{aligned}$$

with the notations of Example 1.2. This is visibly symmetric in a and b, so it confirms Theorem 1.4.

We will prove Theorem 1.4, and many other results we shall soon state, in the next section (Section 2).

Next, even though the products $\mathbf{L}_a\mathbf{L}_b$ and $\mathbf{L}_b\mathbf{L}_a$ are usually distinct for general W, we shall show that certain coefficients of these products are equal. For this, we introduce a notation:

Definition 1.6. Let W be a group. Let $a \in \mathbf{k}[W]$ be an element of its group algebra. Let $w \in W$. Then, the w-coefficient of a will be denoted by $[w]\, a$.

Definition 1.7. Let (W, S) be a Coxeter system with finite Coxeter group W. Let $w \in W$. We say that the element w is *W-length-symmetric* if every commutative ring $\mathbf{k}$ and every $a, b \in \mathbf{k}$ satisfy

$$[w]\,(\mathbf{L}_a\mathbf{L}_b) = [w]\,(\mathbf{L}_b\mathbf{L}_a). \tag{1}$$

Alternatively, W-length-symmetry can be characterized combinatorially as follows:

Proposition 1.8. Let (W, S) be a Coxeter system with finite Coxeter group W. Let $w \in W$. Then, w is W-length-symmetric if and only if for every $p, q \in \mathbb{N}$, we have

$$\begin{aligned}&\left(\text{number of } u \in W \text{ satisfying } \ell(u) = p \text{ and } \ell\left(u^{-1}w\right) = q\right)\\ &= \left(\text{number of } u \in W \text{ satisfying } \ell(u) = p \text{ and } \ell\left(wu^{-1}\right) = q\right).\end{aligned}$$

Proof. By Definition 1.1, we have $\mathbf{L}_a = \sum\limits_{w\in W} a^{\ell(w)} w$ and $\mathbf{L}_b = \sum\limits_{w\in W} b^{\ell(w)} w$ for all $a, b \in \mathbf{k}$. Thus, for all $a, b \in \mathbf{k}$, we have

$$\mathbf{L}_a\mathbf{L}_b = \left(\sum_{w\in W} a^{\ell(w)} w\right)\left(\sum_{w\in W} b^{\ell(w)} w\right) = \sum_{(u,v)\in W\times W} a^{\ell(u)} b^{\ell(v)} uv \qquad \text{and}$$

$$\begin{aligned}\mathbf{L}_b\mathbf{L}_a &= \left(\sum_{w\in W} b^{\ell(w)} w\right)\left(\sum_{w\in W} a^{\ell(w)} w\right) = \sum_{(v,u)\in W\times W} b^{\ell(v)} a^{\ell(u)} vu\\ &= \sum_{(v,u)\in W\times W} a^{\ell(u)} b^{\ell(v)} vu.\end{aligned}$$

Hence, each $w \in W$ and $a, b \in \mathbf{k}$ satisfy

$$[w]\left(\mathbf{L}_a\mathbf{L}_b\right) = \sum_{\substack{(u,v)\in W\times W;\\ uv=w}} a^{\ell(u)} b^{\ell(v)} = \sum_{u\in W} a^{\ell(u)} b^{\ell\left(u^{-1}w\right)}$$

(here, we have substituted $\left(u, u^{-1}w\right)$ for (u, v)) and

$$[w]\left(\mathbf{L}_b\mathbf{L}_a\right) = \sum_{\substack{(v,u)\in W\times W;\\ vu=w}} a^{\ell(u)} b^{\ell(v)} = \sum_{u\in W} a^{\ell(u)} b^{\ell\left(wu^{-1}\right)}$$

(here, we have substituted $\left(wu^{-1}, u\right)$ for (v, u)). Hence, the equality (1) can be rewritten as

$$\sum_{u\in W} a^{\ell(u)} b^{\ell\left(u^{-1}w\right)} = \sum_{u\in W} a^{\ell(u)} b^{\ell\left(wu^{-1}\right)}. \tag{2}$$

This equality is a polynomial identity in a and b. Thus, it holds for all commutative rings $\mathbf{k}$ and all $a, b \in \mathbf{k}$ if and only if it holds as a polynomial identity, i.e., if and only if we have

$$\sum_{u\in W} X^{\ell(u)} Y^{\ell\left(u^{-1}w\right)} = \sum_{u\in W} X^{\ell(u)} Y^{\ell\left(wu^{-1}\right)} \tag{3}$$

in the polynomial ring $\mathbb{Z}[X, Y]$ (because (2) can be derived from (3) by evaluating both sides at $X = a$ and $Y = b$, whereas (3) is the particular case of (2) for $\mathbf{k} =$

$\mathbb{Z}[X,Y]$ and $a = X$ and $b = Y$). But the identity (3), in turn, is equivalent to saying that for every $p, q \in \mathbb{N}$, we have

$$\begin{aligned} &\left(\text{number of } u \in W \text{ satisfying } \ell(u) = p \text{ and } \ell\left(u^{-1}w\right) = q\right) \\ &= \left(\text{number of } u \in W \text{ satisfying } \ell(u) = p \text{ and } \ell\left(wu^{-1}\right) = q\right) \end{aligned} \tag{4}$$

(since the former number is the X^pY^q-coefficient on the left-hand side of (3), whereas the latter number is the X^pY^q-coefficient on the right-hand side of (3)).

Combining these equivalences, we conclude that the equality (1) holds for all commutative rings $\mathbf{k}$ and all $a, b \in \mathbf{k}$ if and only if the equality (4) holds for every $p, q \in \mathbb{N}$. But the former equality is exactly the definition of "W-length-symmetric"; thus, we have shown that w is W-length-symmetric if and only if the equality (4) holds for every $p, q \in \mathbb{N}$. This proves Proposition 1.8. □

Remark 1.9. The characterization of W-length-symmetry in Proposition 1.8 makes perfect sense even if W is infinite, as long as S is finite (since the number of $u \in W$ satisfying $\ell(u) = p$ is still finite in this case). However, we will not concern ourselves with infinite Coxeter groups here.

Remark 1.10. As we have already noticed, the equality (2) is a polynomial identity in a and b. In other words, the equality (1) is a polynomial identity in a and b. Thus, in order for it to hold for all $\mathbf{k}$ and a, b, it suffices that it hold when $\mathbf{k} = \mathbb{Q}$ and a and b are positive integers (since two polynomials over $\mathbb{Q}$ are identical if they agree on all positive integers). Hence, an element $w \in W$ is W-length-symmetric if and only if it satisfies (1) for $\mathbf{k} = \mathbb{Q}$ and for all positive integers a and b. This gives yet another combinatorial way to think of W-length-symmetry.

Example 1.11. Consider the symmetric group S_n as a Coxeter group. It is easy to see (and follows from Corollary 1.13 below) that each element of S_n is S_n-length-symmetric for all $n \leq 3$. For $n = 4$, all elements of S_4 are S_4-length-symmetric except for the two permutations with one-line notations $(2,4,1,3)$ and $(3,1,4,2)$. These two permutations are precisely the two patterns that the *separable permutations* are defined to avoid. And we shall indeed see that every separable permutation $w \in S_n$ is S_n-length-symmetric, although the converse is not true. (We shall also see that every involution $w \in S_n$ is S_n-length-symmetric, but not every involution is separable.)

Remark 1.12. One might wonder how many W-length-symmetric elements exist for a Coxeter group W. For instance, for each $n \in \mathbb{N}$, let ω_n be the number of S_n-length-symmetric elements of the Coxeter group S_n. Here are the first few values:

n	0	1	2	3	4	5	6	7
ω_n	1	1	2	6	22	94	424	2032

This sequence is not in the OEIS as of 2026-06-05, even without ω_0. (Unverified code generated by GPT-5.5 additionally reports that $\omega_8 = 9980$.)

Somewhat related is OEIS sequence A180389, which counts the permutations $w \in S_n$ satisfying $\operatorname{des} w = \operatorname{des}\left(w^{-1}\right)$, where $\operatorname{des} u$ denotes the number of descents of u (that is, of simple reflections $s \in S$ satisfying $\ell(us) < \ell(u)$). Indeed, any W-length-symmetric element $w \in S_n$ must satisfy $\operatorname{des} w = \operatorname{des}\left(w^{-1}\right)$, as can easily be seen from the $p = 1$ part of Proposition 1.8; but the converse is quite far from truth.

We shall now give several criteria for when elements of a Coxeter group are W-length-symmetric. The first one is an easy consequence of Theorem 1.4:

Corollary 1.13. Let (W, S) be a Coxeter system, where W is finite and $|S| \leq 2$. Then, each $w \in W$ is W-length-symmetric.

We shall now find W-length-symmetric elements in arbitrary finite Coxeter groups. Some are fairly simple:

Proposition 1.14. Let (W, S) be a Coxeter system, where W is finite. Then, each involution $w \in W$ (that is, each $w \in W$ satisfying $w^2 = 1$) is W-length-symmetric.

Proposition 1.15. Let (W, S) be a Coxeter system, where W is finite. Let $w \in W$ satisfy $\ell(w) \leq 2$. Then, w is W-length-symmetric.

Proposition 1.16. Let (W, S) be a Coxeter system, where W is finite. Let $w \in W$. Then, w is W-length-symmetric if and only if w^{-1} is W-length-symmetric.

Proposition 1.17. Let (W, S) be a Coxeter system, where W is finite. Let $w \in W$. Then, w is W-length-symmetric if and only if $w_0 w$ is W-length-symmetric.

We can generalize Proposition 1.14 from involutions to "twisted involutions":

Proposition 1.18. Let (W, S) be a Coxeter system, where W is finite. Let $\phi : W \to W$ be an automorphism of Coxeter groups (that is, $\phi(S) = S$). Let $w \in W$ be such that $w^{-1} = \phi(w)$. Then, w is W-length-symmetric.

Example 1.19. Let (W, S) be a Coxeter system, where W is finite. Then, conjugation by w_0 (that is, the map $W \to W,\ w \mapsto w_0 w w_0^{-1}$) is an automorphism of Coxeter groups. The elements $w \in W$ satisfying $w^{-1} = w_0 w w_0^{-1}$ are sometimes called the *twisted involutions*. Proposition 1.18 shows that they are W-length-symmetric.

Next, we shall show that W-length-symmetry does not really depend on W, in the sense that an element of a standard parabolic subgroup W_I of W is W-length-symmetric if and only if it is W_I-length-symmetric. First we recall the definition of standard parabolic subgroups:

Definition 1.20. Let (W, S) be a Coxeter system. If I is a subset of S, then W_I denotes the subgroup of W generated by I. This is called a *standard parabolic subgroup* of W. It is known (see, e.g., [Luszti14, Proposition 9.5]) that W_I is again a Coxeter group, part of the Coxeter system (W_I, I).

Proposition 1.21. Let (W, S) be a Coxeter system, where W is finite. Let I be a subset of S. Let $w \in W_I$. Then, w is W-length-symmetric if and only if w is W_I-length-symmetric.

Next, we discuss W-length-symmetry for reducible Coxeter groups. A Coxeter group is said to be *reducible* if it has an orthogonal decomposition; this is defined as follows:

Definition 1.22. Let (W, S) be a Coxeter system. Let $m(i, j)$ denote the order of ij in W for all $i, j \in S$. An *orthogonal decomposition* of (W, S) means a pair (I, J) of two disjoint nonempty subsets I and J of S such that $S = I \cup J$ and such that all $i \in I$ and $j \in J$ satisfy $m(i, j) = 2$ (thus $ij = ji$). We shall denote this decomposition as $S = I \oplus J$.

It is easy to see that if $S = I \oplus J$ is an orthogonal decomposition, then $W = W_I W_J \cong W_I \times W_J$.

Proposition 1.23. Let (W, S) be a Coxeter system, where W is finite. Let $S = I \oplus J$ be an orthogonal decomposition. Let $u \in W_I$ and $v \in W_J$. Assume that u is W_I-length-symmetric and v is W_J-length-symmetric. Then, uv is W-length-symmetric.

Remark 1.24. The converse of Proposition 1.23 is false. A large source of counterexamples comes from Proposition 1.18. Namely, let (W, S) be a Coxeter system that is a "direct product" of two isomorphic Coxeter systems – i.e., that has an orthogonal decomposition $S = I \oplus J$ and an isomorphism $\rho : (W_I, I) \to (W_J, J)$ of Coxeter groups. Let $\phi : (W, S) \to (W, S)$ be the automorphism of Coxeter groups that is induced by ρ (that is, sends I to J via ρ, and sends J to I via ρ^{-1}). Fix any $u \in W_I$, and let $v := \rho\left(u^{-1}\right) \in W_J$. Then, the element $uv \in W$ satisfies $(uv)^{-1} = \phi(uv)$ (since the definition of ϕ yields $\phi(uv) = \underbrace{\rho(u)}_{=v^{-1}} \underbrace{\rho^{-1}(v)}_{=u^{-1}} = v^{-1}u^{-1} = (uv)^{-1}$), and thus is W-length-symmetric by Proposition 1.18. But u is not necessarily W_I-length-symmetric, since it can be any element of W_I.

Remark 1.25. One can also define a variant of W-length-symmetry in the Hecke algebra. Let (W, S) be a Coxeter system with W finite, and let $\mathcal{H}$ be its Hecke algebra (see, e.g., [Humphr90, §7.4]) over the commutative ring $\mathbf{k}$ with a parameter

$q \in \mathbf{k}$. For any $r \in \mathbf{k}$, define

$$\mathbf{L}_r^{\mathcal{H}} := \sum_{w \in W} r^{\ell(w)} T_w \in \mathcal{H}.$$

Then, an element $w \in W$ can be called *Hecke-W-length-symmetric* if the coefficient of T_w in $\mathbf{L}_a^{\mathcal{H}} \mathbf{L}_b^{\mathcal{H}}$ equals the coefficient of T_w in $\mathbf{L}_b^{\mathcal{H}} \mathbf{L}_a^{\mathcal{H}}$ for arbitrary $\mathbf{k}$ and $q, a, b \in \mathbf{k}$. This is a stronger property than W-length-symmetry; in particular, the permutations $[2431]$, $[3241]$, $[4132]$, $[4213]$ in S_4 (written in one-line notation) are S_4-length-symmetric but not Hecke-S_4-length-symmetric.

Finding Hecke analogues of the above results on W-length-symmetry appears doable but not always trivial. An easy first step is generalizing Theorem 1.4: If $|S| \leq 2$, then

$$\mathbf{L}_a^{\mathcal{H}} \mathbf{L}_b^{\mathcal{H}} = \mathbf{L}_b^{\mathcal{H}} \mathbf{L}_a^{\mathcal{H}} \qquad \text{for all } q, a, b \in \mathbf{k}. \tag{5}$$

This can be proved in roughly the same way as we prove Theorem 1.4 below (viz., by replacing each w by T_w in Lemma 2.2, and slightly modifying its proof). Furthermore, Proposition 1.14, Proposition 1.16 and Proposition 1.18 also remain true if we replace "W-length-symmetric" by "Hecke-W-length-symmetric", and the proof we give below generalizes to this as well (since $\mathcal{H}$ also has an "antipode", i.e., an involutive $\mathbf{k}$-algebra anti-automorphism that sends each T_w to $T_{w^{-1}}$). On the other hand, a version of Proposition 1.17 appears to be missing for $\mathcal{H}$. The generalizability of Proposition 1.21 is an interesting question.

2. Proofs

2.1. Coefficients

Before we step to the proofs of the above claims, we establish some basic facts.

We begin with some basic properties of the notation from Definition 1.6:

Lemma 2.1. Let W be a group. Let $w, g, h \in W$ and $a \in \mathbf{k}[W]$. Then,

$$[w](gah) = \left[g^{-1} w h^{-1}\right](a).$$

Proof. Both sides of the claimed equality are $\mathbf{k}$-linear in a. Thus, we WLOG assume that a is an element of the basis W of the $\mathbf{k}$-module $\mathbf{k}[W]$ (since a is clearly a $\mathbf{k}$-linear combination of these basis elements). In other words, $a \in W$. Hence, $gah \in W$ as well, so that

$$[w](gah) = \begin{cases} 1, & \text{if } w = gah; \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

(because any $p, q \in W$ satisfy $[p]\, q = \begin{cases} 1, & \text{if } p = q; \\ 0, & \text{otherwise} \end{cases}$) and similarly

$$\left[g^{-1} w h^{-1}\right](a) = \begin{cases} 1, & \text{if } g^{-1} w h^{-1} = a; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

However, $w = gah$ holds if and only if $g^{-1} w h^{-1} = a$. Hence, the right-hand sides of the equalities (6) and (7) are equal. Thus, their left-hand sides are equal as well. In other words, $[w]\,(gah) = \left[g^{-1} w h^{-1}\right](a)$. This proves Lemma 2.1. □

2.2. Dihedral groups

Dihedral groups (Coxeter groups with two generators) are simple but form a crucial building block of Coxeter group theory. The following lemma is likely known in some form, but we could not find it in the literature.

Lemma 2.2. Let (W, S) be a Coxeter system, where $S = \{s, t\}$ is a two-element set. (Thus, W is a dihedral group.) For each $i \in \mathbb{N}$, we let $(st)_i$ denote the product $ststs \cdots$ with i factors (that is, $(st)^{i/2}$ if i is even, and $(st)^{(i-1)/2} s$ if i is odd), and we let $(ts)_i$ denote the product $tstst \cdots$ with i factors. Let B be the $\mathbf{k}$-subalgebra of the group algebra $\mathbf{k}[W]$ generated by $s + t$. Then, each $i \in \mathbb{N}$ satisfies

$$(st)_i + (ts)_i \in B.$$

Proof. We prove this by strong induction on i. The *base cases* $i = 0$ and $i = 1$ are obvious (since $(st)_0 + (ts)_0 = 1 + 1 = 2$ and $(st)_1 + (ts)_1 = s + t$). For the *induction step*, we observe that each $i \geq 2$ satisfies

$$\begin{aligned}
&(s+t)\left((st)_{i-1} + (ts)_{i-1}\right) \\
&= s \underbrace{(st)_{i-1}}_{=s(ts)_{i-2}} + \underbrace{s\,(ts)_{i-1}}_{=(st)_i} + \underbrace{t\,(st)_{i-1}}_{=(ts)_i} + t \underbrace{(ts)_{i-1}}_{=t(st)_{i-2}} \\
&= \underbrace{ss}_{=s^2=1} (ts)_{i-2} + (st)_i + (ts)_i + \underbrace{tt}_{=t^2=1} (st)_{i-2} \\
&= (ts)_{i-2} + (st)_i + (ts)_i + (st)_{i-2} = (st)_i + (ts)_i + (st)_{i-2} + (ts)_{i-2}
\end{aligned}$$

and thus

$$\begin{aligned}
(st)_i + (ts)_i &= \underbrace{(s+t)}_{\in B} \underbrace{\left((st)_{i-1} + (ts)_{i-1}\right)}_{\substack{\in B \\ \text{(by the induction hypothesis)}}} - \underbrace{\left((st)_{i-2} + (ts)_{i-2}\right)}_{\substack{\in B \\ \text{(by the induction hypothesis)}}} \\
&\in B \qquad (\text{since } B \text{ is a subalgebra}).
\end{aligned}$$

This completes the induction step, and thus the proof of the lemma. □

Proof of Theorem 1.4. Let $a, b \in \mathbf{k}$. We must prove that $\mathbf{L}_a \mathbf{L}_b = \mathbf{L}_b \mathbf{L}_a$.

If $|S| \leq 1$, then the $\mathbf{k}$-algebra $\mathbf{k}[W]$ is commutative (since it is generated by S), and thus the claim $\mathbf{L}_a \mathbf{L}_b = \mathbf{L}_b \mathbf{L}_a$ holds for obvious reasons. Thus, we WLOG assume that $|S| > 1$ from now on.

Hence, $|S| = 2$ (since we assumed that $|S| \leq 2$). Write this 2-element set S as $S = \{s, t\}$.

Use the notations from Lemma 2.2. In particular, define the $\mathbf{k}$-subalgebra B of $\mathbf{k}[W]$ as in Lemma 2.2. Note that this algebra B is commutative, since it is generated by a single element.

However, W is a dihedral group, so it has a well-known description: If we let m denote the order of st in the group W, then the elements of W are precisely the products $(st)_i$ and $(ts)_i$ for all $i \in \{0, 1, \ldots, m\}$, which are furthermore all distinct except that $(st)_0 = (ts)_0$ and $(st)_m = (ts)_m$. Moreover, the products $(st)_i$ and $(ts)_i$ have length i for all $i \in \{0, 1, \ldots, m\}$. Altogether, we see that the elements of W (listed without repetitions) are

$$\underbrace{1}_{\text{length } 0}, \underbrace{(st)_1, (ts)_1}_{\text{length } 1}, \underbrace{(st)_2, (ts)_2}_{\text{length } 2}, \ldots, \underbrace{(st)_{m-1}, (ts)_{m-1}}_{\text{length } m-1}, \underbrace{(st)_m}_{\text{length } m}.$$

Thus, the definition $\mathbf{L}_a = \sum\limits_{w \in W} a^{\ell(w)} w$ of $\mathbf{L}_a$ can be rewritten as

$$\begin{aligned} \mathbf{L}_a &= 1 + \sum_{i=1}^{m-1} \left(a^i (st)_i + a^i (ts)_i \right) + a^m (st)_m \\ &= 1 + \sum_{i=1}^{m-1} a^i \left((st)_i + (ts)_i \right) + a^m (st)_m . \end{aligned} \tag{8}$$

Let us WLOG assume that $\mathbf{k}$ is a polynomial ring over $\mathbb{Q}$ (we can do this, since we are proving a polynomial identity, so we can assume that $\mathbf{k}$, a and b are generic, as explained in Remark 1.10). Thus, 2 is invertible in $\mathbf{k}$. Hence, in (8), we can rewrite $(st)_m$ as $\dfrac{(st)_m + (ts)_m}{2}$ (since $(st)_m = (ts)_m$), and thus (8) becomes

$$\mathbf{L}_a = 1 + \sum_{i=1}^{m-1} a^i \left((st)_i + (ts)_i \right) + a^m \frac{(st)_m + (ts)_m}{2}.$$

This shows that $\mathbf{L}_a$ belongs to B (since Lemma 2.2 shows that all the sums $(st)_i + (ts)_i$ belong to B). Similarly, $\mathbf{L}_b$ belongs to B as well. Hence, $\mathbf{L}_a$ and $\mathbf{L}_b$ commute (since B is commutative). That is, $\mathbf{L}_a \mathbf{L}_b = \mathbf{L}_b \mathbf{L}_a$. Theorem 1.4 is thus proved. $\square$

Proof of Corollary 1.13. Let $w \in W$. For all $a, b \in \mathbf{k}$, we have $\mathbf{L}_a \mathbf{L}_b = \mathbf{L}_b \mathbf{L}_a$ by Theorem 1.4, and therefore $[w](\mathbf{L}_a \mathbf{L}_b) = [w](\mathbf{L}_b \mathbf{L}_a)$. In other words, w is W-length-symmetric. This proves Corollary 1.13. $\square$

2.3. The antipode, inverses and involutions

For any group G, there is a $\mathbf{k}$-linear map $\mathbf{k}[G] \to \mathbf{k}[G]$ that sends each group element $g \in G$ to g^{-1}. This map is called the *antipode* of the group algebra $\mathbf{k}[G]$, and will be denoted by $a \mapsto a^*$. It is an involution (i.e., each $a \in \mathbf{k}[G]$ satisfies $a^{**} = a$), and is a $\mathbf{k}$-algebra anti-morphism (i.e., each $a, b \in \mathbf{k}[G]$ satisfy $(ab)^* = b^* a^*$). Thus, it is a $\mathbf{k}$-algebra anti-isomorphism. Clearly, all $w \in G$ and $a \in \mathbf{k}[G]$ satisfy

$$[w](a) = \left[w^{-1}\right](a^*). \tag{9}$$

Lemma 2.3. Let (W, S) be a Coxeter system, where W is finite. Let $q \in \mathbf{k}$. Then, $\mathbf{L}_q^* = \mathbf{L}_q$.

Proof. This is clear, since each $w \in W$ satisfies $\ell\left(w^{-1}\right) = \ell(w)$. □

Lemma 2.4. Let (W, S) be a Coxeter system, where W is finite. Let $a, b \in \mathbf{k}$ and $w \in W$. Then, $[w](\mathbf{L}_a \mathbf{L}_b) = \left[w^{-1}\right](\mathbf{L}_b \mathbf{L}_a)$.

Proof. We have $(\mathbf{L}_a \mathbf{L}_b)^* = \mathbf{L}_b^* \mathbf{L}_a^*$ (since the antipode is a $\mathbf{k}$-algebra anti-morphism). But Lemma 2.3 yields $\mathbf{L}_a^* = \mathbf{L}_a$ and $\mathbf{L}_b^* = \mathbf{L}_b$. Hence, $(\mathbf{L}_a \mathbf{L}_b)^* = \underbrace{\mathbf{L}_b^*}_{=\mathbf{L}_b} \underbrace{\mathbf{L}_a^*}_{=\mathbf{L}_a} = \mathbf{L}_b \mathbf{L}_a$.

Now, (9) yields

$$[w](\mathbf{L}_a \mathbf{L}_b) = \left[w^{-1}\right]\left((\mathbf{L}_a \mathbf{L}_b)^*\right) = \left[w^{-1}\right](\mathbf{L}_b \mathbf{L}_a) \qquad \left(\text{since } (\mathbf{L}_a \mathbf{L}_b)^* = \mathbf{L}_b \mathbf{L}_a\right),$$

and thus Lemma 2.4 is proved. □

Proof of Proposition 1.16. Let $a, b \in \mathbf{k}$ be arbitrary. Then,

$$[w](\mathbf{L}_a \mathbf{L}_b) = \left[w^{-1}\right](\mathbf{L}_b \mathbf{L}_a) \qquad (\text{by Lemma 2.4}), \qquad \text{and}$$
$$[w](\mathbf{L}_b \mathbf{L}_a) = \left[w^{-1}\right](\mathbf{L}_a \mathbf{L}_b) \qquad (\text{by Lemma 2.4, with the roles of } a \text{ and } b \text{ swapped}).$$

Hence, the equality $[w](\mathbf{L}_a \mathbf{L}_b) = [w](\mathbf{L}_b \mathbf{L}_a)$ holds if and only if the equality $\left[w^{-1}\right](\mathbf{L}_b \mathbf{L}_a) = \left[w^{-1}\right](\mathbf{L}_a \mathbf{L}_b)$ holds. But the former equality (stated for all $\mathbf{k}$ and $a, b \in \mathbf{k}$) says that w is W-length-symmetric, whereas the latter equality says that w^{-1} is W-length-symmetric. Hence, w is W-length-symmetric if and only if w^{-1} is W-length-symmetric. This proves Proposition 1.16. □

Proof of Proposition 1.14. Let $w \in W$ be an involution, and $a, b \in \mathbf{k}$ be arbitrary. Then, w is an involution, so that $w^{-1} = w$. However, Lemma 2.4 yields

$$[w](\mathbf{L}_a \mathbf{L}_b) = \left[w^{-1}\right](\mathbf{L}_b \mathbf{L}_a) = [w](\mathbf{L}_b \mathbf{L}_a) \qquad \left(\text{since } w^{-1} = w\right).$$

This shows that w is W-length-symmetric. This proves Proposition 1.14. □

Proof of Proposition 1.18. Forget that we fixed w. The group automorphism ϕ of W induces (by linearity) a $\mathbf{k}$-algebra automorphism $\mathbf{k}[\phi]$ of $\mathbf{k}[W]$, which sends each $w \in W$ to $\phi(w)$.

Moreover, the automorphism ϕ of W is a Coxeter group automorphism. Thus, it preserves the length of any element $w \in W$; that is, we have

$$\ell(\phi(w)) = \ell(w) \qquad \text{for each } w \in W. \tag{10}$$

Thus, for each $q \in \mathbf{k}$, we have

$$\begin{aligned}
(\mathbf{k}[\phi])(\mathbf{L}_q) &= (\mathbf{k}[\phi])\left(\sum_{w\in W} q^{\ell(w)} w\right) \qquad \left(\text{since } \mathbf{L}_q = \sum_{w\in W} q^{\ell(w)} w\right) \\
&= \sum_{w\in W} \underbrace{q^{\ell(w)}}_{\substack{=q^{\ell(\phi(w))} \\ \text{(since (10) yields } \ell(w)=\ell(\phi(w)))}} \phi(w) = \sum_{w\in W} q^{\ell(\phi(w))} \phi(w) \\
&= \sum_{w\in W} q^{\ell(w)} w \qquad \left(\begin{array}{c}\text{here, we have substituted } w \text{ for } \phi(w) \text{ in} \\ \text{the sum, since } \phi : W \to W \text{ is a bijection}\end{array}\right) \\
&= \mathbf{L}_q.
\end{aligned}$$

In other words, $\mathbf{k}[\phi]$ fixes $\mathbf{L}_q$ for each $q \in \mathbf{k}$.

Let $a, b \in \mathbf{k}$ be arbitrary. Thus, the $\mathbf{k}$-algebra automorphism $\mathbf{k}[\phi]$ of $\mathbf{k}[W]$ fixes the elements $\mathbf{L}_a$ and $\mathbf{L}_b$ (since it fixes $\mathbf{L}_q$ for each $q \in \mathbf{k}$). Being a $\mathbf{k}$-algebra automorphism, it therefore also fixes the product $\mathbf{L}_b\mathbf{L}_a$. In other words,

$$(\mathbf{k}[\phi])(\mathbf{L}_b\mathbf{L}_a) = \mathbf{L}_b\mathbf{L}_a. \tag{11}$$

Moreover, by its very definition, it satisfies

$$[\phi(w)]((\mathbf{k}[\phi])(\alpha)) = [w](\alpha) \tag{12}$$

for all $w \in W$ and $\alpha \in \mathbf{k}[W]$.

Now, let $w \in W$ be such that $w^{-1} = \phi(w)$. Now, Lemma 2.4 yields

$$\begin{aligned}
[w](\mathbf{L}_a\mathbf{L}_b) &= \left[w^{-1}\right](\mathbf{L}_b\mathbf{L}_a) \\
&= [\phi(w)] \underbrace{(\mathbf{L}_b\mathbf{L}_a)}_{\substack{=(\mathbf{k}[\phi])(\mathbf{L}_b\mathbf{L}_a) \\ \text{(by (11))}}} \qquad \left(\text{since } w^{-1} = \phi(w)\right) \\
&= [\phi(w)]((\mathbf{k}[\phi])(\mathbf{L}_b\mathbf{L}_a)) = [w](\mathbf{L}_b\mathbf{L}_a) \qquad \text{(by (12))}.
\end{aligned}$$

This shows that w is W-length-symmetric. This proves Proposition 1.18. □

2.4. Cosets and parabolic subgroups

Our next lemma is concerned with cosets and their representatives. Due to the perennial vexatiousness of their chirality, we recall their definitions:

Definition 2.5. Let H be a subgroup of a group G.

- The *left cosets* of H in G are the subsets of the form uH for $u \in G$. The set of these left cosets is denoted by G/H.

  A *left transversal* of H in G means a subset L of G that intersects each of these left cosets in exactly one element (i.e., satisfies $|uH \cap L| = 1$ for each $u \in G$). Equivalently, it is a subset L of G with the property that each $g \in G$ can be written uniquely as xh with $h \in H$ and $x \in L$.

- The *right cosets* of H in G are the subsets of the form Hu for $u \in G$. The set of these right cosets is denoted by $H \backslash G$.

  A *right transversal* of H in G means a subset R of G that intersects each of these right cosets in exactly one element (i.e., satisfies $|Hu \cap R| = 1$ for each $u \in G$). Equivalently, it is a subset R of G with the property that each $g \in G$ can be written uniquely as hx with $h \in H$ and $x \in R$.

The left and right versions here are intimately related: If L is a left transversal of H in G, then $\left\{x^{-1} \mid x \in L\right\}$ is a right transversal of H in G. An analogous statement holds with "left" and "right" interchanged.

Lemma 2.6. Let G be a finite group, and let H be a subgroup of G. Let R be a right transversal of H in G. For each $x \in R$, let $p_x, q_x \in \mathbf{k}$ be two scalars. Set

$$p = \sum_{x \in R} p_x x \qquad \text{and} \qquad q = \sum_{x \in R} q_x x^{-1}$$

in $\mathbf{k}[G]$. Let $w \in H$ and $\alpha, \beta \in \mathbf{k}[H]$ (where $\mathbf{k}[H]$ is embedded in $\mathbf{k}[G]$ in the obvious way). Then,

$$[w](\alpha p q \beta) = \left(\sum_{x \in R} p_x q_x\right)[w](\alpha\beta). \tag{13}$$

Proof. The claim is $\mathbf{k}$-linear in each of α and β. Thus, we WLOG assume that $\alpha, \beta \in H$ (since $\alpha, \beta \in \mathbf{k}[H]$ are clearly $\mathbf{k}$-linear combinations of elements of H). Furthermore, let us set $u := \alpha^{-1} w \beta^{-1} \in H$ (since $w, \alpha, \beta \in H$).

Hence, $w, \alpha, \beta \in H \subseteq G$. Thus, Lemma 2.1 (applied to α, β and pq instead of g, h and a) yields

$$[w](\alpha p q \beta) = \left[\alpha^{-1} w \beta^{-1}\right](pq) = [u](pq) \tag{14}$$

(since $\alpha^{-1} w \beta^{-1} = u$). Moreover, Lemma 2.1 (applied to α, β and 1 instead of g, h and a) yields

$$[w](\alpha\beta) = \left[\alpha^{-1} w \beta^{-1}\right] 1 = [u] 1 \tag{15}$$

(since $\alpha^{-1}w\beta^{-1} = u$). Using (14) and (15), we can rewrite our claim (13) as

$$[u](pq) = \left(\sum_{x\in R} p_x q_x\right)[u](1). \tag{16}$$

So it remains to prove (16).

But the definitions of p and q yield

$$\begin{aligned} pq &= \left(\sum_{x\in R} p_x x\right)\left(\sum_{x\in R} q_x x^{-1}\right) = \sum_{(x,y)\in R\times R} p_x q_y xy^{-1} \\ &= \sum_{\substack{(x,y)\in R\times R;\\ x=y}} p_x q_y xy^{-1} + \sum_{\substack{(x,y)\in R\times R;\\ x\neq y}} p_x q_y xy^{-1} \end{aligned}$$

(here, we have split the sum into the part with $x = y$ and the part with $x \neq y$). In view of

$$\sum_{\substack{(x,y)\in R\times R;\\ x=y}} p_x q_y xy^{-1} = \sum_{x\in R} p_x q_x \underbrace{xx^{-1}}_{=1} = \sum_{x\in R} p_x q_x 1,$$

we can rewrite this as

$$pq = \sum_{x\in R} p_x q_x 1 + \sum_{\substack{(x,y)\in R\times R;\\ x\neq y}} p_x q_y xy^{-1}. \tag{17}$$

Comparing coefficients of u on both sides of this equality, we obtain

$$[u](pq) = \sum_{x\in R} p_x q_x [u] 1 + \sum_{\substack{(x,y)\in R\times R;\\ x\neq y}} p_x q_y [u]\left(xy^{-1}\right). \tag{18}$$

However, we claim that if two elements $x, y \in R$ satisfy $x \neq y$, then

$$[u]\left(xy^{-1}\right) = 0. \tag{19}$$

[*Proof:* Let $x, y \in R$ be two elements satisfying $x \neq y$. Then, x and y are two distinct elements of R. Since R is a right transversal of H, this entails that the right cosets Hx and Hy are distinct. Thus, the two elements u and xy^{-1} of G are distinct (because if they were equal, then we would have $xy^{-1} = u \in H$, so that $x \in Hy$ and therefore $Hx = Hy$, contradicting the distinctness of Hx and Hy). Therefore, $[u]\left(xy^{-1}\right) = 0$ (because $[g]h = 0$ for any two distinct elements g and h of G). This proves (19).]

Now, (18) becomes

$$\begin{aligned} [u](pq) &= \sum_{x\in R} p_x q_x [u] 1 + \sum_{\substack{(x,y)\in R\times R;\\ x\neq y}} p_x q_y \underbrace{[u]\left(xy^{-1}\right)}_{\substack{=0\\ \text{(by (19))}}} = \sum_{x\in R} p_x q_x [u] 1 \\ &= \left(\sum_{x\in R} p_x q_x\right)[u](1). \end{aligned}$$

This proves (16). Thus, Lemma 2.6 is proved. $\square$

Proof of Proposition 1.21. Given any $q \in \mathbf{k}$, let us denote the element $\mathbf{L}_q(W) \in \mathbf{k}[W]$ by $\mathbf{L}_q$, while the analogous element $\mathbf{L}_q(W_I) \in \mathbf{k}[W_I]$ shall be denoted by $\mathbf{L}'_q$. Thus, w is W-length-symmetric if and only if all $\mathbf{k}$ and $a, b \in \mathbf{k}$ satisfy $[w](\mathbf{L}_a\mathbf{L}_b) = [w](\mathbf{L}_b\mathbf{L}_a)$, whereas w is W_I-length-symmetric if and only if all $\mathbf{k}$ and $a, b \in \mathbf{k}$ satisfy $[w]\left(\mathbf{L}'_a\mathbf{L}'_b\right) = [w]\left(\mathbf{L}'_b\mathbf{L}'_a\right)$. In both cases, we can restrict ourselves to the "generic" case, i.e., to the case when $\mathbf{k}$ is the polynomial ring $\mathbb{Z}[a, b]$ and when a and b are the two indeterminates (as in the proof of Proposition 1.8). Let us do so. Thus, w is W-length-symmetric if and only if $[w](\mathbf{L}_a\mathbf{L}_b) = [w](\mathbf{L}_b\mathbf{L}_a)$, whereas w is W_I-length-symmetric if and only if $[w]\left(\mathbf{L}'_a\mathbf{L}'_b\right) = [w]\left(\mathbf{L}'_b\mathbf{L}'_a\right)$.

Each right coset of W_I in W has a unique representative of minimum length (by [Luszti14, Lemma 9.7 (a)]). Let R be the set of all these representatives. Thus, R is a right transversal of W_I in W. In other words, each $g \in W$ can be written uniquely as hx with $h \in W_I$ and $x \in R$. That is, the map $W_I \times R \to W$, $(h, x) \mapsto hx$ is a bijection. Moreover, [Luszti14, Lemma 9.7 (b)] says that any $x \in R$ and any $h \in W_I$ satisfy

$$\ell(hx) = \ell(h) + \ell(x) \tag{20}$$

(since $x \in R$ means that x is a representative of minimum length for some right coset $W_I a$ of W_I in W). Set

$$\mathbf{X}_a := \sum_{x \in R} a^{\ell(x)} x \qquad \text{and} \qquad \mathbf{X}_b := \sum_{x \in R} b^{\ell(x)} x.$$

Then, the definition of $\mathbf{L}_a = \mathbf{L}_a(W)$ yields

$$\begin{aligned}
\mathbf{L}_a &= \sum_{w \in W} a^{\ell(w)} w = \underbrace{\sum_{(h,x) \in W_I \times R}}_{= \sum_{h \in W_I} \sum_{x \in R}} \underbrace{a^{\ell(hx)}}_{\substack{= a^{\ell(h)+\ell(x)} \\ \text{(by (20))}}} hx \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } hx \text{ for } w \text{ in the sum,} \\ \text{since the map } W_I \times R \to W,\ (h, x) \mapsto hx \text{ is a bijection} \end{array} \right) \\
&= \sum_{h \in W_I} \sum_{x \in R} a^{\ell(h)+\ell(x)} hx = \underbrace{\left(\sum_{h \in W_I} a^{\ell(h)} h \right)}_{\substack{= \mathbf{L}'_a \\ \text{(by the definition of } \mathbf{L}'_a = \mathbf{L}_a(W_I))}} \underbrace{\left(\sum_{x \in R} a^{\ell(x)} x \right)}_{\substack{= \mathbf{X}_a \\ \text{(by the definition of } \mathbf{X}_a)}} \\
&= \mathbf{L}'_a \mathbf{X}_a.
\end{aligned} \tag{21}$$

Similarly, $\mathbf{L}_b = \mathbf{L}'_b\mathbf{X}_b$. Applying the antipode to this latter equality, we obtain

$$\mathbf{L}_b^* = \left(\mathbf{L}'_b\mathbf{X}_b\right)^* = \mathbf{X}_b^*\mathbf{L}'^*_b \tag{22}$$

(since the antipode is a $\mathbf{k}$-algebra anti-morphism). But Lemma 2.3 yields $\mathbf{L}_b^* = \mathbf{L}_b$ and $\mathbf{L}'^*_b = \mathbf{L}'_b$. Hence, we can rewrite (22) as

$$\mathbf{L}_b = \mathbf{X}_b^*\mathbf{L}'_b. \tag{23}$$

However, $w \in W_I$ and $\mathbf{L}'_a, \mathbf{L}'_b \in \mathbf{k}[W_I]$ and

$$\mathbf{X}_a = \sum_{x \in R} a^{\ell(x)} x \qquad \text{and} \qquad \mathbf{X}^*_b = \sum_{x \in R} b^{\ell(x)} x^{-1}$$

(the latter equality follows from $\mathbf{X}_b = \sum\limits_{x \in R} b^{\ell(x)} x$ by the definition of the antipode). Hence, Lemma 2.6 (applied to $G = W$ and $H = W_I$ and $p_x = a^{\ell(x)}$ and $q_x = b^{\ell(x)}$ and $p = \mathbf{X}_a$ and $q = \mathbf{X}^*_b$ and $\alpha = \mathbf{L}'_a$ and $\beta = \mathbf{L}'_b$) yields

$$[w]\left(\mathbf{L}'_a \mathbf{X}_a \mathbf{X}^*_b \mathbf{L}'_b\right) = \left(\sum_{x \in R} a^{\ell(x)} b^{\ell(x)}\right) [w]\left(\mathbf{L}'_a \mathbf{L}'_b\right).$$

Using (21) and (23), we can rewrite this as

$$[w]\left(\mathbf{L}_a \mathbf{L}_b\right) = \left(\sum_{x \in R} a^{\ell(x)} b^{\ell(x)}\right) [w]\left(\mathbf{L}'_a \mathbf{L}'_b\right). \tag{24}$$

The same argument (with the roles of a and b interchanged) shows that

$$[w]\left(\mathbf{L}_b \mathbf{L}_a\right) = \left(\sum_{x \in R} b^{\ell(x)} a^{\ell(x)}\right) [w]\left(\mathbf{L}'_b \mathbf{L}'_a\right). \tag{25}$$

Clearly, the factors $\sum\limits_{x \in R} a^{\ell(x)} b^{\ell(x)}$ and $\sum\limits_{x \in R} b^{\ell(x)} a^{\ell(x)}$ on the right-hand sides of the two equalities (24) and (25) are equal, and are not zero divisors (since the polynomial ring $\mathbb{Z}[a,b]$ is an integral domain, and since R is nonempty, so that $\sum\limits_{x \in R} a^{\ell(x)} b^{\ell(x)} \neq 0$ in $\mathbb{Z}[a,b]$). Hence, these two equalities show that $[w]\left(\mathbf{L}_a \mathbf{L}_b\right) = [w]\left(\mathbf{L}_b \mathbf{L}_a\right)$ if and only if $[w]\left(\mathbf{L}'_a \mathbf{L}'_b\right) = [w]\left(\mathbf{L}'_b \mathbf{L}'_a\right)$.

In other words, w is W-length-symmetric if and only if w is W_I-length-symmetric (since we know that w is W-length-symmetric if and only if $[w]\left(\mathbf{L}_a \mathbf{L}_b\right) = [w]\left(\mathbf{L}_b \mathbf{L}_a\right)$, whereas w is W_I-length-symmetric if and only if $[w]\left(\mathbf{L}'_a \mathbf{L}'_b\right) = [w]\left(\mathbf{L}'_b \mathbf{L}'_a\right)$). This proves Proposition 1.21. □

Proof of Proposition 1.15. From $\ell(w) \leq 2$, we know that we can write w as a product of at most two simple reflections. Let I be the set of all simple reflections appearing in this product. Then, $w \in W_I$ and $|I| \leq 2$. Hence, Corollary 1.13 (applied to the Coxeter system (W_I, I) instead of (W, S)) yields that w is W_I-length-symmetric. By Proposition 1.21, this entails that w is W-length-symmetric. Thus, Proposition 1.15 is proved. □

2.5. Orthogonal decompositions

Proof of Proposition 1.23. Let $\mathbf{k}$ be any commutative ring, and $a, b \in \mathbf{k}$.

Since $S = I \oplus J$, we have $W \cong W_I \times W_J$; more specifically, the map

$$\begin{aligned} W_I \times W_J &\to W, \\ (x, y) &\mapsto xy \end{aligned} \tag{26}$$

is a group isomorphism. This isomorphism entails that the $\mathbf{k}$-linear map

$$\begin{aligned} \Phi : \mathbf{k}[W_I] \otimes \mathbf{k}[W_J] &\to \mathbf{k}[W], \\ x \otimes y &\mapsto xy \end{aligned} \tag{27}$$

is a $\mathbf{k}$-algebra isomorphism (because there is a $\mathbf{k}$-algebra isomorphism

$$\begin{aligned} \mathbf{k}[W_I \times W_J] &\to \mathbf{k}[W_I] \otimes \mathbf{k}[W_J], \\ (x, y) &\mapsto x \otimes y, \end{aligned}$$

and if we identify $\mathbf{k}[W_I] \otimes \mathbf{k}[W_J]$ with $\mathbf{k}[W_I \times W_J]$ along this isomorphism, then (27) is the linearization of the group isomorphism (26)). The construction of Φ shows that any $\alpha \in \mathbf{k}[W_I]$ and $\beta \in \mathbf{k}[W_J]$ satisfy

$$[uv](\Phi(\alpha \otimes \beta)) = [u](\alpha) \cdot [v](\beta). \tag{28}$$

(Indeed, this equality is $\mathbf{k}$-linear in α and β, and thus needs only be checked for all $\alpha \in W_I$ and $\beta \in W_J$; but then it boils down to "$\alpha\beta = uv$ if and only if $\alpha = u$ and $\beta = v$".)

Moreover, any element of W_I commutes with any element of W_J; thus, for any $x \in W_I$ and $y \in W_J$, we have

$$\ell(xy) = \ell(x) + \ell(y) \tag{29}$$

(since any reduced expression for xy can be transformed into a reduced expression for x times a reduced expression for y by commuting its I-factors to the left of all its J-factors[1]). But Definition 1.1 yields

$$\mathbf{L}_a(W_I) = \sum_{x \in W_I} a^{\ell(x)} x \qquad \text{and} \qquad \mathbf{L}_a(W_J) = \sum_{y \in W_J} a^{\ell(y)} y,$$

so that

$$\begin{aligned}
\mathbf{L}_a(W_I) \otimes \mathbf{L}_a(W_J) &= \left(\sum_{x \in W_I} a^{\ell(x)} x \right) \otimes \left(\sum_{y \in W_J} a^{\ell(y)} y \right) \\
&= \sum_{(x,y) \in W_I \times W_J} \underbrace{a^{\ell(x)} a^{\ell(y)}}_{\substack{= a^{\ell(x)+\ell(y)} = a^{\ell(xy)} \\ \text{(by (29))}}} x \otimes y = \sum_{(x,y) \in W_I \times W_J} a^{\ell(xy)} x \otimes y.
\end{aligned}$$

Applying the linear map Φ to this equality, we obtain

$$\begin{aligned}
\Phi\left(\mathbf{L}_a(W_I) \otimes \mathbf{L}_a(W_J)\right) &= \sum_{(x,y) \in W_I \times W_J} a^{\ell(xy)} \underbrace{\Phi(x \otimes y)}_{= xy} = \sum_{(x,y) \in W_I \times W_J} a^{\ell(xy)} xy \\
&= \sum_{w \in W} a^{\ell(w)} w \qquad \left(\begin{array}{c} \text{here, we have substituted } w \text{ for } xy, \\ \text{since the map (26) is a bijection} \end{array} \right) \\
&= \mathbf{L}_a(W).
\end{aligned}$$

Similarly, $\Phi\left(\mathbf{L}_b(W_I) \otimes \mathbf{L}_b(W_J)\right) = \mathbf{L}_b(W)$. Since Φ is an algebra morphism, we now have

$$\begin{aligned}
&\Phi\left(\left(\mathbf{L}_a(W_I) \otimes \mathbf{L}_a(W_J)\right)\left(\mathbf{L}_b(W_I) \otimes \mathbf{L}_b(W_J)\right)\right) \\
&= \underbrace{\Phi\left(\mathbf{L}_a(W_I) \otimes \mathbf{L}_a(W_J)\right)}_{= \mathbf{L}_a(W)} \cdot \underbrace{\Phi\left(\mathbf{L}_b(W_I) \otimes \mathbf{L}_b(W_J)\right)}_{= \mathbf{L}_b(W)} \\
&= \mathbf{L}_a(W) \cdot \mathbf{L}_b(W),
\end{aligned}$$

[1]In more detail: Let $x \in W_I$ and $y \in W_J$. Let $s_1 s_2 \cdots s_p$ be any reduced expression for xy. Then, $p = \ell(xy)$. Moreover, each s_i belongs to $S = I \sqcup J$, hence lies in either I or J. Since each $i \in I$ commutes with each $j \in J$ (because $S = I \oplus J$ is an orthogonal decomposition), we can freely move all the s_i that belong to I in the reduced expression $s_1 s_2 \cdots s_p$ past the s_i that belong to J. In particular, we can move all the former s_i to the left end of the reduced expression, while letting the latter s_i accumulate on the right end. As a consequence, our reduced expression assumes the form $t_1 t_2 \cdots t_p$, where the first k factors $t_1, t_2, \ldots, t_k$ (for some $k \in \{0, 1, \ldots, p\}$) belong to I while the remaining $p - k$ factors $t_{k+1}, t_{k+2}, \ldots, t_p$ belong to J. Thus, of course, $t_1 t_2 \cdots t_k \in W_I$ and $t_{k+1} t_{k+2} \cdots t_p \in W_J$. Hence,

$$xy = s_1 s_2 \cdots s_p = t_1 t_2 \cdots t_p = \underbrace{(t_1 t_2 \cdots t_k)}_{\in W_I} \underbrace{(t_{k+1} t_{k+2} \cdots t_p)}_{\in W_J}.$$

In other words, the map (26) sends the pairs $(x, y) \in W_I \times W_J$ and $(t_1 t_2 \cdots t_k,\ t_{k+1} t_{k+2} \cdots t_p) \in W_I \times W_J$ to the same image. Since this map is injective (being a group isomorphism), it thus follows that these pairs are equal. That is, $x = t_1 t_2 \cdots t_k$ and $y = t_{k+1} t_{k+2} \cdots t_p$. Hence, $\ell(x) \leq k$ and $\ell(y) \leq p - k$. Summing these two inequalities, we obtain $\ell(x) + \ell(y) \leq k + (p - k) = p = \ell(xy)$. Combining this with $\ell(xy) \leq \ell(x) + \ell(y)$ (which is a general property of Coxeter lengths in any Coxeter group), we obtain $\ell(xy) = \ell(x) + \ell(y)$. This proves (29).

so that

$$\begin{aligned}
&\mathbf{L}_a(W)\cdot\mathbf{L}_b(W)\\
&=\Phi\left(\underbrace{\left(\mathbf{L}_a(W_I)\otimes\mathbf{L}_a(W_J)\right)\left(\mathbf{L}_b(W_I)\otimes\mathbf{L}_b(W_J)\right)}_{=\mathbf{L}_a(W_I)\cdot\mathbf{L}_b(W_I)\otimes\mathbf{L}_a(W_J)\cdot\mathbf{L}_b(W_J)}\right)\\
&=\Phi\left(\mathbf{L}_a(W_I)\cdot\mathbf{L}_b(W_I)\otimes\mathbf{L}_a(W_J)\cdot\mathbf{L}_b(W_J)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&[uv]\left(\mathbf{L}_a(W)\cdot\mathbf{L}_b(W)\right)\\
&=[uv]\left(\Phi\left(\mathbf{L}_a(W_I)\cdot\mathbf{L}_b(W_I)\otimes\mathbf{L}_a(W_J)\cdot\mathbf{L}_b(W_J)\right)\right)\\
&=[u]\left(\mathbf{L}_a(W_I)\cdot\mathbf{L}_b(W_I)\right)\cdot[v]\left(\mathbf{L}_a(W_J)\cdot\mathbf{L}_b(W_J)\right) \qquad (30)
\end{aligned}$$

(by (28), applied to $\alpha=\mathbf{L}_a(W_I)\cdot\mathbf{L}_b(W_I)$ and $\beta=\mathbf{L}_a(W_J)\cdot\mathbf{L}_b(W_J)$). The same argument (with the roles of a and b interchanged) shows that

$$\begin{aligned}
&[uv]\left(\mathbf{L}_b(W)\cdot\mathbf{L}_a(W)\right)\\
&=[u]\left(\mathbf{L}_b(W_I)\cdot\mathbf{L}_a(W_I)\right)\cdot[v]\left(\mathbf{L}_b(W_J)\cdot\mathbf{L}_a(W_J)\right). \qquad (31)
\end{aligned}$$

However, $[u]\left(\mathbf{L}_a(W_I)\cdot\mathbf{L}_b(W_I)\right)=[u]\left(\mathbf{L}_b(W_I)\cdot\mathbf{L}_a(W_I)\right)$ (since u is W_I-length-symmetric) and $[v]\left(\mathbf{L}_a(W_J)\cdot\mathbf{L}_b(W_J)\right)=[v]\left(\mathbf{L}_b(W_J)\cdot\mathbf{L}_a(W_J)\right)$ (since v is W_J-length-symmetric). Thus, the right-hand sides of (30) and (31) are equal (factor by factor). Consequently, so are the left-hand sides. In other words,

$$[uv]\left(\mathbf{L}_a(W)\cdot\mathbf{L}_b(W)\right)=[uv]\left(\mathbf{L}_b(W)\cdot\mathbf{L}_a(W)\right).$$

This shows that uv is W-length-symmetric. This proves Proposition 1.23. □

2.6. The longest word

Lemma 2.7. Let (W,S) be a Coxeter system, where W is finite. Let $q\in\mathbf{k}$ be invertible. Then,

$$w_0\mathbf{L}_q=\mathbf{L}_q w_0=q^{\ell(w_0)}\mathbf{L}_{1/q}.$$

Proof. As we know, w_0 is an involution; thus, $w_0w_0=1$ and $w_0^{-1}=w_0$. Moreover,

$$\ell(w_0w)=\ell(w_0)-\ell(w) \qquad (32)$$

for any $w\in W$ (see, e.g., [Luszti14, §11.2 (b)] for a proof). But the definition of $\mathbf{L}_{1/q}$ yields

$$\mathbf{L}_{1/q}=\sum_{w\in W}(1/q)^{\ell(w)}w. \qquad (33)$$

Now, the definition of $\mathbf{L}_q$ yields

$$\begin{aligned}
w_0 \mathbf{L}_q &= w_0 \sum_{w \in W} q^{\ell(w)} w = \sum_{w \in W} q^{\ell(w)} w_0 w \\
&= \sum_{w \in W} \underbrace{q^{\ell(w_0 w)}}_{\substack{= q^{\ell(w_0) - \ell(w)} \\ \text{(by (32))}}} \underbrace{w_0 w_0}_{=1} w \qquad \left(\begin{array}{c} \text{here, we have substituted } w_0 w \\ \text{for } w \text{ in the sum} \end{array} \right) \\
&= \sum_{w \in W} \underbrace{q^{\ell(w_0) - \ell(w)}}_{\substack{= q^{\ell(w_0)} / q^{\ell(w)} \\ = q^{\ell(w_0)} (1/q)^{\ell(w)}}} w = q^{\ell(w_0)} \underbrace{\sum_{w \in W} (1/q)^{\ell(w)} w}_{\substack{= \mathbf{L}_{1/q} \\ \text{(by (33))}}} = q^{\ell(w_0)} \mathbf{L}_{1/q}.
\end{aligned}$$

Applying the antipode to both sides of this equality, we obtain

$$\left(w_0 \mathbf{L}_q \right)^* = q^{\ell(w_0)} \mathbf{L}_{1/q}^*.$$

Since $\left(w_0 \mathbf{L}_q \right)^* = \mathbf{L}_q^* w_0^*$ (because the antipode is an algebra anti-morphism), we can rewrite this as

$$\mathbf{L}_q^* w_0^* = q^{\ell(w_0)} \mathbf{L}_{1/q}^*.$$

Using Lemma 2.3 and the obvious fact that $w_0^* = w_0^{-1} = w_0$, we can rewrite this as

$$\mathbf{L}_q w_0 = q^{\ell(w_0)} \mathbf{L}_{1/q}.$$

Combining this with $w_0 \mathbf{L}_q = q^{\ell(w_0)} \mathbf{L}_{1/q}$, we obtain the whole claim of Lemma 2.7. □

Proof of Proposition 1.17. Recall that w_0 is an involution, so that $w_0 = w_0^{-1}$ and $w_0 w_0 = 1$. Hence, $w_0 (w_0 w) = \underbrace{w_0 w_0}_{=1} w = w$. Thus, the relation between w and $w_0 w$ is mutual. Hence, we only need to prove the "only if" direction of Proposition 1.17 (as the "if" direction will then follow by applying the "only if" direction to $w_0 w$ instead of w).

So let us assume that w is W-length-symmetric. We must show that $w_0 w$ is W-length-symmetric. Set $u := w_0 w$. Thus, we must show that u is W-length-symmetric.

Fix a commutative ring $\mathbf{k}$ and $a, b \in \mathbf{k}$. We must show that $[u] (\mathbf{L}_a \mathbf{L}_b) = [u] (\mathbf{L}_b \mathbf{L}_a)$. We WLOG assume that $\mathbf{k}$ is a ring of Laurent polynomials and a and b are two indeterminates (since the claim we are proving is a polynomial identity in a and b). Thus, a and b are invertible in $\mathbf{k}$. Since w is W-length-symmetric, we have $[w] (\mathbf{L}_a \mathbf{L}_b) = [w] (\mathbf{L}_b \mathbf{L}_a)$ and likewise

$$[w] \left(\mathbf{L}_a \mathbf{L}_{1/b} \right) = [w] \left(\mathbf{L}_{1/b} \mathbf{L}_a \right).$$

From Lemma 2.7, we have

$$w_0 \mathbf{L}_a = \mathbf{L}_a w_0 = a^{\ell(w_0)} \mathbf{L}_{1/a}$$

and
$$w_0 \mathbf{L}_b = \mathbf{L}_b w_0 = b^{\ell(w_0)} \mathbf{L}_{1/b}.$$

From $u = w_0 w = w_0^{-1} w$ (since $w_0 = w_0^{-1}$), we obtain

$$[u] \left(\mathbf{L}_a \mathbf{L}_b\right) = \left[w_0^{-1} w\right] \left(\mathbf{L}_a \mathbf{L}_b\right) = [w] \left(w_0 \mathbf{L}_a \mathbf{L}_b\right)$$

(since Lemma 2.1 (applied to w_0, 1 and $\mathbf{L}_a \mathbf{L}_b$ instead of g, h and a) yields $[w] \left(w_0 \mathbf{L}_a \mathbf{L}_b\right) = \left[w_0^{-1} w\right] \left(\mathbf{L}_a \mathbf{L}_b\right)$). Therefore,

$$[u] \left(\mathbf{L}_a \mathbf{L}_b\right) = [w] \left(\underbrace{w_0 \mathbf{L}_a}_{= \mathbf{L}_a w_0} \mathbf{L}_b\right) = [w] \left(\mathbf{L}_a w_0 \mathbf{L}_b\right).$$

The same argument (with the roles of a and b swapped) shows that

$$[u] \left(\mathbf{L}_b \mathbf{L}_a\right) = [w] \left(\mathbf{L}_b w_0 \mathbf{L}_a\right).$$

The right-hand sides of these two equalities are equal, since

$$\begin{aligned} [w] \left(\mathbf{L}_a \underbrace{w_0 \mathbf{L}_b}_{= b^{\ell(w_0)} \mathbf{L}_{1/b}}\right) &= b^{\ell(w_0)} \underbrace{[w] \left(\mathbf{L}_a \mathbf{L}_{1/b}\right)}_{= [w](\mathbf{L}_{1/b} \mathbf{L}_a)} = b^{\ell(w_0)} [w] \left(\mathbf{L}_{1/b} \mathbf{L}_a\right) \\ &= [w] \left(\underbrace{b^{\ell(w_0)} \mathbf{L}_{1/b}}_{= \mathbf{L}_b w_0} \mathbf{L}_a\right) = [w] \left(\mathbf{L}_b w_0 \mathbf{L}_a\right). \end{aligned}$$

Hence, so are the left-hand sides. That is, $[u] \left(\mathbf{L}_a \mathbf{L}_b\right) = [u] \left(\mathbf{L}_b \mathbf{L}_a\right)$. This shows that u is W-length-symmetric. This proves the "only if" direction of Proposition 1.17, and thus (as explained above) the whole proposition. □

3. Application: Separable permutations (TODO)

We now apply the above results to prove a theorem we promised in the introduction:

Theorem 3.1. Let $n \in \mathbb{N}$. Consider the symmetric group S_n as a Coxeter group, as in Example 1.2. A permutation $w \in S_n$ is said to be *separable* if and only if there exist no four elements $a < b < c < d$ of $\{1, 2, \ldots, n\}$ satisfying

$$\left(w(b) < w(d) < w(a) < w(c) \qquad \text{or} \qquad w(c) < w(a) < w(d) < w(b)\right).$$

Then, each separable permutation $w \in S_n$ is S_n-length-symmetric.

Proof sketch. Among the many characterizations of separable permutations given in [Ghys16, "From a pruned tree to a polynomial interchange and a separable permutation", Theorem], we will need the fourth one: A permutation $w \in S_n$ is separable if and only if it can be obtained from several copies of the trivial permutation on one object (i.e., the permutation $\mathrm{id} \in S_1$) by successive $\oplus$ and $\ominus$ operations. Here, the operation $\oplus$ sends two permutations $u \in S_k$ and $v \in S_\ell$ to the permutation $u \oplus v \in S_{k+\ell}$ that acts as u on the numbers $1, 2, \ldots, k$ and as (a shifted copy of) v on the numbers $k+1, k+2, \ldots, k+\ell$, whereas the operation $\ominus$ sends two permutations $u \in S_k$ and $v \in S_\ell$ to the permutation $((uw_{0,k}) \oplus (vw_{0,\ell}))\, w_{0,k+\ell} \in S_{k+\ell}$, where $w_{0,m}$ denotes the longest word of S_m. This recursive definition allows for proving the claim by induction, using Proposition 1.23 (to show that $u \oplus v$ is S_n-length-symmetric whenever u and v are) and Proposition 1.17 and Proposition 1.16 (to show that ww_0 is S_n-length-symmetric whenever w is). We omit the details.

TODO: give the details... □

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