Hyperplane arrangements and descent algebras
Franco V Saliola
`saliola - DesAlgLectureNotes.pdf`
version of 10 January 2006
Errata and addenda by Darij Grinberg

I will refer to the results appearing in the article "A Hyperplane arrangements and descent algebras" by the numbers under which they appear in this article.

6. Errata

- **Various places (for example, §2.1):** You use the notations $\subseteq$ and $\subset$ synonymously. It might be better if you consistently keep to one of them, as the appearance of both of them in your notes suggests that $\subset$ means proper inclusion (but it does not).
- **Page 1, §1.1:** Replace "$v_1 + v_2 + v_3 = 0$" by "$v_1 + v_2 + \cdots + v_n = 0$".
- **Page 3:** Replace "the nonempty intersections of the open half spaces" by "a nonempty intersection of open half spaces". (Maybe also add "(one for each hyperplane)" at the end of the sentence.)
- **Page 4, Figure 3:** I think the "$(+0-)$" label is wrong, and should be a "$(-0-)$" label instead.
- **Page 4:** Replace "and that the closure" by "and that the closures".
- **Page 5, §1.3:** Your claim that "the *join* $X \vee Y$ of X and Y is $X + Y$" is generally false (even when $\mathcal{A}$ is the braid arrangement)[1]. I don't think the join can be characterized this easily. (Of course, the existence of a join follows from the existence of the meet using the fact that any finite meet-semilattice having a greatest element is a lattice.)
- **Page 5, §1.3:** I don't think your claim that "The rank of $X \in \mathcal{L}$ is the dimension of the subspace $X \subset \mathbb{R}^d$" is true.
- **Page 8, Exercise 2:** I think it would be useful to add the following claim between (2) and (3): "$x \leq xy$".
- **Page 9:** In the formula for $\sigma_{H_{ij}}(BC)$, why do you write "$C(j) < C(i)$" instead of "$C(i) > C(j)$"? Of course, this is equivalent, but it looks out of place.

[1] For a counterexample, set $n = 4$, $X = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_2 \text{ and } x_3 = x_4\right\}$ and $Y = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_3 \text{ and } x_2 = x_4\right\}$. Then, the join $X \vee Y$ is the whole space, while the sum $X + Y$ is the hyperplane $\left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - x_2 - x_3 + x_4 = 0\right\}$ (which is not an element of $\mathcal{L}$).

- **Page 10, Example 4:** Replace "subset of the" by "subset of".
- **Page 11, §1.6:** You write: "and with (s, t)-entry the probability of moving to the state s from the state t". I think you want to interchange the words "from" and "to" here, since otherwise (I believe) this definition does not match the equations further below.
- **Page 12, Theorem 1.4:** Replace "left" by "let".
- **Page 12:** Replace "defined in the next section" by "defined in the previous section".
- **Page 12:** Replace "$\sum\limits_{c \in \mathcal{F}}$" by "$\sum\limits_{c \in \mathcal{C}}$".
- **Page 12:** Replace "Since $k\mathcal{C}$" by "Since $\mathbb{R}\mathcal{C}$".
- **Page 13, §2.1:** It would be good to explain how you define the descent algebra[2]. You use this notion in the proof of Theorem 2.1, yet before you show that the x_J span an algebra, and it is not immediately clear whether you mean the span of the x_J or the subalgebra they generate or something else.
- **Page 13, §2.2:** Replace "endomorphisms" by "automorphisms".
- **Page 13, §2.2:** Replace "endomorphism" by "automorphism".
- **Page 13, §2.2:** Replace "if $a, a' \in A$" by "if $a, a' \in A^G$".
- **Page 14, §2.3:** Since you have recalled the definition of an automorphism previously, it would seem reasonable to also give the definition of an anti-isomorphism and what it means for two algebras to be anti-isomorphic.
- **Page 14, proof of Theorem 2.1:** Replace "that $w(a) = a$" by "that $\omega(a) = a$".
- **Page 14, proof of Theorem 2.1:** Replace "This gives an algebra homomorphism" by "Thus, $a \mapsto f_a$ gives an algebra homomorphism".
- **Page 15:** Replace both "End_{S_n}"'s by "$\operatorname{End}_{kS_n}$"'s.
- **Page 15:** Replace each "ψ" and each "ψ^{-1}" appearing on page 15 by "ψ^{-1}" and "ψ", respectively.

[2] along the lines of: "We define $\mathcal{D}(S_n)$ to be the vector subspace of kS_n spanned by the x_J with $J \subseteq [n-1]$. We call $\mathcal{D}(S_n)$ the *descent algebra* of S_n, although we do not yet know that it is an algebra (we shall see this in the proof of Theorem 2.1)."

- **Page 15:** When you introduce a_B, it would be useful to point out that a_B is the sum of all set compositions of $[n]$ having the form $(C_1, C_2, \ldots, C_m)$ which satisfy $|C_i| = |B_i|$ for each $i \in \{1, 2, \ldots, m\}$. This alternative description of a_B is what is used on page 16 to find $a_B(1, 2, \ldots, n)$.

- **Page 16, proof of Theorem 2.1:** You need to WLOG apply $n \geq 1$ in the last paragraph (otherwise, the descent algebra is not of dimension 2^{n-1}).

- **Page 16, proof of Theorem 2.1:** Replace "and the sum" by "and the sums".

- **Page 16, §2.3:** I know this flies in the face of the underlying philosophy of your article, but methinks it wouldn't hurt to point out that the geometric language you are using (i.e., the language of hyperplane arrangements, faces and chambers) wasn't necessary for the proof of Theorem 2.1. In fact, Theorem 2.1 (and, with it, the fact that $\mathcal{D}(S_n)$ is a subalgebra of kS_n) becomes a purely elementary combinatorial statement if we just define $\mathcal{F}$ as the set of all set compositions of $[n]$ (and we define the action of S_n on $\mathcal{F}$ by setting

$$\omega((B_1, B_2, \ldots, B_m)) = (\omega(B_1), \omega(B_2), \ldots, \omega(B_m)) \qquad \text{for all } \omega \in S_n \text{ and } (B_1, B_2, \ldots, B_m) \in \mathcal{F} \tag{1}$$

). Moreover, your proof of Theorem 2.1 becomes a purely combinatorial proof of this combinatorial statement if we make the following changes:

  - We define $\mathcal{F}$ as the set of all set compositions of $[n]$.
  - We define the action of S_n on $\mathcal{F}$ by setting (1).
  - For any two set compositions $(B_1, B_2, \ldots, B_l)$ and $(C_1, C_2, \ldots, C_m)$ in $\mathcal{F}$, we define the product $(B_1, B_2, \ldots, B_l)(C_1, C_2, \ldots, C_m)$ by

    $$\begin{aligned}&(B_1, B_2, \ldots, B_l)(C_1, C_2, \ldots, C_m)\\ &= (B_1 \cap C_1, B_1 \cap C_2, \ldots, B_1 \cap C_m,\\ &\qquad B_2 \cap C_1, B_2 \cap C_2, \ldots, B_2 \cap C_m,\\ &\qquad \ldots,\\ &\qquad B_l \cap C_1, B_l \cap C_2, \ldots, B_l \cap C_m)^{✄},\end{aligned}$$

    where ✄ means "delete empty intersections from the list".
  - We define $\mathcal{C}$ as the set of all set compositions of $[n]$ into singleton blocks. This is a subset of $\mathcal{F}$.
  - We replace "faces of the chamber $(1, 2, \ldots, n)$" by "set compositions of $[n]$ having the form

    $$(\{1, 2, \ldots, i_1\}, \{i_1 + 1, i_1 + 2, \ldots, i_2\}, \{i_2 + 1, i_2 + 2, \ldots, i_3\}, \ldots, \{i_k + 1, i_k + 2, \ldots, n\})$$

where $i_1, i_2, \ldots, i_k$ are elements of $[n-1]$ satisfying $i_1 < i_2 < \cdots < i_k$". (This includes both the set composition $(\{1, 2, \ldots, n\})$, which is obtained for $k = 0$, and the set composition $(1, 2, \ldots, n) = (\{1\}, \{2\}, \ldots, \{n\})$, which is obtained for $k = n-1$ and $i_j = j$.)

- **Page 17:** When you speak of cosets, it might be helpful to explain whether they are left or right cosets.

- **Page 17, Theorem 2.2:** The definition of $\mathcal{C}$ should be moved from the proof of Theorem 2.2 into the statement of Theorem 2.2.

- **Page 17, proof of Theorem 2.2:** Replace "any kW-endomorphism commuting with the action of W" by "any kW-endomorphism of kW" (or "any endomorphism of kW commuting with the action of W").

- **Page 17, proof of Theorem 2.2:** Replace "be reversing" by "by reversing".

- **Page 18, §2.5:** At the beginning of §2.5, it would be good to explicitly say that you are returning to the setting in which $\mathcal{A}$ is the braid arrangement.

- **Page 18, §2.5:** You say that the elements x_λ (with λ ranging over the integer compositions of n) form a basis of $(k\mathcal{F})^{S_n}$. At this point it is natural to observe that we have already seen these elements x_λ: namely, the elements a_B defined in the proof of Theorem 2.1 satisfy $a_B = x_{\lambda(B)}$ for all set compositions B of $[n]$.

- **Page 18, proof of Proposition 2.3:** Replace "The coefficient of C is exactly" by "The coefficient of C in the left hand side is exactly".

- **Page 19, proof of Corollary 2.4:** Replace "$\sum_{i=1}^{k-1} \gamma_i$" by "$\sum_{i=1}^{k-1} \gamma_i + 1$".

- **Page 19, proof of Corollary 2.4:** After "give the integer composition γ", add "(when $AB = C$)".

- **Page 19, Exercise 5:** It would be good to clarify that $\lambda(J)$ is defined to be (n) when J is the empty set. (Your definition of $\lambda(J)$ is slightly unclear in this case.)

- **Page 20, Exercise 5:** Replace "on n" by "of n".

- **Page 20, Exercise 5:** I think this exercise is wrong as stated. For instance (it would be good if you could double-check me), if $K = \varnothing$, then

$c_{\lambda(K),\lambda(J),\lambda(L)} = \delta_{J,L}$, whereas

$$\begin{aligned}&\left|\left\{\omega \in X_J^{-1} \cap \underbrace{X_K}_{=\{\mathrm{id}\}} \ : \ L = \underbrace{K}_{=\varnothing} \cap \omega^{-1}(J)\right\}\right| \\ &= \left|\left\{\omega \in X_J^{-1} \cap \{\mathrm{id}\} \ : \ L = \varnothing \cap \omega^{-1}(J)\right\}\right| \\ &= \left|\{\omega \in \{\mathrm{id}\} \ : \ L = \varnothing\}\right| = \delta_{L,\varnothing}.\end{aligned}$$

Is Theorem 1 in `arXiv:0706.2714v1` what you are trying to get at?

- **Page 20, §3.1:** Replace "if e is idempotent" by "if e is a nonzero idempotent".

- **Page 20, §3.1:** The claim "Moreover, if M is any A-module, then there is an A-module decomposition of M given by the idempotents: $M \cong \bigoplus_{i \in I} Me_i$" is false as stated. In truth, if M is a left A-module, then $M = \bigoplus_{i \in I} e_i M$ (not Me_i, which makes no sense), but this is only a decomposition of k-vector spaces, not of A-modules (unless M is an (A, A)-bimodule, in which case this is a decomposition of right A-modules).

  An example for this is when A is a matrix algebra $k^{n \times n}$ and M is the vector space k^n (with the obvious left A-action by matrix-vector multiplication). The diagonal unit matrices $e_i := E_{i,i} \in A$ for $i \in \{1, 2, \ldots, n\}$ form a complete system of primitive orthogonal idempotents for A, but M is indecomposable as an A-module, and its subspaces $e_i M$ are only k-vector subspaces, not A-submodules.

- **Page 20, §3.1:** It would be good to give references for all claims about primitive idempotents made here (in the first two paragraphs of §3.1). One such reference is §I.4 and Corollary I.5.17 in: Ibrahim Assem, Daniel Simson, Andrzej Skowroński, *Elements of the Representation Theory of Associative Algebras, Volume 1: Techniques of Representation Theory*, Cambridge University Press 2006.

- **Page 21, Example 5:** Two of the minus signs in the computation of $\left(e_{\{13,2\}}\right)^2$ should be plus signs (namely, the last minus signs on the second and on the third line of the computation).

- **Page 22, proof of Lemma 3.1:** Replace "Proposition 2 (2)" by "Exercise 2 (2)".

- **Page 22, proof of Lemma 3.1:** Replace "$W \vee X$" by "$X \vee W$" twice. (Of course, "$W \vee X$" is correct, too, but it helps to keep notations consistent.)

- **Page 22, proof of Lemma 3.1:** In your place, I would explain how you get $z = \sum_{Y \geq X \vee W} ze_Y$. (Namely, you have $ze_{X \vee W} = e_{X \vee W} = z - \sum_{Y > X \vee W} ze_Y$, so that $z = ze_{X \vee W} + \sum_{Y > X \vee W} ze_Y = \sum_{Y \geq X \vee W} ze_Y$).
- **Page 22, proof of Lemma 3.1:** Replace "Proposition 2 (5)" by "Exercise 2 (5)".
- **Page 22, proof of Theorem 3.2:** Replace "Proposition 2 (5)" by "Exercise 2 (5)".
- **Page 22, proof of Theorem 3.2:** In the "Idempotent" part of the proof, please define x and y. (Namely, x is the element of support X that was chosen while defining e_X, and y is the element of support Y that was chosen while defining e_Y.)
- **Page 22, proof of Theorem 3.2:** After you observe that "$e_Y z = e_Y$ for any z with $\operatorname{supp}(z) \leq Y$", it would be helpful to point out that this, in particular, shows that $e_Y y = e_Y$. (You use this equality a few lines later.)
- **Page 23, proof of Theorem 3.2:** In the "Idempotent" part of the proof, it would help to clarify why $\sum_{Y > X} xe_Y (ye_X)$. (Indeed, this is because every $Y > X$ satisfies $ye_X = 0$ (by Lemma 3.1, applied to $w = y$)).
- **Page 23, proof of Theorem 3.2:** In the "Orthogonal" part of the proof, please define x. (Namely, x is the element of support X that was chosen while defining e_X.)
- **Page 23, proof of Theorem 3.2:** In the "Orthogonal" part of the proof, you are writing "$e_X e_Y = e_X x e_Y$". It would be good to explain why this holds. (Namely, it follows from the equality $e_X = e_X x$. This equality can be proven just as the equality $e_Y = e_Y y$ in the "Idempotent" part of the proof. In my opinion it wouldn't hurt to explicitly state both equalities $x = e_X x$ and $x = xe_X$ as a lemma, given that you are applying them several times.)
- **Page 23, proof of Theorem 3.2:** In the "Primitive" part of the proof, you are using some notations which, in my opinion, you should define:
  - You extend the map $\operatorname{supp} : \mathcal{F} \to \mathcal{L}$ to a k-linear map $k\mathcal{F} \to k\mathcal{L}$, and denote the latter map again by supp. This allows you to speak of $\operatorname{supp} w$ for arbitrary $w \in k\mathcal{F}$, not only for $w \in \mathcal{F}$.
  - You extend the binary operation $\vee : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ to a k-bilinear binary operation $k\mathcal{L} \times k\mathcal{L} \to k\mathcal{L}$, and denote this latter operation again by $\vee$. This operation $\vee$ turns $k\mathcal{L}$ into a commutative k-algebra. (This allows you to speak of the E_X as being idempotents.) The map $\operatorname{supp} : k\mathcal{F} \to k\mathcal{L}$ becomes a surjective k-algebra homomorphism.

- **Page 23, proof of Theorem 3.2:** In the "Primitive" part of the proof, you write: "Then the above arguments show that the elements E_X are orthogonal idempotents in $k\mathcal{L}$ summing to 1". This looks a bit like a non-sequitur (although I understand what you apparently want to say). In my opinion, it would be easier to first prove that the elements e_X of $k\mathcal{F}$ lift the elements E_X of $k\mathcal{L}$ (that is, $\operatorname{supp}(e_X) = E_X$ for every $X \in \mathcal{L}$), and then conclude that the E_X are orthogonal idempotents in $k\mathcal{L}$ summing up to 1 (since the e_X are orthogonal idempotents in $k\mathcal{F}$ summing up to 1). The fact that the elements e_X of $k\mathcal{F}$ lift the elements E_X of $k\mathcal{L}$ is proven in the next paragraph of your proof of Theorem 3.2.

- **Page 23, proof of Theorem 3.2:** In the "Primitive" part of the proof, replace "orthognal" by "orthogonal".

- **Page 23, proof of Theorem 3.2:** In the "Primitive" part of the proof, please define x. (Namely, x is the element of support X that was chosen while defining e_X.)

- **Page 23, proof of Theorem 3.2:** In the "Primitive" part of the proof, replace "$\sum\limits_{Y \geq X} E_X$" by "$\sum\limits_{Y \geq X} E_Y$".

- **Page 23, proof of Theorem 3.2:** In the "Primitive" part of the proof, replace "$X - \sum\limits_{Y > X} E_Y = E_Y$" by "$X - \sum\limits_{Y > X} E_Y = E_X$".

- **Page 23, proof of Theorem 3.2:** In the "Primitive" part of the proof, you claim that "This kernel is nilpotent" (speaking of the kernel of supp). This is correct, but I don't see any previous statement from which this would follow easily. Let me outline my proof of this nilpotency.

  Theorem 3.1a. The kernel of the k-algebra homomorphism $\operatorname{supp} : k\mathcal{F} \to k\mathcal{L}$ (which is defined by extending the map $\operatorname{supp} : \mathcal{F} \to \mathcal{L}$, as above) is nilpotent.

  Proof sketch. Let us denote this kernel by P. Let us furthermore define a few more notations:

  For every $X \in \mathcal{L}$, we define the *corank* of X to be the largest $\ell \in \mathbb{N}$ such that there exist elements $X_0, X_1, \ldots, X_\ell \in \mathcal{L}$ with $X_0 = X$ and $X_0 < X_1 < \cdots < X_\ell$. Notice that this is well-defined (because such an ℓ exists (namely, $\ell = 0$ fits the bill), but the finiteness of X forces any such ℓ to be $< |X|$). We denote the corank of X by $\operatorname{corank} X$. The following property of coranks is obvious: If X and Y are two elements of $\mathcal{L}$ such that $X < Y$, then

  $$\operatorname{corank} X > \operatorname{corank} Y. \tag{2}$$

  For every $N \in \mathbb{Z}$, we define a subset $\mathcal{L}_N$ of $\mathcal{L}$ by

  $$\mathcal{L}_N = \{X \in \mathcal{L} \mid \operatorname{corank} X < N\}.$$

Then, $\mathcal{L}_0 = \varnothing$ (since no $X \in \mathcal{L}$ has corank < 0), while $\mathcal{L}_{|X|} = \mathcal{L}$ (since each $X \in \mathcal{L}$ has corank $< |X|$).

For every $N \in \mathbb{Z}$, we define a subset $\mathcal{F}_N$ of $\mathcal{F}$ by $\mathcal{F}_N = \operatorname{supp}^{-1} \mathcal{L}_N$ (where supp here means the map $\operatorname{supp} : \mathcal{F} \to \mathcal{L}$, not the map $\operatorname{supp} : k\mathcal{F} \to k\mathcal{L}$). Clearly, $\mathcal{F}_0 = \operatorname{supp}^{-1} \underbrace{\mathcal{L}_0}_{=\varnothing} = \operatorname{supp}^{-1} \varnothing = \varnothing$ and $\mathcal{F}_{|X|} = \operatorname{supp}^{-1} \underbrace{\mathcal{L}_{|X|}}_{=\mathcal{L}} = \operatorname{supp}^{-1} \mathcal{L} = \mathcal{F}$.

For every $N \in \mathbb{N}$, the set $\mathcal{F}_N$ is a subset of $\mathcal{F}$, and thus $k\mathcal{F}_N$ becomes a k-vector subspace of $k\mathcal{F}$. We are now going to show that

$$(k\mathcal{F}_N) \cdot P \subseteq k\mathcal{F}_{N-1} \qquad \text{for every } N \in \mathbb{Z}. \tag{3}$$

Proof of (3): Let $N \in \mathbb{Z}$. We need to prove (3). It is clearly enough to show that $xp \in k\mathcal{F}_{N-1}$ for each $x \in k\mathcal{F}_N$ and $p \in P$. So fix $x \in k\mathcal{F}_N$ and $p \in P$. We need to show that $xp \in k\mathcal{F}_{N-1}$. Since this relation is k-linear in x, we can WLOG assume that $x \in \mathcal{F}_N$. Assume this.

Every element $u \in \mathcal{F}_N$ satisfying $\operatorname{supp} u \not\leq \operatorname{supp} x$ satisfies

$$xu \in \mathcal{F}_{N-1}. \tag{4}$$

[3]

We can write p in the form $p = \sum_{u \in \mathcal{F}} \lambda_u u$ for some family $(\lambda_u)_{u \in \mathcal{F}}$ of elements of k. Consider this family $(\lambda_u)_{u \in \mathcal{F}}$. We have

$$\sum_{\substack{u \in \mathcal{F}; \\ \operatorname{supp} u = U}} \lambda_u = 0 \qquad \text{for every } U \in \mathcal{L} \tag{5}$$

[3]*Proof of (4):* Let u be an element of $\mathcal{F}_N$ satisfying $\operatorname{supp} u \not\leq \operatorname{supp} x$. Then, $\operatorname{supp} u \vee \operatorname{supp} x \neq \operatorname{supp} x$ (because otherwise, we would have $\operatorname{supp} u \leq \operatorname{supp} u \vee \operatorname{supp} x = \operatorname{supp} x$, which would contradict $\operatorname{supp} u \not\leq \operatorname{supp} x$). Combined with $\operatorname{supp} x \leq \operatorname{supp} u \vee \operatorname{supp} x$, this yields $\operatorname{supp} x < \operatorname{supp} u \vee \operatorname{supp} x$. Hence, (2) (applied to $X = \operatorname{supp} x$ and $Y = \operatorname{supp} u \vee \operatorname{supp} x$) yields $\operatorname{corank}(\operatorname{supp} x) > \operatorname{corank}(\operatorname{supp} u \vee \operatorname{supp} x)$. Since $\operatorname{supp} u \vee \operatorname{supp} x = \operatorname{supp} x \vee \operatorname{supp} u = \operatorname{supp}(xu)$, this rewrites as $\operatorname{corank}(\operatorname{supp} x) > \operatorname{corank}(\operatorname{supp}(xu))$. Hence, $\operatorname{corank}(\operatorname{supp}(xu)) < \operatorname{corank}(\operatorname{supp} x)$ and thus

$$\operatorname{corank}(\operatorname{supp}(xu)) \leq \underbrace{\operatorname{corank}(\operatorname{supp} x)}_{\substack{<N \\ (\text{since } \operatorname{supp} x \in \mathcal{L}_N \\ (\text{since } x \in \mathcal{F}_N = \operatorname{supp}^{-1} \mathcal{L}_N))}} - 1 < N - 1.$$

In other words, $\operatorname{supp}(xu) \in \mathcal{L}_{N-1}$, so that $xu \in \operatorname{supp}^{-1} \mathcal{L}_{N-1} = \mathcal{F}_{N-1}$. This proves (4).

[4]. Hence,

$$\sum_{\substack{u\in\mathcal{F};\\ \operatorname{supp} u\leq \operatorname{supp} x}} \lambda_u = \sum_{\substack{U\in\mathcal{L};\\ U\leq \operatorname{supp} x}} \underbrace{\sum_{\substack{u\in\mathcal{F};\\ \operatorname{supp} u=U}} \lambda_u}_{\substack{=0\\ \text{(by (5))}}} = 0. \tag{6}$$

Now, multiplying both sides of the equality $p = \sum_{u\in\mathcal{F}} \lambda_u u$ by x from the left, we obtain

$$\begin{aligned} xp &= x\sum_{u\in\mathcal{F}} \lambda_u u = \sum_{u\in\mathcal{F}} \lambda_u xu \\ &= \sum_{\substack{u\in\mathcal{F};\\ \operatorname{supp} u\leq \operatorname{supp} x}} \lambda_u \underbrace{xu}_{\substack{=x\\ \text{(by Exercise 2 (5),}\\ \text{since } \operatorname{supp} u\leq \operatorname{supp} x)}} + \sum_{\substack{u\in\mathcal{F};\\ \operatorname{supp} u\not\leq \operatorname{supp} x}} \lambda_u \underbrace{xu}_{\substack{\in\mathcal{F}_{N-1}\\ \text{(by (4))}}} \\ &\in \underbrace{\sum_{\substack{u\in\mathcal{F};\\ \operatorname{supp} u\leq \operatorname{supp} x}} \lambda_u}_{\substack{=0\\ \text{(by (6))}}} x + \underbrace{\sum_{\substack{u\in\mathcal{F};\\ \operatorname{supp} u\not\leq \operatorname{supp} x}} \lambda_u \mathcal{F}_{N-1}}_{\subseteq k\mathcal{F}_{N-1}} \subseteq 0x + k\mathcal{F}_{N-1} = k\mathcal{F}_{N-1}. \end{aligned}$$

This finishes the proof of (3).

Now, for every $M\in\mathbb{N}$ and every $N\in\mathbb{Z}$, we have

$$(k\mathcal{F}_N)\cdot P^M \subseteq k\mathcal{F}_{N-M}. \tag{7}$$

(In fact, this can be proven by straightforward induction over M using (3).) Applying (7) to $N=|X|$ and $M=|X|$, we obtain

$$\left(k\mathcal{F}_{|X|}\right)\cdot P^{|X|} \subseteq k\underbrace{\mathcal{F}_{|X|-|X|}}_{=\mathcal{F}_0=\varnothing} = k\varnothing = 0.$$

[4]*Proof of (5):* We have $p\in P$. In other words, p belongs to the kernel of the k-algebra homomorphism $\operatorname{supp}: k\mathcal{F}\to k\mathcal{L}$ (since P is the kernel of the k-algebra homomorphism $\operatorname{supp}: k\mathcal{F}\to k\mathcal{L}$). In other words, $\operatorname{supp} p = 0$. Hence,

$$\begin{aligned} 0 &= \operatorname{supp} \underbrace{p}_{=\sum_{u\in\mathcal{F}} \lambda_u u} = \operatorname{supp}\left(\sum_{u\in\mathcal{F}} \lambda_u u\right) \\ &= \sum_{u\in\mathcal{F}} \lambda_u \operatorname{supp} u = \sum_{U\in\mathcal{L}} \sum_{\substack{u\in\mathcal{F};\\ \operatorname{supp} u=U}} \lambda_u U = \sum_{U\in\mathcal{L}} \left(\sum_{\substack{u\in\mathcal{F};\\ \operatorname{supp} u=U}} \lambda_u\right) U. \end{aligned}$$

Thus, the element $\sum_{U\in\mathcal{L}} \left(\sum_{\substack{u\in\mathcal{F};\\ \operatorname{supp} u=U}} \lambda_u\right) U$ of $k\mathcal{L}$ is 0. Since the elements U of $\mathcal{L}$ are k-linearly independent in $k\mathcal{L}$, this shows that $\sum_{\substack{u\in\mathcal{F};\\ \operatorname{supp} u=U}} \lambda_u = 0$ for every $U\in\mathcal{L}$. This proves (5).

Since $\mathcal{F}_{|X|} = \mathcal{F}$, this rewrites as $(k\mathcal{F}) \cdot P^{|X|} \subseteq 0$. Now, $P^{|X|} = \underbrace{1}_{\in k\mathcal{F}} \cdot P^{|X|} \subseteq (k\mathcal{F}) \cdot P^{|X|} \subseteq 0$, so that $P^{|X|} = 0$. Thus, P is nilpotent. This proves Theorem 3.1a.

- **Page 23, proof of Theorem 3.2:** In the "Primitive" part of the proof, replace "for some $n \geq 0$" by "for some $n \geq 1$" (since you end up using $e_1 = e_1^n$ on the next line).
- **Page 24, Remark 3.3:** Replace "in for" by "for".
- **Page 24, Remark 3.3:** One further observation needs to be checked to ensure that the proofs still hold if x is replaced by $\widetilde{x}$: Namely, it should be checked that $\widetilde{x}y\widetilde{x} = \widetilde{x}y$ for every $y \in \mathcal{F}$. This, fortunately, is easy.[5]
- **Page 24, Remark 3.4:** I think "left regular band" should be replaced by "finite left regular band" in the first sentence; or at least my proof of the existence of L and supp only works in the case when S is finite.[6]

---

[5] *Proof.* Let $y \in \mathcal{F}$. Then,

$$\begin{aligned}
\widetilde{x}y\widetilde{x} &= \left(\sum_{\substack{x \in \mathcal{F};\\ \operatorname{supp} x = X}} \lambda_x x \right) y \left(\sum_{\substack{x \in \mathcal{F};\\ \operatorname{supp} x = X}} \lambda_x x \right) \qquad \left(\text{since } \widetilde{x} = \sum_{\substack{x \in \mathcal{F};\\ \operatorname{supp} x = X}} \lambda_x x \right) \\
&= \left(\sum_{\substack{x \in \mathcal{F};\\ \operatorname{supp} x = X}} \lambda_x x \right) y \left(\sum_{\substack{x' \in \mathcal{F};\\ \operatorname{supp}(x') = X}} \lambda_{x'} x' \right) \\
&= \sum_{\substack{x \in \mathcal{F};\\ \operatorname{supp} x = X}} \lambda_x \sum_{\substack{x' \in \mathcal{F};\\ \operatorname{supp}(x') = X}} \lambda_{x'} \underbrace{xyx'}_{\substack{= xy \\ \text{(by Exercise 2 (5) (applied to } xy \text{ and } x' \text{ instead of } x \text{ and } y\text{),} \\ \text{since } \operatorname{supp}(x') = X = \operatorname{supp} x \leq \operatorname{supp} x \vee \operatorname{supp} y = \operatorname{supp}(xy))}} \\
&= \sum_{\substack{x \in \mathcal{F};\\ \operatorname{supp} x = X}} \lambda_x \underbrace{\sum_{\substack{x' \in \mathcal{F};\\ \operatorname{supp}(x') = X}} \lambda_{x'}}_{=1} xy = \underbrace{\sum_{\substack{x \in \mathcal{F};\\ \operatorname{supp} x = X}} \lambda_x x}_{= \widetilde{x}} y = \widetilde{x}y,
\end{aligned}$$

qed.

[6] Here is a rough sketch of my proof (but I expect it is the same as yours):

We define a binary relation $\equiv$ on the set S by setting

$$(x \equiv y) \iff (xy = x \text{ and } yx = y)$$

for any $x \in S$ and $y \in S$. It is easy to see that this relation $\equiv$ is transitive, reflexive and symmetric (the transitivity follows from arguing that if $x \equiv y$ and $y \equiv z$, then $\underbrace{x}_{=xy} z = x \underbrace{yz}_{=y} = xy = x$ and $\underbrace{z}_{=zy} x = z \underbrace{yx}_{=y} = zy = z$). Thus, $\equiv$ is an equivalence relation. Also, if x, y, x' and y' are four elements of S satisfying $x \equiv y$ and $x' \equiv y'$, then $xx' \equiv yy'$

- **Page 24, proof of Corollary 3.5:** The letter "x" is used in two different meanings here: Up until "$\operatorname{supp}(yx) = \operatorname{supp}(y) \vee \operatorname{supp}(x) = X$", it means the element of support X that was chosen while defining e_X; but from "the face y" on, it means an arbitrary element of $\mathcal{F}$.

- **Page 24, proof of Corollary 3.5:** You write: "xe_X lifts the primitive idempotent E_X". It might be helpful to justify why this is the case. (In fact, it is because

$$\operatorname{supp}(xe_X) = \underbrace{\operatorname{supp} x}_{=X} \vee \underbrace{\operatorname{supp}(e_X)}_{=E_X} = X \vee E_X = X \vee \sum_{Y \geq X} \mu(X,Y) Y$$
$$= \sum_{Y \geq X} \mu(X,Y) \underbrace{X \vee Y}_{\substack{=Y \\ (\text{since } Y \geq X)}} = \sum_{Y \geq X} \mu(X,Y) Y = E_X.$$

)

- **Page 24, proof of Corollary 3.5:** Replace "Corollary 3.2" by "Theorem 3.2".

- **Page 24, §3.2:** At some point here you should require that the field k have characteristic 0.

- **Page 25:** Once you have defined the new family $(e_X)_{X \in \mathcal{L}}$ of idempotents (the one that relies on the normalized sums $\widehat{X}$), it would be good to show an example. Here is one possible example: For the braid arrangement for

(because $\underbrace{xy}_{=xyx} x'y' = xy \underbrace{xx'}_{=x} y' = \underbrace{xyx}_{=xy} y' = x \underbrace{yy'}_{=y} = xy$ and similarly $x'y'xy \equiv x'y'$). In other words, the relation $\equiv$ respects the multiplication of S. Hence, the quotient set $S/\equiv$ (which is well-defined since $\equiv$ is an equivalence relation) is a semigroup. We denote this semigroup by L, and we denote the canonical projection $S \to L$ by supp. The semigroup L is clearly a left regular band and therefore, in particular, idempotent. Moreover, any $x \in S$ and $y \in S$ satisfy $xy \equiv yx$ (because $x \underbrace{yy}_{=y} x = xyx = xy$ and similarly $yxxy = yx$). Thus, the semigroup $L = S/\equiv$ is commutative. We regard this semigroup L as a join-semilattice, with join operation $\vee$ defined to be a product. (This is a well-defined semilattice because S is a commutative idempotent semigroup.) Then, L is a lattice (since every finite join-semilattice is a lattice). Moreover, it is clear that $\operatorname{supp}(xy) = \operatorname{supp} x \vee \operatorname{supp} y$ for all $x \in S$ and $y \in S$ (since supp is the canonical projection $S \to L$, and since $\vee$ is the multiplication in L). Finally, for all $x, y \in S$, we have $xy = x$ if and only if $\operatorname{supp}(y) \leq \operatorname{supp}(x)$ (this is easy to check).

$n = 3$, we have

$$\begin{aligned}
e_{\{1,2,3\}} &= \{\widehat{1,2,3}\} \\
&= \frac{1}{6}\left((1,2,3) + (1,3,2) + (2,1,3) + (2,3,1) + (3,1,2) + (3,2,1)\right); \\
e_{\{12,3\}} &= \frac{1}{2}\left((12,3) + (3,12)\right) - \frac{1}{2}\left((12,3) + (3,12)\right) e_{\{1,2,3\}} \\
&\qquad \left(\text{since } \{\widehat{12,3}\} = \frac{1}{2}\left((12,3) + (3,12)\right)\right) \\
&= \frac{1}{2}\left((12,3) + (3,12) - \frac{1}{2}\left((1,2,3) + (2,1,3) + (3,1,2) + (3,2,1)\right)\right); \\
e_{\{13,2\}} &= \frac{1}{2}\left((13,2) + (2,13) - \frac{1}{2}\left((1,3,2) + (3,1,2) + (2,1,3) + (2,3,1)\right)\right); \\
e_{\{1,23\}} &= \frac{1}{2}\left((1,23) + (23,1) - \frac{1}{2}\left((1,2,3) + (1,3,2) + (2,3,1) + (3,2,1)\right)\right); \\
e_{\{123\}} &= (123) - e_{\{12,3\}} - e_{\{13,2\}} - e_{\{1,23\}} - e_{\{1,2,3\}} \\
&= (123) - \frac{1}{2}\left((12,3) + (3,12) + (13,2) + (2,13) + (1,23) + (23,1)\right) \\
&\qquad + \frac{1}{3}\left((1,2,3) + (1,3,2) + (2,1,3) + (2,3,1) + (3,1,2) + (3,2,1)\right).
\end{aligned}$$

- **Page 25:** You say that "it is not difficult to show that these are also primitive". I didn't find it too easy either; let me sketch my argument:

  The surjective k-algebra homomorphism $\operatorname{supp} : k\mathcal{F} \to k\mathcal{L}$ restricts to a k-algebra homomorphism $\operatorname{supp} : (k\mathcal{F})^W \to (k\mathcal{L})^W$ (which is also surjective, even though we will not use this). With respect to this latter homomorphism, the idempotent $\varepsilon_X = \sum_{Y\in[X]} e_Y$ of $(k\mathcal{F})^W$ lifts the idempotent $\sum_{Y\in[X]} E_Y$ of $(k\mathcal{L})^W$ (since each e_Y lifts the corresponding E_Y with respect to $\operatorname{supp} : k\mathcal{F} \to k\mathcal{L}$). Hence, in order to prove that the idempotent ε_X of $(k\mathcal{F})^W$ is primitive, it is enough to show that the idempotent $\sum_{Y\in[X]} E_Y$ of $(k\mathcal{L})^W$ is primitive (since, at the end of the proof of Theorem 3.2, we argued that a lift of a primitive idempotent with respect to a k-algebra homomorphism with nilpotent kernel must always be primitive). But the latter can be proven similarly to how we showed the primitivity of the E_Y: The elements $\sum_{Y\in[X]} E_Y$ of $(k\mathcal{L})^W$ of $k\mathcal{L}$ (with $[X]$ ranging over all the W-orbits on $\mathcal{L}$, with each orbit only counting once) are orthogonal idempotents and sum to 1 (because they are the images of the elements ε_X under supp, and we know that this holds for the elements ε_X). But their number is the number of all W-orbits on $\mathcal{L}$; this is the same as the dimension of $(k\mathcal{L})^W$.

Hence, these elements $\sum_{Y\in[X]} E_Y$ form a basis of the k-module $(k\mathcal{L})^W$, and each W-orbit $[X]$ on $\mathcal{L}$ satisfies $(k\mathcal{L})^W \left(\sum_{Y\in[X]} E_Y\right) = \operatorname{span}_k \left(\sum_{Y\in[X]} E_Y\right) \cong k$, which is an indecomposable $(k\mathcal{L})^W$-module. Hence, each of the idempotents $\sum_{Y\in[X]} E_Y$ is primitive, and this completes our proof.

- **Page 26:** Replace "elements ε_X sum to 1" by "elements ε_X (with $[X]$ ranging over all the W-orbits on $\mathcal{L}$, with each orbit only counting once)".

- **Page 26, Corollary 3.8:** This is not quite correct: The elements mentioned here are not idempotent, but rather quasi-idempotent[7]. Also, in order for them to form a basis of $(k\mathcal{F})^W$, you need to remove repetitions (that is, not the whole family $\left(\sum_{w\in W} w\left(xe_{\operatorname{supp}(x)}\right)\right)_{x\in\mathcal{F}}$, but only its subfamily $\left(\sum_{w\in W} w\left(xe_{\operatorname{supp}(x)}\right)\right)_{x\in T}$ forms a basis of $(k\mathcal{F})^W$, where T is any system of distinct representatives for the W-orbits on $\mathcal{F}$) [8].

  Let me sketch a proof of the quasi-idempotency of the elements described in Corollary 3.8:

  Corollary 3.8a. (a) We have $\varepsilon_X \in (k\mathcal{F})^W$ for every $X \in \mathcal{L}$.

  (b) We have $\sum_{w\in W} w\left(xe_{\operatorname{supp}(x)}\right) = \left(\sum_{w\in W} w(x)\right) \varepsilon_{[\operatorname{supp}(x)]}$ for every $x \in \mathcal{F}$.

  (c) For every $X \in \mathcal{L}$, let $\operatorname{Stab}_W X$ denote the stabilizer of X with respect to the W-action on $\mathcal{L}$. (This is the subset $\{w \in W \mid w(X) = X\}$ of W.) Let $x \in \mathcal{F}$. Let $\mathfrak{E}_x$ denote the element $\sum_{w\in W} w\left(xe_{\operatorname{supp}(x)}\right)$ of $k\mathcal{F}$. Then,

  $$\mathfrak{E}_x x e_{\operatorname{supp}(x)} = \sum_{w\in\operatorname{Stab}_W(\operatorname{supp}(x))} w\left(xe_{\operatorname{supp}(x)}\right) \tag{8}$$

  and

  $$\mathfrak{E}_x^2 = \left|\operatorname{Stab}_W(\operatorname{supp}(x))\right| \mathfrak{E}_x. \tag{9}$$

  Before we prove Corollary 3.8a, let us first show three very simple lemmas:

  Lemma 3.8b. Let $w \in \mathcal{F}$, $p \in k\mathcal{F}$ and $X \in \mathcal{L}$ be such that $\operatorname{supp}(w) \not\leq X$. Then, $wpe_X = 0$.

[7] For example, for $x = 1 = \widehat{0}$, the element $\sum_{w\in W} w\left(xe_{\operatorname{supp}(x)}\right)$ equals $|W| = n!$.

[8] Indeed, the element $\sum_{w\in W} w\left(xe_{\operatorname{supp}(x)}\right)$ only depends on the W-orbit on x, not on x itself, and so should only be picked once per W-orbit.

Proof of Lemma 3.8b. The equality that we want to prove (namely, $wpe_X = 0$) is k-linear in p. Thus, we can WLOG assume that $p \in \mathcal{F}$ (since $\mathcal{F}$ is a basis of the k-vector space $k\mathcal{F}$). Assume this. Exercise 2 (6) (applied to w and p instead of x and y) yields that $\operatorname{supp}(wp) = \operatorname{supp}(w) \vee \operatorname{supp}(p) \geq \operatorname{supp}(w)$, so that $\operatorname{supp}(w) \leq \operatorname{supp}(wp)$. Hence, if we had $\operatorname{supp}(wp) \leq X$, then we would have $\operatorname{supp}(w) \leq \operatorname{supp}(wp) \leq X$, which would contradict $\operatorname{supp}(w) \not\leq X$. Thus, we cannot have $\operatorname{supp}(wp) \leq X$. In other words, we have $\operatorname{supp}(wp) \not\leq X$. Hence, Lemma 3.1 (applied to wp instead of w) yields $wpe_X = 0$. This proves Lemma 3.8b.

Lemma 3.8c. Let X and Y be two distinct elements of $\mathcal{L}$ which belong to one and the same W-orbit. Then, $Y \not\leq X$.

Proof of Lemma 3.8c. Assume the contrary. Thus, $Y \leq X$. In other words, the set partition Y is obtained from X by repeated merging of blocks. Therefore, either the number of blocks of Y is smaller than the number of blocks of X, or we have $Y = X$. Since we cannot have $Y = X$ (because X and Y are distinct), this shows that the number of blocks of Y is smaller than the number of blocks of X.

But X and Y belong to one and the same W-orbit. Thus, $Y = w(X)$ for some $w \in W$. Consequently, the blocks of Y are obtained by applying w (pointwise) to the blocks of X. Hence, the number of blocks of Y equals the number of blocks of X. This contradicts the fact that the number of blocks of Y is smaller than the number of blocks of X. This contradiction shows that our assumption was wrong. Lemma 3.8c is thus proven.

Lemma 3.8d. Let $w \in \mathcal{F}$, $p \in k\mathcal{F}$ and $t \in \mathcal{F}$ be such that $\operatorname{supp}(t) \leq \operatorname{supp}(w)$. Then, $wpt = wp$.

Proof of Lemma 3.8d. The equality that we want to prove (namely, $wpt = wp$) is k-linear in p. Thus, we can WLOG assume that $p \in \mathcal{F}$ (since $\mathcal{F}$ is a basis of the k-vector space $k\mathcal{F}$). Assume this. Exercise 2 (6) (applied to w and p instead of x and y) yields that $\operatorname{supp}(wp) = \operatorname{supp}(w) \vee \operatorname{supp}(p) \geq \operatorname{supp}(w)$, so that $\operatorname{supp}(w) \leq \operatorname{supp}(wp)$. Thus, $\operatorname{supp}(t) \leq \operatorname{supp}(w) \leq \operatorname{supp}(wp)$. But Exercise 2 (5) (applied to wp and t instead of x and y) yields that $wpt = wp$ if and only if $\operatorname{supp}(t) \leq \operatorname{supp}(wp)$. Hence, $wpt = wp$ (since $\operatorname{supp}(t) \leq \operatorname{supp}(wp)$). This proves Lemma 3.8d.

Proof of Corollary 3.8a. **(a)** Let $X \in \mathcal{L}$. Let $w \in W$. The set $[X]$ is a W-orbit. Hence, the action of w on $\mathcal{L}$ restricts to a permutation of this set $[X]$. Consequently, we can substitute $w(Y)$ for Y in the sum $\sum_{Y \in [X]} e_Y$. As a result, we obtain

$$\sum_{Y\in[X]} e_Y = \sum_{Y\in[X]} \underbrace{e_{w(Y)}}_{\substack{=w(e_Y) \\ \text{(since Lemma 3.6} \\ \text{(applied to } Y \text{ instead of} \\ X\text{) yields } w(e_Y)=e_{w(Y)})}} = \sum_{Y\in[X]} w\left(e_Y\right) = w\left(\sum_{Y\in[X]} e_Y\right).$$

In other words, $\varepsilon_{[X]} = w\left(\varepsilon_{[X]}\right)$ (since $\varepsilon_{[X]} = \sum_{Y\in[X]} e_Y$).

Let us now forget that we fixed w. We thus have shown that $\varepsilon_{[X]} = w\left(\varepsilon_{[X]}\right)$ for every $w \in W$. In other words, $\varepsilon_{[X]} \in (k\mathcal{F})^W$. This proves Corollary 3.8a **(a)**.

(b) We have $x\varepsilon_{[\operatorname{supp}(x)]} = xe_{\operatorname{supp}(x)}$ [9]. Hence, every $w \in W$ satisfies

[9] *Proof.* Let $Y \in [\operatorname{supp}(x)]$ be such that $Y \neq \operatorname{supp}(x)$. Then, the elements Y and $\operatorname{supp}(x)$ of $\mathcal{L}$ are distinct (since $Y \neq \operatorname{supp}(x)$) and belong to one and the same W-orbit (namely, to the W-orbit $[\operatorname{supp}(x)]$). Hence, $\operatorname{supp}(x) \not\leq Y$ (by Lemma 3.8c, applied to Y and $\operatorname{supp}(x)$ instead of X and Y). Therefore, $xe_Y = 0$ (by Lemma 3.1, applied to x and Y instead of w and X).

Let us now forget that we fixed Y. We thus have shown that every $Y \in [\operatorname{supp}(x)]$ satisfying $Y \neq \operatorname{supp}(x)$ satisfies $xe_Y = 0$. Thus, $\sum_{\substack{Y\in[\operatorname{supp}(x)]; \\ Y\neq\operatorname{supp}(x)}} \underbrace{xe_Y}_{=0} = \sum_{\substack{Y\in[\operatorname{supp}(x)]; \\ Y\neq\operatorname{supp}(x)}} 0 = 0$.

Now, the definition of $\varepsilon_{[\operatorname{supp}(x)]}$ yields $\varepsilon_{[\operatorname{supp}(x)]} = \sum_{Y\in[\operatorname{supp}(x)]} e_Y$. Hence,

$$\begin{aligned} x \underbrace{\varepsilon_{[\operatorname{supp}(x)]}}_{=\sum_{Y\in[\operatorname{supp}(x)]} e_Y} &= x \sum_{Y\in[\operatorname{supp}(x)]} e_Y = \sum_{Y\in[\operatorname{supp}(x)]} xe_Y \\ &= \sum_{\substack{Y\in[\operatorname{supp}(x)]; \\ Y=\operatorname{supp}(x)}} x \underbrace{e_Y}_{\substack{=e_{\operatorname{supp}(x)} \\ \text{(since } Y=\operatorname{supp}(x))}} + \sum_{\substack{Y\in[\operatorname{supp}(x)]; \\ Y\neq\operatorname{supp}(x)}} \underbrace{xe_Y}_{=0} = \sum_{\substack{Y\in[\operatorname{supp}(x)]; \\ Y=\operatorname{supp}(x)}} xe_{\operatorname{supp}(x)} + \underbrace{\sum_{\substack{Y\in[\operatorname{supp}(x)]; \\ Y\neq\operatorname{supp}(x)}} 0}_{=0} \\ &= \sum_{\substack{Y\in[\operatorname{supp}(x)]; \\ Y=\operatorname{supp}(x)}} xe_{\operatorname{supp}(x)} = xe_{\operatorname{supp}(x)} \qquad \left(\text{since } \operatorname{supp}(x) \in [\operatorname{supp}(x)]\right), \end{aligned}$$

qed.

$w(x)\,\varepsilon_{[\operatorname{supp}(x)]} = w\left(xe_{\operatorname{supp}(x)}\right)$ [10]. Now,

$$\left(\sum_{w\in W} w(x)\right)\varepsilon_{[\operatorname{supp}(x)]} = \sum_{w\in W} \underbrace{w(x)\,\varepsilon_{[\operatorname{supp}(x)]}}_{=w\left(xe_{\operatorname{supp}(x)}\right)} = \sum_{w\in W} w\left(xe_{\operatorname{supp}(x)}\right).$$

This proves Corollary 3.8a **(b)**.

(c) Let us first fix some $w \in W$ such that $w(\operatorname{supp}(x)) = \operatorname{supp}(x)$. Now, $\operatorname{supp}(wx) = w(\operatorname{supp}(x))$. Hence, $\operatorname{supp}(wx) = w(\operatorname{supp}(x)) = \operatorname{supp}(x)$, so that $\operatorname{supp}(x) = \operatorname{supp}(wx) \leq \operatorname{supp}(wx)$. Therefore, Lemma 3.8d (applied to wx, $we_{\operatorname{supp}(x)}$ and x instead of w, p and t) yields $(wx)\left(we_{\operatorname{supp}(x)}\right)x = (wx)\left(we_{\operatorname{supp}(x)}\right)$.

But Lemma 3.6 (applied to $X = \operatorname{supp}(x)$) yields $we_{\operatorname{supp}(x)} = e_{w(\operatorname{supp}(x))} = e_{\operatorname{supp}(x)}$ (since $w(\operatorname{supp}(x)) = \operatorname{supp}(x)$). Thus,

$$(wx)\left(we_{\operatorname{supp}(x)}\right)x = (wx)\underbrace{\left(we_{\operatorname{supp}(x)}\right)}_{=e_{\operatorname{supp}(x)}} = (wx)\,e_{\operatorname{supp}(x)}.$$

Since $(wx)\left(we_{\operatorname{supp}(x)}\right) = w\left(xe_{\operatorname{supp}(x)}\right)$ (because $w \in W$ acts on $k\mathcal{F}$ by an algebra homomorphism), this rewrites as $w\left(xe_{\operatorname{supp}(x)}\right)x = (wx)\,e_{\operatorname{supp}(x)}$. Thus,

$$\underbrace{w\left(xe_{\operatorname{supp}(x)}\right)x}_{=(wx)e_{\operatorname{supp}(x)}} e_{\operatorname{supp}(x)} = (wx)\underbrace{e_{\operatorname{supp}(x)}e_{\operatorname{supp}(x)}}_{\substack{=e_{\operatorname{supp}(x)} \\ \text{(since } e_{\operatorname{supp}(x)} \text{ is an idempotent)}}} = wx\underbrace{e_{\operatorname{supp}(x)}}_{\substack{=we_{\operatorname{supp}(x)} \\ \text{(since } we_{\operatorname{supp}(x)}=e_{\operatorname{supp}(x)})}}$$
$$= (wx)\left(we_{\operatorname{supp}(x)}\right) = w\left(xe_{\operatorname{supp}(x)}\right).$$

Let us now forget that we fixed w. We thus have proven that

$$w\left(xe_{\operatorname{supp}(x)}\right)xe_{\operatorname{supp}(x)} = w\left(xe_{\operatorname{supp}(x)}\right) \tag{10}$$

[10] *Proof.* Let $w \in W$. Then, $w\left(\varepsilon_{[\operatorname{supp}(x)]}\right) = \varepsilon_{[\operatorname{supp}(x)]}$ (since $\varepsilon_{[\operatorname{supp}(x)]} \in (k\mathcal{F})^W$ (by Corollary 3.8a **(a)**, applied to $X = \operatorname{supp}(x)$)). But $w \in W$ acts on $k\mathcal{F}$ by an algebra homomorphism, and therefore we have

$$w\left(x\varepsilon_{[\operatorname{supp}(x)]}\right) = w(x)\underbrace{w\left(\varepsilon_{[\operatorname{supp}(x)]}\right)}_{=\varepsilon_{[\operatorname{supp}(x)]}} = w(x)\,\varepsilon_{[\operatorname{supp}(x)]}.$$

Hence, $w(x)\,\varepsilon_{[\operatorname{supp}(x)]} = w\left(\underbrace{x\varepsilon_{[\operatorname{supp}(x)]}}_{=xe_{\operatorname{supp}(x)}}\right) = w\left(xe_{\operatorname{supp}(x)}\right)$, qed.

for every $w \in W$ such that $w(\operatorname{supp}(x)) = \operatorname{supp}(x)$.

On the other hand, let us fix some $w \in W$ such that $w(\operatorname{supp}(x)) \neq \operatorname{supp}(x)$. Thus, $\operatorname{supp}(x)$ and $w(\operatorname{supp}(x))$ are two distinct elements of $\mathcal{L}$ which belong to one and the same W-orbit. Thus, Lemma 3.8c (applied to $X = \operatorname{supp}(x)$ and $Y = w(\operatorname{supp}(x))$) yields $w(\operatorname{supp}(x)) \not\leq \operatorname{supp}(x)$. Thus, $\operatorname{supp}(wx) = w(\operatorname{supp}(x)) \not\leq \operatorname{supp}(x)$. Hence, Lemma 3.8b (applied to wx, $\left(we_{\operatorname{supp}(x)}\right)x$ and $\operatorname{supp}(x)$ instead of w, p and X) yields $(wx)\left(we_{\operatorname{supp}(x)}\right)xe_{\operatorname{supp}(x)} = 0$. Since $(wx)\left(we_{\operatorname{supp}(x)}\right) = w\left(xe_{\operatorname{supp}(x)}\right)$ (because $w \in W$ acts on $k\mathcal{F}$ by an algebra homomorphism), this rewrites as $w\left(xe_{\operatorname{supp}(x)}\right)xe_{\operatorname{supp}(x)} = 0$.

Let us now forget that we fixed w. We thus have proven that

$$w\left(xe_{\operatorname{supp}(x)}\right)xe_{\operatorname{supp}(x)} = 0 \tag{11}$$

for every $w \in W$ such that $w(\operatorname{supp}(x)) \neq \operatorname{supp}(x)$.

But recall that $\mathfrak{E}_x = \sum_{w \in W} w\left(xe_{\operatorname{supp}(x)}\right)$. Multiplying both sides of this equality with $xe_{\operatorname{supp}(x)}$ from the right, we obtain

$$\begin{aligned}
\mathfrak{E}_x xe_{\operatorname{supp}(x)} &= \left(\sum_{w \in W} w\left(xe_{\operatorname{supp}(x)}\right)\right)xe_{\operatorname{supp}(x)} = \sum_{w \in W} w\left(xe_{\operatorname{supp}(x)}\right)xe_{\operatorname{supp}(x)} \\
&= \underbrace{\sum_{\substack{w \in W;\\ w(\operatorname{supp}(x))=\operatorname{supp}(x)}}}_{\substack{=\sum_{w \in \operatorname{Stab}_W(\operatorname{supp}(x))} \\ \text{(since } \operatorname{Stab}_W(\operatorname{supp}(x)) \\ =\{w \in W \mid w(\operatorname{supp}(x))=\operatorname{supp}(x)\} \\ \text{(by the definition of } \operatorname{Stab}_W(\operatorname{supp}(x))))}} \underbrace{w\left(xe_{\operatorname{supp}(x)}\right)xe_{\operatorname{supp}(x)}}_{\substack{=w\left(xe_{\operatorname{supp}(x)}\right) \\ \text{(by (10))}}} \\
&\qquad + \sum_{\substack{w \in W;\\ w(\operatorname{supp}(x))\neq\operatorname{supp}(x)}} \underbrace{w\left(xe_{\operatorname{supp}(x)}\right)xe_{\operatorname{supp}(x)}}_{\substack{=0 \\ \text{(by (11))}}} \\
&= \sum_{w \in \operatorname{Stab}_W(\operatorname{supp}(x))} w\left(xe_{\operatorname{supp}(x)}\right) + \underbrace{\sum_{\substack{w \in W;\\ w(\operatorname{supp}(x))\neq\operatorname{supp}(x)}} 0}_{=0} \\
&= \sum_{w \in \operatorname{Stab}_W(\operatorname{supp}(x))} w\left(xe_{\operatorname{supp}(x)}\right).
\end{aligned}$$

This proves (8).

Now, recall that $\mathfrak{E}_x = \sum\limits_{w \in W} w\left(xe_{\operatorname{supp}(x)}\right) = \sum\limits_{p \in W} p\left(xe_{\operatorname{supp}(x)}\right)$ (here, we renamed the summation index w as p). But every $p \in W$ satisfies

$$\begin{aligned}
p\left(\underbrace{\mathfrak{E}_x}_{= \sum\limits_{w \in W} w\left(xe_{\operatorname{supp}(x)}\right)}\right) &= p\left(\sum_{w \in W} w\left(xe_{\operatorname{supp}(x)}\right)\right) = \sum_{w \in W} \underbrace{p\left(w\left(xe_{\operatorname{supp}(x)}\right)\right)}_{=(pw)\left(xe_{\operatorname{supp}(x)}\right)} \\
&= \sum_{w \in W} (pw)\left(xe_{\operatorname{supp}(x)}\right) = \sum_{w \in W} w\left(xe_{\operatorname{supp}(x)}\right) \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } w \text{ for } pw \text{ in the sum,} \\ \text{since the map } W \to W,\ w \mapsto pw \text{ is a bijection} \\ \text{(because } W \text{ is a group)}\end{array}\right) \\
&= \mathfrak{E}_x. \qquad (12)
\end{aligned}$$

Now,

$$\begin{aligned}
\mathfrak{E}_x^2 &= \mathfrak{E}_x \underbrace{\mathfrak{E}_x}_{=\sum_{p\in W} p\left(xe_{\operatorname{supp}(x)}\right)} = \mathfrak{E}_x \sum_{p\in W} p\left(xe_{\operatorname{supp}(x)}\right) \\
&= \sum_{p\in W} \underbrace{\mathfrak{E}_x}_{\substack{=p(\mathfrak{E}_x)\\ \text{(by (12))}}} p\left(xe_{\operatorname{supp}(x)}\right) = \sum_{p\in W} \underbrace{p\left(\mathfrak{E}_x\right) p\left(xe_{\operatorname{supp}(x)}\right)}_{\substack{=p\left(\mathfrak{E}_x xe_{\operatorname{supp}(x)}\right) \\ \text{(since } p\in W \text{ acts on } k\mathcal{F} \text{ by} \\ \text{a } k\text{-algebra homomorphism)}}} \\
&= \sum_{p\in W} p\left(\underbrace{\mathfrak{E}_x xe_{\operatorname{supp}(x)}}_{\substack{=\sum_{w\in\operatorname{Stab}_W(\operatorname{supp}(x))} w\left(xe_{\operatorname{supp}(x)}\right) \\ \text{(by (8))}}}\right) = \sum_{p\in W} p\left(\sum_{w\in\operatorname{Stab}_W(\operatorname{supp}(x))} w\left(xe_{\operatorname{supp}(x)}\right)\right) \\
&= \underbrace{\sum_{p\in W}\sum_{w\in\operatorname{Stab}_W(\operatorname{supp}(x))}}_{=\sum_{w\in\operatorname{Stab}_W(\operatorname{supp}(x))}\sum_{p\in W}} \underbrace{p\left(w\left(xe_{\operatorname{supp}(x)}\right)\right)}_{=(pw)\left(xe_{\operatorname{supp}(x)}\right)} \\
&= \sum_{w\in\operatorname{Stab}_W(\operatorname{supp}(x))}\sum_{p\in W} (pw)\left(xe_{\operatorname{supp}(x)}\right) = \sum_{w\in\operatorname{Stab}_W(\operatorname{supp}(x))} \underbrace{\sum_{p\in W} p\left(xe_{\operatorname{supp}(x)}\right)}_{=\mathfrak{E}_x} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } p \text{ for } pw \text{ in the second sum,} \\ \text{since the map } W\to W,\ p\mapsto pw \text{ is a bijection} \\ \text{(because } W \text{ is a group)}\end{array}\right) \\
&= \sum_{w\in\operatorname{Stab}_W(\operatorname{supp}(x))} \mathfrak{E}_x = \left|\operatorname{Stab}_W\left(\operatorname{supp}\left(x\right)\right)\right| \mathfrak{E}_x.
\end{aligned}$$

This proves (9).

- **Page 26, Exercise 6:** Replace "an equivalence class" by "an equivalence relation".
- **Page 26, Exercise 6:** I haven't been reading the things pertaining to the general Coxeter-group case anywhere near carefully, but shouldn't x_J be w_J to match your notations in §2.4?
- **Page 26, Exercise 6:** Replace "as recursively" by "recursively".