# Powers of matrices with all principal minors equal to 1

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**Abstract**

Consider a square matrix $A$ whose all principal minors are equal to 1. Over a field, this property is inherited by any power of $A$, but this is not the case over an arbitrary commutative ring. We show that it is the case over any reduced ring, and also over the ring $\mathbb{Z}/d$ for any integer $d$, and in some other settings (quotients of Prüfer domains and principal quotients of normal domains). This generalizes Problem B5 of the 2021 Putnam contest.

Over arbitrary commutative rings, we identify a stronger property that is always inherited by powers: We say that a matrix $A = (a_{i,j})_{i,j\in[n]}$ is 1-nullcyclic if all its diagonal entries are 1 and if all the cyclic products $a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_k,i_1}$ with $k > 1$ and distinct $i_1, i_2, \ldots, i_k$ vanish. We show that if $A$ is 1-nullcyclic, then so is $A^m$ for any integer $m$. Furthermore, every 1-nullcyclic matrix is 1-principled over any commutative ring, while the converse holds over any reduced ring or any quotient of a ring by an integrally closed ideal.

Along the way, we prove versions of these results that don't require the diagonal entries to be 1. Here, the principal minors of $A$ are required not to equal 1 but rather to equal the respective products of diagonal entries of $A$.

A crucial auxiliary result, which holds for any $n\times n$-matrix $A$, is that the cyclic products $a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_k,i_1}$ (with $k > 1$ and distinct $i_1, i_2, \ldots, i_k$) are integral over the ideal generated by the principal minors of $A$ minus the corresponding products of diagonal entries of $A$.

## 1 Definitions and the main result

Throughout this note, rings are commutative, associative and unital. For $n \in \mathbb{N}$, we set $[n] := \{1, 2, \ldots, n\}$.

If $A = (a_{i,j})_{i,j\in[n]}$ is an $n \times n$-matrix over a ring $R$ and if $S \subseteq [n]$, then $A_S$ denotes the principal submatrix of $A$ with row and column set $S$. That is, $A_S = \operatorname{sub}_S^S A$ in the

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notation of [2]. For instance, $\begin{pmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{pmatrix}_{\{1,3\}} = \begin{pmatrix} a & c \\ a'' & c'' \end{pmatrix}$. The *principal minors* of an $n \times n$-matrix $A$ are its minors $\det A_S$ for all $S \subseteq [n]$. We use the convention that the empty determinant is 1.

Problem B5 of the 2021 Putnam contest [1, Problem B5] asserts that if $A \in \mathbb{Z}^{n\times n}$ is an integer matrix whose all principal minors are odd, then all powers $A^m$ of $A$ have the same property. By reducing the matrix modulo 2, this can be restated as follows: If $A \in (\mathbb{Z}/2)^{n\times n}$ is a matrix over the two-element field $\mathbb{Z}/2$ whose all principal minors equal 1, then all its powers $A^m$ have the same property. This suggests a generalization to arbitrary commutative rings instead of $\mathbb{Z}/2$; however, this generalization was disproved in [2, §6] for $4 \times 4$-matrices over a certain finite ring.[1]

In this note, we shall show that this generalization is nevertheless true if we replace $\mathbb{Z}/2$ by any ring of the form $\mathbb{Z}/d$ with $d \in \mathbb{Z}$. More generally, it is true over every quotient $D/I$ whose kernel ideal $I$ is integrally closed in $D$. This includes quotients of Prüfer domains by arbitrary ideals and quotients of normal domains by principal ideals.

We introduce some terminology for the type of matrices we will study.

**Definition 1.1.** Let $R$ be a commutative ring, and let $A = (a_{i,j})_{i,j\in[n]}$ be an $n\times n$-matrix over $R$. We say that $A$ is 1-*principled* if

$$\det A_S = 1 \qquad \text{for every } S \subseteq [n] \,.$$

Note that the diagonal entries of an $n \times n$-matrix are its $1 \times 1$ principal minors. Hence, the diagonal entries of a 1-principled matrix are 1.

Also, each 1-principled matrix $A$ has determinant $\det A = 1$ (since $\det A = \det A_{[n]}$ is a principal minor of $A$), and thus is invertible (since a matrix with invertible determinant is invertible). Thus, its powers $A^m$ are defined for all integers $m$.

Our main result is the following theorem, which recovers Problem B5 of the 2021 Putnam contest when we take $D = \mathbb{Z}$ and $I = 2\mathbb{Z}$ and $m \geq 0$.

**Theorem 1.2.** *Let $D$ be a commutative ring, let $I$ be an integrally closed ideal of $D$, and set $R := D/I$. Let $A$ be a 1-principled matrix over $R$. Then $A^m$ is 1-principled for every $m \in \mathbb{Z}$.*

The notion of an integrally closed ideal will be recalled in Section 4.

Our proof of Theorem 1.2 will be obtained by a combination of several results involving other properties of matrices. We shall introduce them one by one now.

**Definition 1.3.** An $n \times n$-matrix $A = (a_{i,j})_{i,j\in[n]}$ is said to be 1-*diagonal* if all its diagonal entries $a_{i,i}$ are 1.

[1] That said, a part of the generalization is true over any $R$ (see [2, Theorem 5.2]): If $A \in R^{n\times n}$ is a matrix whose all principal minors equal 1, then all **diagonal entries** of its powers $A^m$ equal 1 as well (even though some principal minors of $A^m$ may differ from 1).

Note that [2, Theorem 5.2] says that if $A$ is a 1-principled matrix and $m \in \mathbb{N}$, then $A^m$ is 1-diagonal.

**Definition 1.4.** Let $R$ be a commutative ring, and let $A = (a_{i,j})_{i,j\in[n]}$ be an $n\times n$-matrix over $R$. We say that $A$ is *principled* if

$$\det A_S = \prod_{i\in S} a_{i,i} \qquad \text{for every } S \subseteq [n]\,.$$

The following simple proposition is an exercise in unfolding definitions:

**Proposition 1.5.** *Let $R$ be a commutative ring, and let $A = (a_{i,j})_{i,j\in[n]}$ be an $n \times n$-matrix over $R$. Then, $A$ is 1-principled if and only if $A$ is 1-diagonal and principled.*

*Proof.* $\Longrightarrow$: Assume that $A$ is 1-principled. Then, each diagonal entry $a_{i,i}$ of $A$ equals the principal $1 \times 1$-minor $\det A_{\{i\}}$, which is 1 since $A$ is 1-principled. Thus, $A$ is 1-diagonal. Furthermore, for every $S \subseteq [n]$, we have $\prod_{i\in S} a_{i,i} = 1$ (since all factors $a_{i,i}$ of this product are 1, as we just saw) and $\det A_S = 1$ (since $A$ is 1-principled), so that $\det A_S = \prod_{i\in S} a_{i,i}$ (by comparing the preceding two equalities). This shows that $A$ is principled. Thus, we have proved that $A$ is 1-diagonal and principled.

$\Longleftarrow$: Assume that $A$ is 1-diagonal and principled. Then, for every $S \subseteq [n]$, we have $\det A_S = \prod_{i\in S} a_{i,i}$ (since $A$ is principled). But all the factors $a_{i,i}$ of the latter product are 1 (since $A$ is 1-diagonal), so this equality rewrites as $\det A_S = \prod_{i\in S} 1 = 1$. This shows that $A$ is 1-principled. □

Our last two matrix properties rely on a simple combinatorial notion.

**Definition 1.6.** Let $S$ be a set. A *cycle* on $S$ will mean a $k$-tuple

$$C = (i_1, i_2, \ldots, i_k)$$

where $k > 0$ and where $i_1, i_2, \ldots, i_k$ are distinct elements of $S$. To be more precise, the cycle is not this $k$-tuple itself, but rather its equivalence class under cyclic rotation (i.e., we count $(i_1, i_2, \ldots, i_k)$ and $(i_2, i_3, \ldots, i_k, i_1)$ as being the same cycle). This cycle is said to have *length* $k$, *vertices* $i_1, i_2, \ldots, i_k$ and *arcs* $(i_1, i_2)\,,\ (i_2, i_3)\,,\ \ldots,\ (i_k, i_1)$; furthermore, we call it *nontrivial* if $k > 1$ (that is, if the cycle has more than one arc).

**Definition 1.7.** Let $A = (a_{i,j})_{i,j\in[n]}$ be an $n \times n$-matrix over a commutative ring $R$.

**(a)** The *$A$-weight* of a cycle $C = (i_1, i_2, \ldots, i_k)$ on $[n]$ is defined to be

$$w_A(C) := a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_{k-1},i_k} a_{i_k,i_1}.$$

**(b)** We say that $A$ is *nullcyclic* if

$$w_A(C) = 0 \qquad \text{for each nontrivial cycle } C \text{ on } [n]$$

(that is, each nontrivial cycle on $[n]$ has $A$-weight 0).

**(c)** We say that $A$ is 1-*nullcyclic* if $A$ is 1-diagonal and nullcyclic.

**Example 1.8.** If a matrix $A = (a_{i,j})_{i,j\in[n]}$ is triangular, then $A$ is nullcyclic. Indeed, any nontrivial cycle on $[n]$ has an arc $(i,j)$ with $i<j$ and an arc $(i,j)$ with $i>j$, and at least one of these arcs will satisfy $a_{i,j}=0$; thus, the $A$-weight of the cycle is 0.

Thus, if a matrix $A = (a_{i,j})_{i,j\in[n]}$ is unitriangular (i.e., triangular and satisfies $a_{i,i}=1$ for all $i\in[n]$), then $A$ is 1-nullcyclic.

However, there are 1-nullcyclic matrices that are not unitriangular, such as

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \text{over } R = \mathbb{Z}/4.$$

We then claim the following twin to Theorem 1.2 (but only for $m \geq 0$, since the invertibility of $A$ is no longer guaranteed):

**Theorem 1.9.** *Let $D$ be a commutative ring, let $I$ be an integrally closed ideal of $D$, and set $R := D/I$. Let $A$ be a principled matrix over $R$. Then $A^m$ is principled for every $m\in\mathbb{N}$.*

The proof of Theorem 1.9 relies on three independent results, all of which hold over any (commutative) ring: First, every nullcyclic matrix is principled (Proposition 2.1), although the converse does not always hold. Second, nullcyclic matrices are stable under powers (Proposition 2.3). Third, every nontrivial cycle's $A$-weight is integral over the ideal generated by the principal-minor defects (i.e., the principal minors minus the corresponding products of diagonal entries) (Theorem 4.6). In the setting of Theorem 1.9, this forces these weights to belong to $I$, since $I$ is integrally closed.

The proof of Theorem 1.2 will reuse these auxiliary results, partly in modified form.

After we prove all the results we have described, we shall discuss some examples of integrally closed ideals.

# 2 Nullcyclic matrices

We begin with the theory of nullcyclic matrices.

**Proposition 2.1.** *Every nullcyclic matrix over a commutative ring is principled.*

*Proof.* Let $A = (a_{i,j})_{i,j\in[n]}$ be a nullcyclic $n\times n$-matrix. Let $S\subseteq[n]$. We expand

$$\det A_S = \sum_{\pi\in\mathfrak{S}_S} \operatorname{sgn}(\pi) \prod_{i\in S} a_{i,\pi(i)} \tag{2.1}$$

(where $\mathfrak{S}_S$ is the group of all permutations of $S$). The identity permutation $\mathrm{id}\in\mathfrak{S}_S$ contributes the addend

$$\operatorname{sgn}(\mathrm{id}) \prod_{i\in S} a_{i,\mathrm{id}(i)} = \prod_{i\in S} a_{i,i}$$

to the right-hand side of (2.1). The remaining addends on the right-hand side of (2.1) correspond to non-identity permutations $\pi \in \mathfrak{S}_S$. Every non-identity permutation $\pi \in \mathfrak{S}_S$ has at least one nontrivial cycle $(i_1, i_2, \ldots, i_k)$ in its cycle decomposition. The corresponding addend on the right-hand side of (2.1) therefore contains, as a factor, the $A$-weight $a_{i_1,i_2} a_{i_2,i_3} \cdots a_{i_{k-1},i_k} a_{i_k,i_1}$ of this nontrivial cycle. This factor is 0, since $A$ is nullcyclic. Hence all non-identity addends on the right-hand side of (2.1) vanish, and so the equality (2.1) simplifies to

$$\det A_S = \prod_{i \in S} a_{i,i}.$$

That is, $A$ is principled. $\square$

We shall use some basic graph theory (see, e.g., [4, Chapter 4]). Let $K_n^{\rightarrow}$ be the simple digraph (i.e., directed graph) with $n$ vertices $1, 2, \ldots, n$ and $n^2$ arcs $(i, j)$ for all $i, j \in [n]$. What we called "cycles on $[n]$" above are exactly the cycles of $K_n^{\rightarrow}$ (considered up to cyclic rotation). The nontrivial cycles on $[n]$ are then the cycles of $K_n^{\rightarrow}$ having length $> 1$. We make the following elementary observation about walks.

**Lemma 2.2.** *Let*

$$i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_m = i_0$$

*be a closed walk of a simple digraph. If this closed walk is not the stationary walk* $i_0 \rightarrow i_0 \rightarrow \cdots \rightarrow i_0$*, then it contains a nontrivial cycle. ("Contains" means that each arc of the cycle is an arc of the walk.)*

*Proof.* Delete all loops[2] from the closed walk. Since the original walk is not stationary, some arcs remain upon this deletion. Starting at any remaining arc and following the walk cyclically, eventually a vertex is repeated. The part of the walk between the first occurrence of this vertex and its next occurrence is a closed walk with no repeated internal vertices. This closed walk is therefore a cycle, and moreover a nontrivial cycle (since all loops have been deleted). $\square$

**Proposition 2.3.** *Let $A$ be a nullcyclic matrix over a commutative ring $R$. Then $A^m$ is nullcyclic for every $m \in \mathbb{N}$.*

*Proof.* Let $m \in \mathbb{N}$. The case $m = 0$ is clear, since $A^0 = I_n$. Assume $m > 0$.

Write the $n \times n$-matrix $A$ as $A = (a_{i,j})_{i,j \in [n]}$. We also use $B_{i,j}$ to refer to the $(i, j)$-th entry of any matrix $B$; thus, $A_{i,j} = a_{i,j}$ for all $i, j \in [n]$.

It is well-known that for any two vertices $i$ and $j$ of $K_n^{\rightarrow}$, we have[3]

$$(A^m)_{i,j} = \sum_{\substack{i = i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_m = j \\ \text{is a walk of } K_n^{\rightarrow}}} a_{i_0,i_1} a_{i_1,i_2} \cdots a_{i_{m-1},i_m}. \tag{2.2}$$

[2] Recall that a *loop* means an arc of the form $(v, v)$ for some vertex $v$.

[3] The formula (2.2) (generalized to arbitrary digraphs) appears, e.g., in [8, last sentence of §1] and in [7, Theorem 4.7.1] (and also – in an unweighted version – in [4, Theorem 4.5.10] as well, where a more detailed proof can be found, which can be easily adapted to the weighted case). Anyway, the reader can easily prove it by induction on $m$ using $A^m = AA^{m-1}$.

Let us refer to the product $a_{i_0,i_1}a_{i_1,i_2}\cdots a_{i_{m-1},i_m}$ in this sum as the *A-weight* of the walk $i = i_0 \to i_1 \to \cdots \to i_m = j$. Thus, (2.2) says that

$$\left(A^m\right)_{i,j} = (\text{sum of the } A\text{-weights of all length-}m \text{ walks from } i \text{ to } j)\,. \tag{2.3}$$

We must prove that $A^m$ is nullcyclic. In other words, we must prove that every nontrivial cycle on $[n]$ has $A^m$-weight 0. Let

$$C = (i_1, i_2, \ldots, i_k)$$

be a nontrivial cycle on $[n]$, with length $k > 1$. We must show that $w_{A^m}(C) = 0$.

Note that $i_1 \neq i_2$ (by the definition of a cycle, since $k > 1$).

Set $i_{k+1} = i_1$ (that is, read the indices cyclically modulo $k$). Then,

$$\begin{aligned} w_{A^m}(C) &= \left(A^m\right)_{i_1,i_2}\left(A^m\right)_{i_2,i_3}\cdots\left(A^m\right)_{i_k,i_1} = \prod_{j=1}^{k}\left(A^m\right)_{i_j,i_{j+1}} \\ &= \prod_{j=1}^{k}(\text{sum of the } A\text{-weights of all length-}m \text{ walks from } i_j \text{ to } i_{j+1}) \end{aligned}$$

(by (2.3), applied to $i_j$ and $i_{j+1}$ instead of $i$ and $j$). Expanding this product, we obtain a sum over all $k$-tuples $(W_1, W_2, \ldots, W_k)$ of length-$m$ walks from $i_j$ to $i_{j+1}$ for each $j \in [k]$. The addend corresponding to such a $k$-tuple is the product of the $A$-weights of all these walks $W_1, W_2, \ldots, W_k$; but this is, of course, the $A$-weight of the closed walk $W_1W_2\cdots W_k$ (of length $km$) obtained by concatenating these $k$ walks. This closed walk $W_1W_2\cdots W_k$ is not stationary (since $i_1 \neq i_2$ are two distinct vertices on it). Hence it contains a nontrivial cycle $C'$ (by Lemma 2.2). The $A$-weight of this cycle $C'$ is 0 since $A$ is nullcyclic. Thus, the whole closed walk $W_1W_2\cdots W_k$ has $A$-weight 0 as well (since $C'$ is contained in $W_1W_2\cdots W_k$, and thus the $A$-weight of $W_1W_2\cdots W_k$ is a multiple of the $A$-weight of $C'$).

Thus, we have shown that $w_{A^m}(C)$ is a sum of $A$-weights of certain closed walks, but each of these closed walks has $A$-weight 0. Hence,

$$w_{A^m}(C) = 0.$$

As explained above, this completes the proof. □

**Corollary 2.4.** *Let $A$ be a nullcyclic matrix over a commutative ring $R$. Then $A^m$ is principled for every $m \in \mathbb{N}$.*

*Proof.* The matrix $A^m$ is nullcyclic by Proposition 2.3, and therefore is principled by Proposition 2.1. □

**Remark 2.5.** Proposition 2.3 is genuinely a statement about powers, not about products. Even commuting products of 1-nullcyclic matrices need not be nullcyclic.

For an explicit counterexample, let $R = \mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2$. Let

$$J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

so that $J^2 = 0$. Now define the block matrices

$$A = \begin{pmatrix} I_2 & 0 \\ J & I_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} I_2 & J \\ 0 & I_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Both $A$ and $B$ are unitriangular and thus 1-nullcyclic (by Example 1.8).

However,

$$AB = \begin{pmatrix} I_2 & J \\ J & I_2 + J^2 \end{pmatrix} = \begin{pmatrix} I_2 & J \\ J & I_2 \end{pmatrix} = \begin{pmatrix} I_2 + J^2 & J \\ J & I_2 \end{pmatrix} = BA = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

This common product is not nullcyclic, since the nontrivial cycle $(1, 3)$ has $AB$-weight 1.

# 3 1-Nullcyclic matrices

The analogue of Proposition 2.1 for 1-nullcyclic matrices follows easily from Proposition 2.1:

**Proposition 3.1.** *Every 1-nullcyclic matrix over a commutative ring is 1-principled.*

*Proof.* Let $A$ be a 1-nullcyclic matrix. Then, $A$ is nullcyclic and thus principled (by Proposition 2.1). But $A$ is furthermore 1-diagonal (being 1-nullcyclic). Hence, $A$ is 1-principled (by Proposition 1.5). □

Next we derive the analogue of Proposition 2.3 for 1-nullcyclic matrices:

**Proposition 3.2.** *Let $A$ be a 1-nullcyclic matrix over a commutative ring $R$. Then $A^m$ is 1-nullcyclic for every $m \in \mathbb{N}$.*

*Proof.* Let $m \in \mathbb{N}$. The case $m = 0$ is again clear, since $A^0 = I_n$. Assume $m > 0$.

We know from Proposition 2.3 that $A^m$ is nullcyclic; thus, it remains to show that $A^m$ is 1-diagonal. In other words, we must show that all diagonal entries of $A^m$ are 1.

Write the $n \times n$-matrix $A$ as $A = (a_{i,j})_{i,j \in [n]}$. We also use $B_{i,j}$ to refer to the $(i, j)$-th entry of any matrix $B$; thus, $A_{i,j} = a_{i,j}$ for all $i, j \in [n]$.

The matrix $A$ is 1-nullcyclic, hence 1-diagonal. That is, all diagonal entries of $A$ are 1: For each $i \in [n]$, we have

$$a_{i,i} = 1. \tag{3.1}$$

Fix $i \in [n]$. The equality (2.3) shows that $(A^m)_{i,i}$ is the sum of the $A$-weights of all length-$m$ walks from $i$ to $i$. The stationary walk $i = i \to i \to \cdots \to i = i$ contributes 1 to this sum, since its $A$-weight is $a_{i,i}a_{i,i}\cdots a_{i,i} = a_{i,i}^m = 1$ (by (3.1)). Each of the other length-$m$ walks from $i$ to $i$ contains a nontrivial cycle by Lemma 2.2 (since it is a closed walk but not stationary). Since the latter cycle has $A$-weight 0 (because $A$ is nullcyclic), we conclude that the walk that contains it must have $A$-weight 0 as well (indeed, since $R$ is commutative, the $A$-weight of the cycle is a factor of the $A$-weight of the walk). Thus, $(A^m)_{i,i}$ is the sum of a single 1 (corresponding to the stationary walk $i = i \to i \to \cdots \to i = i$) and a lot of 0's (coming from all the other walks). Therefore,

$$(A^m)_{i,i} = 1. \tag{3.2}$$

Forget that we fixed $i$. So we have proved (3.2) for each $i \in [n]$. In other words, all diagonal entries of $A^m$ are 1. This completes the proof. □

**Remark 3.3.** Alternatively, we could have proved Proposition 3.2 as follows: Proposition 3.1 shows that $A$ is 1-principled. Thus, [2, Theorem 5.2] (which says that any power of a 1-principled matrix is 1-diagonal) shows that $A^m$ is 1-diagonal. But Proposition 2.3 shows that $A^m$ is nullcyclic. Combining these two results, we conclude that $A^m$ is 1-nullcyclic.

We have shown in Proposition 3.2 that 1-nullcyclic matrices are stable under non-negative powers. We now extend this to all integer powers. The key observation is that a 1-nullcyclic matrix is unipotent with nilpotent off-diagonal part.

**Lemma 3.4.** *Let $A = (a_{i,j})_{i,j\in[n]}$ be a 1-nullcyclic matrix over a commutative ring $R$, and set $N := A - I_n$. Then:*

- ***(a)*** *Every closed walk $i_0 \to i_1 \to \cdots \to i_m = i_0$ in $K_n^{\to}$ of length $m > 0$ has $N$-weight[4] 0.*
- ***(b)*** *We have $N^n = 0$.*
- ***(c)*** *The matrix $A$ is invertible, and its inverse $A^{-1}$ is 1-nullcyclic.*

*Proof.* We will use the notation $B_{i,j}$ for the $(i,j)$-th entry of any matrix $B$.

The matrix $A$ is 1-nullcyclic, thus 1-diagonal. That is, every diagonal entry $A_{i,i}$ of $A$ is 1. Hence, every diagonal entry $N_{i,i}$ of $N$ is 0 (since $N = A - I_n$ and therefore $N_{i,i} = A_{i,i} - 1$). On the other hand, for any $i \neq j$, we have $N_{i,j} = A_{i,j}$ (since $N = A - I_n$ differs from $A$ only in its diagonal entries). Thus, if a walk of $K_n^{\to}$ has no loops, then its $N$-weight equals its $A$-weight. In particular, any nontrivial cycle on $[n]$ has $N$-weight 0 (because it has no loops, so its $N$-weight equals its $A$-weight; but its $A$-weight is 0 because $A$ is nullcyclic).

[4] The $N$-weight of a walk $i_0 \to i_1 \to \cdots \to i_m$ is defined to be $N_{i_0,i_1}N_{i_1,i_2}\cdots N_{i_{m-1},i_m}$, where $N_{i,j}$ denotes the $(i,j)$-th entry of $N$.

**(a)** Let

$$i_0 \to i_1 \to \cdots \to i_m = i_0$$

be a closed walk of positive length $m$. If this closed walk is stationary, then all its arcs are loops, and its $N$-weight is 0, since every diagonal entry of $N$ is 0.

If the closed walk is not stationary, then it contains a nontrivial cycle by Lemma 2.2. The $N$-weight of this cycle is 0 (since any nontrivial cycle on $[n]$ has $N$-weight 0). Hence the $N$-weight of the whole closed walk is 0 as well, since the $N$-weight of the cycle is a factor of it[5]. This proves part **(a)**.

**(b)** Let $i, j \in [n]$. The walk expansion (2.3) (applied to $N$ and $n$ instead of $A$ and $m$) expresses the entry $(N^n)_{i,j}$ as the sum of the $N$-weights of all length-$n$ walks $i = i_0 \to i_1 \to \cdots \to i_n = j$ of $K_n^{\to}$ from $i$ to $j$. Every such walk has a repeated vertex (by the pigeonhole principle, since $K_n^{\to}$ has only $n$ vertices), and therefore contains a closed subwalk of positive length. By part **(a)**, this closed subwalk has $N$-weight 0. Thus the whole walk has $N$-weight 0 as well. Since $(N^n)_{i,j}$ has been expressed as the sum of the weights of such walks, we thus conclude that $(N^n)_{i,j} = 0$. Hence all entries of $N^n$ are 0, proving $N^n = 0$. This proves part **(b)**.

**(c)** By part **(b)**, we have $N^n = 0$. Hence $A = I_n + N$ is invertible, with inverse

$$A^{-1} = \sum_{r=0}^{n} (-N)^r \tag{3.3}$$

(by the geometric series formula, here truncated to a finite sum since $-N$ is nilpotent). (We could replace the upper limit of this sum by $n-1$, but we have no need for this.)

We first show that $A^{-1}$ is 1-diagonal. Let $i \in [n]$. From (3.3), we obtain

$$\begin{aligned}\left(A^{-1}\right)_{i,i} &= \left(\sum_{r=0}^{n} (-N)^r\right)_{i,i} = \sum_{r=0}^{n} (-1)^r (N^r)_{i,i} \\ &= 1 + \sum_{r=1}^{n} (-1)^r (N^r)_{i,i}. \end{aligned} \tag{3.4}$$

For each $r \geq 1$, the walk expansion (2.3) (applied to $N$ and $r$ instead of $A$ and $m$) expresses $(N^r)_{i,i}$ as a sum of $N$-weights of positive-length closed walks. These weights are all 0 by part **(a)**. Hence, (3.4) simplifies to $(A^{-1})_{i,i} = 1$. This shows that $A^{-1}$ is 1-diagonal.

It remains to show that $A^{-1}$ is nullcyclic. Let $C = (i_1, i_2, \ldots, i_k)$ be a nontrivial cycle on $[n]$, and set $i_{k+1} := i_1$. We must show that $w_{A^{-1}}(C) = 0$.

For each $j \in [k]$, we can compare $(i_j, i_{j+1})$-entries in (3.3) to obtain

$$\begin{aligned}\left(A^{-1}\right)_{i_j,i_{j+1}} &= \left(\sum_{r=0}^{n} (-N)^r\right)_{i_j,i_{j+1}} = \sum_{r=0}^{n} (-1)^r (N^r)_{i_j,i_{j+1}} \\ &= \sum_{r=0}^{n} (-1)^r \text{(sum of the } N\text{-weights of all length-}r \text{ walks from } i_j \text{ to } i_{j+1}) \end{aligned}$$

[5]Here we are using the commutativity of $R$.

(by (2.3), applied to $N$ and $r$ instead of $A$ and $m$). We can rewrite this further as

$$\left(A^{-1}\right)_{i_j,i_{j+1}} = \left(\text{sum of the signed } N\text{-weights of all length-}\leq n \text{ walks from } i_j \text{ to } i_{j+1}\right),$$

where the "signed $N$-weight" of a length-$r$ walk is defined to be $(-1)^r$ times the $N$-weight of this walk. Substituting this into

$$w_{A^{-1}}(C) = \left(A^{-1}\right)_{i_1,i_2} \left(A^{-1}\right)_{i_2,i_3} \cdots \left(A^{-1}\right)_{i_k,i_1} = \prod_{j=1}^{k} \left(A^{-1}\right)_{i_j,i_{j+1}},$$

and then expanding the product, we express $w_{A^{-1}}(C)$ as a sum over $k$-tuples $(W_1, W_2, \ldots, W_k)$ of walks from $i_j$ to $i_{j+1}$ for each $j \in [k]$; the addends of the sum are the products of the signed $N$-weights of these walks $W_1, W_2, \ldots, W_k$. As in the proof of Proposition 2.3, we can replace these $k$-tuples $(W_1, W_2, \ldots, W_k)$ by their concatenations $W_1 W_2 \cdots W_k$, and then the addends are the signed $N$-weights of these concatenations. But each concatenation $W_1 W_2 \cdots W_k$ is a closed walk of positive length (since it contains the two distinct vertices $i_1$ and $i_2$), so its $N$-weight is 0 by part **(a)**. Therefore all addends in the expansion of $w_{A^{-1}}(C)$ vanish, and thus $w_{A^{-1}}(C) = 0$. This proves that $A^{-1}$ is nullcyclic. Since $A^{-1}$ is also 1-diagonal, we conclude that $A^{-1}$ is 1-nullcyclic. This completes the proof of part **(c)**. ☐

**Proposition 3.5.** *Let $A$ be a 1-nullcyclic matrix over a commutative ring $R$. Then $A^m$ is 1-nullcyclic for every integer $m \in \mathbb{Z}$.*

*Proof.* For $m \in \mathbb{N}$, this is Proposition 3.2. Thus, it remains to treat negative $m$.

Let $m < 0$. By Lemma 3.4 **(c)**, the matrix $A^{-1}$ is 1-nullcyclic. Also, $-m \in \mathbb{N}$, and $A^m = \left(A^{-1}\right)^{-m}$. Hence Proposition 3.2, applied to $A^{-1}$ and $-m$, shows that $A^m$ is 1-nullcyclic. ☐

**Corollary 3.6.** *Let $A$ be a 1-nullcyclic matrix over a commutative ring $R$. Then $A^m$ is 1-principled for every integer $m \in \mathbb{Z}$.*

*Proof.* The matrix $A^m$ is 1-nullcyclic by Proposition 3.5, and therefore is 1-principled by Proposition 3.1. ☐

# 4 Integral closure and cycle weights

We now turn towards the harder, "upstream" direction, from (1-)principled to (1-)nullcyclic. As we already mentioned, in general, a 1-principled matrix is not always nullcyclic; a counterexample is constructed in the footnote in [2, §6][6]. However, (1-

[6]We recall this counterexample: If $F$ is any field, and if $R$ is the ring $F[x,y] \diagup \left(x^3 + y^3, xy, x^4, y^4\right)$, and if $A$ is the matrix $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & y & x \\ x & 0 & 1 & y \\ y & 0 & x & 1 \end{pmatrix}$ over $R$, then $A$ is 1-principled, but $A^2$ is not, since $\det\left(A^2\right)_{\{2,3\}} = 1 - x^3 \neq 1$ in $R$. This counterexample is minimal in the sense that there are no matrices of size $\leq 3$ that are 1-principled but have non-1-principled powers.

)principledness implies a weaker version of (1-)nullcyclicity, which we will later leverage to obtain the full property under certain conditions.

We will need the classical notion of *integrality* over an ideal, which we will now briefly recall; see [6] for a much more extensive treatment.

**Definition 4.1.** Let $R$ be a commutative ring, let $I$ be an ideal of $R$, and let $x \in R$. We say that $x$ is *integral over* $I$ if there exist a positive integer $d$ and elements $c_j \in I^j$ for all $j \in [d]$ such that

$$x^d + c_1 x^{d-1} + c_2 x^{d-2} + \cdots + c_d x^0 = 0.$$

The set of all elements of $R$ that are integral over $I$ is called the *integral closure* of $I$ and is denoted by $\overline{I}$. The ideal $I$ is said to be *integrally closed* if $\overline{I} = I$.

Thus, an element is integral over the zero ideal if and only if it is nilpotent. We shall use the following standard properties of integral closure of ideals.

**Lemma 4.2.** *Let $R$ be a commutative ring, and let $I$ and $J$ be ideals of $R$. Then:*

***(a)*** *The set $\overline{I}$ is an ideal of $R$ containing $I$ as a subset.*

***(b)*** *If $I \subseteq J$, then $\overline{I} \subseteq \overline{J}$.*

***(c)*** *The ideal $\overline{I}$ is integrally closed; that is, $\overline{\overline{I}} = \overline{I}$.*

***(d)*** *If $I \subseteq J \subseteq \overline{I}$, then $\overline{J} = \overline{I}$.*

*Proof.* Parts **(a)** and **(c)** are [6, Corollary 1.3.1]. Part **(b)** is trivial. Part **(d)** follows from **(b)** and **(c)**: the inclusions $I \subseteq J \subseteq \overline{I}$ yield

$$\overline{I} \subseteq \overline{J} \subseteq \overline{\overline{I}} = \overline{I}. \qquad \square$$

We next need a combinatorial ingredient. A *Hamilton cycle on a finite set $S$* means a cycle on $S$ whose vertices are all the elements of $S$. In other words, it means a Hamilton cycle of $K_S^{\rightarrow}$, where $K_S^{\rightarrow}$ is the digraph whose vertices are the elements of $S$ and whose arcs are all the pairs $(s, t)$ for $s, t \in S$. We observe two obvious facts:

1. Each Hamilton cycle on a subset $S$ of $[n]$ is a cycle on $[n]$. Conversely, each cycle on $[n]$ is a Hamilton cycle on $S$, where $S$ is the set of vertices of this cycle.

2. A cycle on a finite set $S$ is Hamilton if and only if it has length $|S|$.

We identify each cycle with its set of arcs (since the latter set uniquely determines the former cycle). The *multiset union* $H \uplus H'$ of two cycles $H$ and $H'$ is defined to be a multidigraph (i.e., a directed multigraph) whose multiset of arcs is obtained by combining the sets of arcs of $H$ and of $H'$. (If an arc appears in both $H$ and $H'$, then it will appear twice in this multiset union.)

**Lemma 4.3.** *Let $S$ be a finite set, and let $H$ and $H'$ be two distinct Hamilton cycles on $S$. Then the multiset union of (the sets of arcs of) $H$ and $H'$ can be partitioned into at least three cycles, each having length strictly smaller than $|S|$.*

**Example 4.4.** Let $S = \{1, 2, 3, 4, 5\}$, and let

$$H = (1, 2, 3, 4, 5) \qquad \text{and} \qquad H' = (1, 3, 5, 2, 4)$$

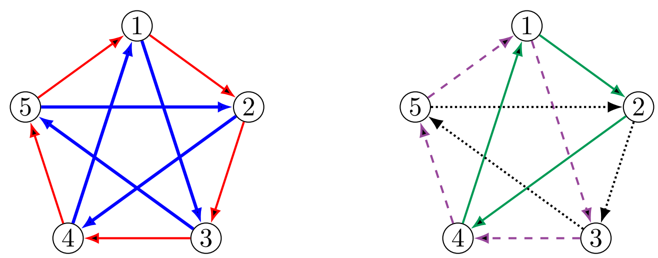
be two Hamilton cycles on $S$. Their multiset union has arcs

$$\underbrace{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)}_{\text{arcs of } H}, \underbrace{(1, 3), (3, 5), (5, 2), (2, 4), (4, 1)}_{\text{arcs of } H'}.$$

These ten arcs can be partitioned into the following three cycles:

$$(1, 2, 4), \qquad (1, 3, 4, 5), \qquad (2, 3, 5).$$

The following two pictures show the same multidigraph $H \uplus H'$ twice. In the left picture, the arcs of $H$ are red and the arcs of $H'$ are blue and thick. In the right picture, the three cycles just listed are drawn in different colors and also different styles.




Indeed, these three cycles use the following arcs:

$$\begin{aligned}(1, 2, 4) &: (1, 2), (2, 4), (4, 1),\\ (1, 3, 4, 5) &: (1, 3), (3, 4), (4, 5), (5, 1),\\ (2, 3, 5) &: (2, 3), (3, 5), (5, 2).\end{aligned}$$

Thus each arc of $H \uplus H'$ is used exactly once. None of the three cycles is Hamilton on $S$, since their lengths are $3, 4, 3$, respectively.

*Proof of Lemma 4.3.* Choose a vertex $v \in S$ whose outgoing arcs in $H$ and $H'$ are distinct.[7] Write these arcs as

$$e = (v, a) \in H \qquad \text{and} \qquad e' = (v, b) \in H',$$

where $a \neq b$. Let $P'$ be the directed path in $H'$ from $a$ back to $v$, and let $P$ be the directed path in $H$ from $b$ back to $v$. Then the walks

$$\begin{aligned}C &:= P'e \qquad \text{(that is, the path } P' \text{ followed by the arc } e) \qquad \text{and}\\ C' &:= Pe' \qquad \text{(that is, the path } P \text{ followed by the arc } e')\end{aligned}$$

[7] Such a vertex must exist, since $H$ and $H'$ are distinct.

are cycles. Both have length at most $|S|-1$. Indeed, if the cycle $C$ had length $\geq |S|$, then it would be Hamilton, and thus the path $P'$ would contain every vertex. Hence the complementary path in $H'$ from $v$ to $a$ would consist of a single arc. This arc would be the outgoing arc $(v,b)$ of $v$ in $H'$, forcing $a=b$. The same argument applies to $C'$.

The cycles $C$ and $C'$ are arc-disjoint in the multidigraph $H \uplus H'$. Indeed:

1. The only $H$-arc used by $C$ is $e$, while the $H$-arcs used by $C'$ lie in $P$, which does not use $e$ because $P$ ends at $v$. Thus, $C$ and $C'$ have no $H$-arcs in common.

2. Similarly, $C$ and $C'$ have no $H'$-arcs in common.

Remove the arcs of the two cycles $C$ and $C'$ from this multidigraph $H \uplus H'$. The remaining multidigraph is balanced[8], because both the original multidigraph $H \uplus H'$ and the removed union $C \cup C'$ are balanced. Hence its arcs can be partitioned into cycles[9]. None of these new cycles contains $v$, since both outgoing arcs from $v$ in $H \uplus H'$ have been removed when we removed the arcs of $C$ and $C'$. Thus every new cycle has length at most $|S|-1$.

We have therefore partitioned the $2\,|S|$ arcs of $H \uplus H'$ into cycles of length at most $|S|-1$. Consequently, the number of these cycles is at least

$$\frac{2\,|S|}{|S|-1} > 2,$$

that is, at least 3 (since it is an integer). This proves the lemma. □

Let $A = (a_{i,j})_{i,j\in[n]}$ be a matrix over a commutative ring $R$. For every integer $r \geq 2$, let $K_{<r}(A)$ denote the ideal of $R$ generated by the $A$-weights of all nontrivial cycles on $[n]$ whose length is strictly smaller than $r$. Thus, $K_{<2}(A) = 0$ (since a nontrivial cycle cannot have length smaller than 2).

**Lemma 4.5.** *Let $S \subseteq [n]$ have size $r \geq 2$, and let $H_1, H_2, \ldots, H_t$ be $t$ distinct Hamilton cycles on $S$, where $t \geq 2$. Then*

$$w_A(H_1)w_A(H_2)\cdots w_A(H_t) \in K_{<r}(A)^t.$$

*Proof.* Pick any $i \neq j$ in $[t]$. By Lemma 4.3, the multiset union of the two distinct Hamilton cycles $H_i$ and $H_j$ on $S$ can be partitioned into at least three cycles of length strictly smaller than $r$. These latter cycles are all nontrivial, since they cannot contain

[8] A multidigraph is said to be *balanced* if for each vertex $v$, the indegree of $v$ equals the outdegree of $v$.

[9] We are using a well-known result saying that the multiset of arcs of a balanced multidigraph can be partitioned into cycles. This can be proved in many ways, e.g.: Start walking at any non-isolated vertex; each time you enter a vertex, the balancedness will ensure that you will be able to exit again; sooner or later you will run into a cycle. Whenever this happens, remove the cycle from the digraph, and repeat the same procedure. Removing a cycle leaves the digraph balanced, so this algorithm will continue until the digraph has no arcs left; at that point, the cycles obtained will form a partition of the set of all arcs.

any loops (indeed, all their arcs must be arcs of the original two Hamilton cycles, but those did not contain any loops). Hence, their $A$-weights belong to $K_{<r}(A)$. Since the product of the $A$-weights is unchanged by repartitioning the same multiset of arcs, this shows that

$$w_A(H_i)w_A(H_j) \in K_{<r}(A)^3. \tag{4.1}$$

Thus, we have proved (4.1) for any $i \neq j$ in $[t]$.

Now, pair off $2\lfloor t/2 \rfloor$ of the $t$ Hamilton cycles $H_1, H_2, \ldots, H_t$. Multiplying the inclusions (4.1) for these pairs, and multiplying by the $A$-weight of the remaining cycle if $t$ is odd, gives

$$w_A(H_1)w_A(H_2)\cdots w_A(H_t) \in K_{<r}(A)^{3\lfloor t/2 \rfloor}.$$

Since $3\lfloor t/2 \rfloor \geq t$ for every $t \geq 2$, the right hand side is contained in $K_{<r}(A)^t$. $\square$

We can now state the universal algebraic result. For any matrix $A = (a_{i,j})_{i,j\in[n]}$ over any commutative ring $R$, we define the *principal-minor-defect ideal* of $A$ to be the ideal

$$J(A) := \left( \det A_S - \prod_{i\in S} a_{i,i} \;\middle|\; S \subseteq [n] \right) \subseteq R$$

(that is, the ideal of $R$ generated by all $2^n$ differences $\det A_S - \prod_{i\in S} a_{i,i}$, where $S$ ranges over the subsets of $[n]$). Note that the generators $\det A_S - \prod_{i\in S} a_{i,i}$ of this ideal for subsets $S$ satisfying $|S| \leq 1$ are zero, since the determinant of a $1\times 1$-matrix or of a $0 \times 0$-matrix is the product of its diagonal entries.

**Theorem 4.6.** *Let $A$ be an $n \times n$-matrix over a commutative ring $R$. Then the $A$-weight of every nontrivial cycle on $[n]$ is integral over $J(A)$. Equivalently, every nontrivial cycle $C$ on $[n]$ satisfies*

$$w_A(C) \in \overline{J(A)}.$$

*Proof.* We must show that each nontrivial cycle $H$ on $[n]$ satisfies $w_A(H) \in \overline{J(A)}$. In other words, we must show that for each $r \geq 2$ and each $r$-element subset $S$ of $[n]$, each Hamilton cycle $H$ on $S$ satisfies $w_A(H) \in \overline{J(A)}$ (since any nontrivial cycle is a Hamilton cycle on a subset of size $\geq 2$).

We use strong induction on $r$. Fix an $r$-element subset $S \subseteq [n]$, where $r \geq 2$, and let $\mathcal{H}_S$ be the set of all Hamilton cycles on $S$. For every $H \in \mathcal{H}_S$, set

$$z_H := (-1)^{r-1} w_A(H).$$

The sign $(-1)^{r-1}$ is the sign of the cyclic permutation corresponding to $H$ (that is, of the permutation of $S$ that sends each vertex of $H$ to the next vertex that follows it on $H$).

Set

$$K := K_{<r}(A) \qquad \text{and} \qquad L := J(A) + K.$$

We first show that each $z_H$ is integral over $L$. Let $e_t$ be the $t$-th elementary symmetric polynomial in the elements $z_H$ for $H \in \mathcal{H}_S$. In particular, $e_0 = 1$ and $e_1 = \sum_{H \in \mathcal{H}_S} z_H$.

We shall now use the determinant expansion (2.1) to show that

$$e_1 \in L. \tag{4.2}$$

*Proof of* (4.2). Each addend on the right-hand side of (2.1) corresponds to some permutation $\pi \in \mathfrak{S}_S$. We can thus classify these addends into three classes:

Class 1 consists of the single addend for the permutation $\pi = \mathrm{id} \in \mathfrak{S}_S$. This addend is

$$\mathrm{sgn}(\mathrm{id}) \prod_{i \in S} a_{i,\mathrm{id}(i)} = \prod_{i \in S} a_{i,i}.$$

Class 2 consists of the addends corresponding to permutations $\pi$ that are $r$-cycles. Such permutations $\pi$ are in a one-to-one correspondence with the Hamilton cycles $H \in \mathcal{H}_S$ (indeed, each such permutation $\pi$ gives rise to the Hamilton cycle $(\pi^0(s), \pi^1(s), \ldots, \pi^{r-1}(s))$ for an arbitrary $s \in S$; the choice of $s$ is immaterial). The addend corresponding to a given Hamilton cycle $H \in \mathcal{H}_S$ is

$$\mathrm{sgn}\,(\pi) \prod_{i \in S} a_{i,\pi(i)} = (-1)^{r-1} \prod_{k=0}^{r-1} a_{\pi^k(s),\pi^{k+1}(s)} = (-1)^{r-1} w_A\,(H) = z_H.$$

Class 3 consists of the addends corresponding to permutations $\pi \neq \mathrm{id}$ that have at least two cycles. For such a permutation $\pi$, all cycles of $\pi$ must have length smaller than $r$ (since a cycle of length $\geq r$ would cover all elements of $S$, thus leaving no room for a second cycle), and at least one of these cycles must be nontrivial (because $\pi \neq \mathrm{id}$). Therefore, the corresponding addend contains the $A$-weight of a nontrivial cycle of length strictly smaller than $r$ as a factor; hence, this addend belongs to $K_{<r}(A) = K \subseteq L$, thus is $\equiv 0 \mod L$.

Hence, reduced modulo $L$, the equality (2.1) becomes

$$\begin{aligned}
\det A_S &\equiv \underbrace{\prod_{i \in S} a_{i,i}}_{\text{sum of addends of Class 1}} + \underbrace{\sum_{H \in \mathcal{H}_S} z_H}_{\text{sum of addends of Class 2}} + \underbrace{0}_{\text{sum of addends of Class 3}} \\
&= \prod_{i \in S} a_{i,i} + \sum_{H \in \mathcal{H}_S} z_H \mod L.
\end{aligned}$$

Therefore,

$$\sum_{H \in \mathcal{H}_S} z_H \equiv \det A_S - \prod_{i \in S} a_{i,i} \equiv 0 \mod L$$

(since the definition of $J(A)$ yields $\det A_S - \prod_{i \in S} a_{i,i} \in J(A) \subseteq L$). That is, $\sum_{H \in \mathcal{H}_S} z_H \in L$. In other words, $e_1 \in L$ (since $e_1 = \sum_{H \in \mathcal{H}_S} z_H$). This proves (4.2). $\square$

Now we shall generalize (4.2) by showing that

$$e_t \in L^t \qquad \text{for each } t \geq 0. \tag{4.3}$$

*Proof of* (4.3). If $t = 0$, then this is obvious (since $L^0 = R$). If $t = 1$, then it follows from (4.2). Thus, assume that $t \geq 2$ henceforth. Now, $e_t$ is defined as the sum of the $t$-wise products of the $z_H$'s with $H \in \mathcal{H}_S$. Each addend in this sum is, up to sign, a product of the $A$-weights of $t$ distinct Hamilton cycles on $S$ (since the $z_H$'s are, up to sign, the $A$-weights of these cycles). But Lemma 4.5 shows that each such product belongs to $K_{<r}(A)^t$. Therefore, their sum $e_t$ belongs to $K_{<r}(A)^t$ as well, and thus also to $L^t$ (since $K_{<r}(A) = K \subseteq L$). This proves (4.3). $\square$

But Viète's formulas show that every $z_H$ is a root of the monic polynomial

$$\prod_{G \in \mathcal{H}_S} (X - z_G) = X^{|\mathcal{H}_S|} - e_1 X^{|\mathcal{H}_S|-1} + e_2 X^{|\mathcal{H}_S|-2} - \cdots \in R[X],$$

whose $X^{|\mathcal{H}_S|-t}$-coefficient belongs to $L^t$ (by (4.3)). Thus every $z_H$ is integral over $L$. Therefore, every $w_A(H)$ is integral over $L$ as well (since $z_H$ is $w_A(H)$ up to sign). In other words,

$$w_A(H) \in \overline{L} \qquad \text{for each } H \in \mathcal{H}_S. \tag{4.4}$$

By the induction hypothesis, the $A$-weights of all nontrivial cycles of length strictly smaller than $r$ belong to $\overline{J(A)}$. Since $\overline{J(A)}$ is an ideal, this yields

$$K \subseteq \overline{J(A)}$$

(since $K = K_{<r}(A)$ is the ideal generated by these $A$-weights). Hence $L = J(A) + K \subseteq \overline{J(A)}$ (since $J(A) \subseteq \overline{J(A)}$ and $K \subseteq \overline{J(A)}$). Thus,

$$J(A) \subseteq L \subseteq \overline{J(A)}.$$

Hence, Lemma 4.2 **(d)** now gives $\overline{L} = \overline{J(A)}$. Thus, (4.4) rewrites as

$$w_A(H) \in \overline{J(A)} \qquad \text{for each } H \in \mathcal{H}_S.$$

In other words, each Hamilton cycle $H$ on $S$ satisfies $w_A(H) \in \overline{J(A)}$. This completes the induction. $\square$

**Corollary 4.7.** *Let $A$ be a principled $n \times n$-matrix over a commutative ring $R$. Then the $A$-weight of every nontrivial cycle on $[n]$ is nilpotent.*

*Proof.* Since $A$ is principled, each subset $S$ of $[n]$ satisfies $\det A_S = \prod_{i \in S} a_{i,i}$ and therefore $\det A_S - \prod_{i \in S} a_{i,i} = 0$. Thus, $J(A) = 0$ (since we have just shown that all generators of $J(A)$ are $0$). Hence, Theorem 4.6 shows that every nontrivial cycle's $A$-weight is integral over the zero ideal, and therefore nilpotent. $\square$

**Corollary 4.8.** *Over a reduced commutative ring, every principled matrix is nullcyclic. Consequently, every power of a principled matrix over a reduced ring is principled.*

*Proof.* A reduced ring has no nonzero nilpotent elements, so the first claim follows from Corollary 4.7. The second then follows from Corollary 2.4. $\square$

**Corollary 4.9.** *Over a reduced commutative ring, every* $1$*-principled matrix is* $1$*-nullcyclic. Consequently, every power of a* $1$*-principled matrix over a reduced ring is* $1$*-principled.*

*Proof.* Every 1-principled matrix is principled (by Proposition 1.5). Hence, the first claim of the corollary follows from the first claim of Corollary 4.8, combined with the fact that every 1-principled matrix is 1-diagonal (see Proposition 1.5). The second claim then follows from Corollary 3.6. $\square$

# 5 Quotients by integrally closed ideals

## 5.1 The general case

The universal theorems from the preceding section have the following immediate consequence.

**Theorem 5.1.** *Let $D$ be a commutative ring, let $I$ be an integrally closed ideal of $D$, and set $R := D/I$. Let $A$ be a principled matrix over $R$. Then $A$ is nullcyclic.*

*Proof.* Choose a matrix $\widetilde{A} = (\widetilde{a}_{i,j})_{i,j\in[n]}$ over $D$ whose image modulo $I$ is $A$. Since $A$ is principled, we have

$$\det \widetilde{A}_S - \prod_{i\in S} \widetilde{a}_{i,i} \in I \qquad \text{for every } S \subseteq [n]\,.$$

Thus, $J(\widetilde{A}) \subseteq I$. By Theorem 4.6, every nontrivial cycle $C$ on $[n]$ satisfies $w_{\widetilde{A}}(C) \in \overline{J(\widetilde{A})} \subseteq \overline{I}$ (by Lemma 4.2 (b), since $J(\widetilde{A}) \subseteq I$) and therefore $w_{\widetilde{A}}(C) \in \overline{I} = I$ (since $I$ is integrally closed). Reducing modulo $I$, we obtain $w_A(C) = 0$ for every nontrivial cycle $C$. Hence $A$ is nullcyclic. $\square$

**Theorem 5.2.** *Let $D$ be a commutative ring, let $I$ be an integrally closed ideal of $D$, and set $R := D/I$. Let $A$ be a* $1$*-principled matrix over $R$. Then $A$ is* $1$*-nullcyclic.*

*Proof.* The matrix $A$ is 1-principled. By Proposition 1.5, this entails that $A$ is 1-diagonal and principled. Hence, Theorem 5.1 shows that $A$ is nullcyclic. Since $A$ is 1-diagonal, we thus conclude that $A$ is 1-nullcyclic. $\square$

*Proof of Theorem 1.2.* By Theorem 5.2, the matrix $A$ is 1-nullcyclic. Hence Corollary 3.6 shows that $A^m$ is 1-principled for every $m \in \mathbb{Z}$. $\square$

*Proof of Theorem 1.9.* By Theorem 5.1, the matrix $A$ is nullcyclic. Hence Corollary 2.4 shows that $A^m$ is principled for every $m \in \mathbb{N}$. $\square$

## 5.2 The rings $\mathbb{Z}/d$

The case of the ring $\mathbb{Z}/d = \mathbb{Z}/d\mathbb{Z}$ admits a particularly elementary application of Theorem 4.6. This relies on the following well-known fact, which we prove here for the sake of completeness:

**Lemma 5.3.** *Every ideal of $\mathbb{Z}$ is integrally closed.*

*Proof.* Every ideal of $\mathbb{Z}$ has the form $d\mathbb{Z}$ for some $d \in \mathbb{N}$. The claim is clear for $d = 0$, since the only nilpotent integer is 0. Assume that $d > 0$, and let $x \in \mathbb{Z}$ be integral over $d\mathbb{Z}$. Thus, for some positive integer $r$, we have

$$x^r + c_1 x^{r-1} + \cdots + c_r x^0 = 0 \qquad \text{with } c_j \in (d\mathbb{Z})^j = d^j\mathbb{Z}. \tag{5.1}$$

Write $c_j = d^j b_j$ with $b_j \in \mathbb{Z}$, and put $y = x/d \in \mathbb{Q}$. Dividing the equation (5.1) by $d^r$ gives

$$y^r + b_1 y^{r-1} + \cdots + b_r y^0 = 0. \tag{5.2}$$

Thus $y$ is a rational root of a monic polynomial in $\mathbb{Z}[X]$. To see directly that $y \in \mathbb{Z}$, write $y = a/b$ in lowest terms with $b > 0$. After multiplication by $b^r$, the equation (5.2) shows that $b \mid a^r$. Since $\gcd(a, b) = 1$, this forces $b = 1$. Hence $y \in \mathbb{Z}$, so $x = dy \in d\mathbb{Z}$. □

**Corollary 5.4.** *Let $d$ be an integer, and let $A$ be a 1-principled matrix over $\mathbb{Z}/d\mathbb{Z}$. Then $A^m$ is 1-principled for every $m \in \mathbb{Z}$.*

*Proof.* The ideal $d\mathbb{Z}$ of $\mathbb{Z}$ is integrally closed by Lemma 5.3. Hence, Theorem 1.2 (applied to $D = \mathbb{Z}$ and $I = d\mathbb{Z}$) shows that $A^m$ is 1-principled for every $m \in \mathbb{Z}$. □

**Remark 5.5.** Our above proof of Corollary 5.4 is self-contained. Indeed, the only result we have used without proof is Lemma 4.2, which in this case is being applied to $R = \mathbb{Z}$; but all claims of this lemma follow from Lemma 5.3 when $R = \mathbb{Z}$.

The reader can easily state and prove an analogue of Corollary 5.4 with "1-principled" replaced by "principled".

## 5.3 Prüfer domains

We next give a broad class of domains for which every ideal is integrally closed.

**Definition 5.6.** Let $D$ be an integral domain with fraction field $K$. A nonzero fractional ideal $L$ of $D$ is called *invertible* if there exists a fractional ideal $M$ such that $LM = D$. The domain $D$ is called a *Prüfer domain* if every nonzero finitely generated ideal of $D$ is invertible.

For background on Prüfer domains and their equivalent characterizations, see [5, §XII.3] and [3].

**Lemma 5.7.** *Every ideal of a Prüfer domain is integrally closed.*

*Proof.* This is part of [5, Theorem XII.3.2], but we give a proof for the sake of completeness.

Let $I$ be an ideal of a Prüfer domain $D$, and let $x \in D$ be integral over $I$. We must show that $x \in I$.

Choose an equation

$$x^r + c_1 x^{r-1} + \cdots + c_r x^0 = 0 \qquad \text{with } c_j \in I^j \tag{5.3}$$

(since $x$ is integral over $I$). Only finitely many elements of $I$ are needed to express all the $c_j$ as sums of products of $j$ elements of $I$. Let $J \subseteq I$ be the finitely generated ideal generated by these elements. Then $c_j \in J^j$ for every $j$, so $x$ is integral over $J$.

Set $L := J + xD$. If $J = 0$, then (5.3) gives $x^r = 0$, whence $x = 0$ since $D$ is a domain; thus the claim is clear. Hence we may assume that $J \neq 0$. Then both $J$ and $L$ are nonzero finitely generated ideals, and therefore invertible. Moreover, (5.3) yields

$$x^r \in JL^{r-1}.$$

Any other product of $r$ generators of $L = J + xD$ already contains a factor from $J$ and thus belongs to $JL^{r-1}$ as well. Hence, $L^r \subseteq JL^{r-1}$. Since the opposite inclusion is obvious, we thus have shown that

$$L^r = JL^{r-1}.$$

Since $L^{r-1}$ is invertible, we may cancel it and obtain $L = J$. In particular, $x \in L = J \subseteq I$. Thus $I$ is integrally closed. □

**Corollary 5.8.** *Let $D$ be a Prüfer domain, let $I$ be an ideal of $D$, and let $A$ be a 1-principled matrix over $D/I$. Then $A^m$ is 1-principled for every $m \in \mathbb{Z}$.*

*Proof.* Lemma 5.7 shows that $I$ is integrally closed; thus, we can apply Theorem 1.2. □

The reader can easily state and prove an analogue of Corollary 5.8 with "1-principled" replaced by "principled".

**Remark 5.9.** Valuation domains are precisely the local Prüfer domains (see [5, §XII.3]). Thus quotients of valuation domains are included in Corollary 5.8.

## 5.4 Normal domains

Recall that a normal domain is a domain that is integrally closed in its fraction field. Every principal ideal of a normal domain is integrally closed; see [6, Proposition 1.5.2]. Therefore Theorem 1.2 yields the following.

**Corollary 5.10.** *Let $D$ be a normal domain, let $f \in D$ be a nonzero nonunit, and let $A$ be a 1-principled matrix over $D/fD$. Then $A^m$ is 1-principled for every $m \in \mathbb{Z}$.*

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