Shuffles in the symmetric group algebra

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/ne2025.pdf

papers: arXiv:2503.17580 arXiv:2212.06274
arXiv:2309.05340 arXiv:2508.00752

Chapter 1

Introduction

References:

- Bruce Sagan, *The symmetric group*, 2nd edition 2001.
- Pavel Etingof et al., *Introduction to representation theory*, AMS 2011, §§5.12–5.17.
- Murray Bremner, Sara Madariaga, Luiz A. Peresi, *Structure theory for the group algebra of the symmetric group, ...*, Commentationes Mathematicae Universitatis Carolinae, 2016.
- Daniel Edwin Rutherford, *Substitutional Analysis*, Edinburgh 1948.
- Darij Grinberg, *An introduction to the symmetric group algebra*, arXiv:2507.20706.

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* Let G be a finite group. (We will only use symmetric groups.)
* Let $\mathbf{k}[G]$ be the group algebra of G over $\mathbf{k}$. Its elements are formal $\mathbf{k}$-linear combinations of elements of G. The multiplication is inherited from G and extended bilinearly.

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  - **Example:** Let G be the symmetric group S_3 on the set $\{1,2,3\}$. For $i \in \{1,2\}$, let $s_i \in S_3$ be the simple transposition that swaps i with $i+1$. Then, in $\mathbf{k}[G] = \mathbf{k}[S_3]$, we have

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(since $s_1^2 = 1$);

$$\begin{aligned}(1 + s_2)(1 + s_1 + s_1 s_2) &= 1 + s_2 + s_1 + s_2 s_1 + s_1 s_2 + s_2 s_1 s_2 \\ &= \sum_{w \in S_3} w.\end{aligned}$$

Finite group algebras: $L(a)$ and $R(a)$

- For each $a \in \mathbf{k}[G]$, we define two $\mathbf{k}$-linear maps

$$L(a) : \mathbf{k}[G] \to \mathbf{k}[G], \qquad x \mapsto ax \qquad (\text{“left multiplication by } a\text{”})$$

and

$$R(a) : \mathbf{k}[G] \to \mathbf{k}[G], \qquad x \mapsto xa \qquad (\text{“right multiplication by } a\text{”}).$$

(So $L(a)(x) = ax$ and $R(a)(x) = xa$.)

Note: The symbol ∗ denotes important points.

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- Both $L(a)$ and $R(a)$ are endomorphisms of the free $\mathbf{k}$-module $\mathbf{k}[G]$. Thus, they can be viewed as $|G| \times |G|$-matrices.
- Hence, $L(a)$ and $R(a)$ are “matrix proxies” for a, allowing to apply linear algebra to studying a.
  (The reason this works is that the maps $a \mapsto L(a)$ and $a \mapsto (R(a))^T$ are two injective $\mathbf{k}$-algebra morphisms from $\mathbf{k}[G]$ to the matrix ring $\operatorname{End}_{\mathbf{k}}(\mathbf{k}[G]) \cong \mathbf{k}^{|G|\times|G|}$.)

* Each $a \in \mathbf{k}[G]$ has a *minimal polynomial*, i.e., a minimum-degree monic polynomial $P \in \mathbf{k}[X]$ such that $P(a) = 0$. It is unique when $\mathbf{k}$ is a field.
  The minimal polynomial of a is also the minimal polynomial of the endomorphisms $L(a)$ and $R(a)$.
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  The minimal polynomial of a is also the minimal polynomial of the endomorphisms $L(a)$ and $R(a)$.
- When $\mathbf{k}$ is a field, we can also study the eigenvectors and eigenvalues of $L(a)$ and $R(a)$.
- **Theorem 1.1.** Assume that $\mathbf{k}$ is a field. Let $a \in \mathbf{k}[G]$. Then, the two linear endomorphisms $L(a)$ and $R(a)$ are conjugate in $\mathrm{End}_{\mathbf{k}}(\mathbf{k}[G])$ (that is, similar as matrices).
  (Thus, they have the same eigenstructure.)
- This is surprisingly nontrivial!

* The *antipode* of the group algebra $\mathbf{k}[G]$ is defined to be the $\mathbf{k}$-linear map

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- **Proposition 1.2.** The antipode S is an involution:

$$a^{**} = a \qquad \text{for all } a \in \mathbf{k}[G],$$

  and a $\mathbf{k}$-algebra anti-automorphism:

$$(ab)^* = b^* a^* \qquad \text{for all } a, b \in \mathbf{k}[G].$$

- **Lemma 1.3.** Assume that $\mathbf{k}$ is a field. Let $a \in \mathbf{k}[G]$. Then, $L(a) \sim L(a^*)$ in $\operatorname{End}_{\mathbf{k}}(\mathbf{k}[G])$.
- *Proof:* Consider the standard basis $(g)_{g \in G}$ of $\mathbf{k}[G]$. The matrices representing the endomorphisms $L(a)$ and $L(a^*)$ in this basis are mutual transposes. But the Taussky–Zassenhaus theorem says that over a field, each matrix A is similar to its transpose A^T.

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- **Lemma 1.4.** Let $a \in \mathbf{k}[G]$. Then, $L(a^*) \sim R(a)$ in $\operatorname{End}_{\mathbf{k}}(\mathbf{k}[G])$.
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- *Proof:* We have $R(a) = S \circ L(a^*) \circ S$ and $S = S^{-1}$.
- *Proof of Theorem 1.1:* Combine Lemma 1.3 with Lemma 1.4.
- **Remark (Martin Lorenz).** Theorem 1.1 generalizes to arbitrary finite-dimensional Frobenius algebras.

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- Now, fix a positive integer n, and let S_n be the *n-th symmetric group*, i.e., the group of permutations of the set $[n]$.
  Multiplication in S_n is composition:

$$(\alpha\beta)(i) = (\alpha \circ \beta)(i) = \alpha(\beta(i)) \qquad \text{for all } \alpha, \beta \in S_n \text{ and } i \in [n].$$

(**Warning:** SageMath has a different opinion!)

- What can we say about the group algebra $\mathbf{k}[S_n]$ that doesn't hold for arbitrary $\mathbf{k}[G]$?
- There is a classical theory ("Young's seminormal form") of the structure of $\mathbf{k}[S_n]$ when $\mathbf{k}$ has characteristic 0. See:
  - Murray Bremner, Sara Madariaga, Luiz A. Peresi, *Structure theory for the group algebra of the symmetric group, ...*, Commentationes Mathematicae Universitatis Carolinae, 2016. (Quick and to the point.)
  - Daniel Edwin Rutherford, *Substitutional Analysis*, Edinburgh 1948. (Dated but careful and quite readable; perhaps the best treatment.)
  - Adriano M. Garsia, Ömer Egecioglu, *Lectures in Algebraic Combinatorics*, Springer 2020. (Messy but full of interesting things.)

- What can we say about the group algebra $\mathbf{k}[S_n]$ that doesn't hold for arbitrary $\mathbf{k}[G]$?
- **Theorem 2.1 (Artin–Wedderburn–Young).** If $\mathbf{k}$ is a field of characteristic 0, then

$$\mathbf{k}[S_n] \cong \prod_{\lambda \text{ is a partition of } n} \underbrace{\mathrm{M}_{f^\lambda}(\mathbf{k})}_{\text{matrix ring}} \qquad (\text{as } \mathbf{k}\text{-algebras}),$$

  where f^λ is the number of standard Young tableaux of shape λ.
- *Proof:* This follows from Young's seminormal form. For the shortest readable proof, see Theorem 1.45 in Bremner/Madariaga/Peresi.
  Or, for a different proof, see my *introduction to the symmetric group algebra* (§5.14).

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- The structure of $\mathbf{k}[S_n]$ for $0 < \operatorname{char} \mathbf{k} \leq n$ is far less straightforward. See, e.g.,
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- **Remark.** If $\mathbf{k}$ is a field of characteristic 0, then each $a \in \mathbf{k}[S_n]$ satisfies $a \sim a^*$ in $\mathbf{k}[S_n]$.
  But not for general $\mathbf{k}$.
- From now on, we shall focus on concrete elements in $\mathbf{k}[S_n]$.

* For any distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, let $\mathrm{cyc}_{i_1,i_2,\ldots,i_k}$ be the permutation in S_n that cyclically permutes $i_1 \mapsto i_2 \mapsto i_3 \mapsto \cdots \mapsto i_k \mapsto i_1$ and leaves all other elements of $[n]$ unchanged.
* **Note.** We have $\mathrm{cyc}_i = \mathrm{id}$, whereas $\mathrm{cyc}_{i,j}$ is the transposition $t_{i,j}$.

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  - **Note.** We have $\operatorname{cyc}_i = \operatorname{id}$, whereas $\operatorname{cyc}_{i,j}$ is the transposition $t_{i,j}$.
- For each $k \in [n]$, we define the *k-th Young–Jucys–Murphy (YJM) element*

$$J_k := \operatorname{cyc}_{1,k} + \operatorname{cyc}_{2,k} + \cdots + \operatorname{cyc}_{k-1,k} \in \mathbf{k}\left[S_n\right].$$

  - **Note.** We have $J_1 = 0$. Also, $J_k^* = J_k$ for each $k \in [n]$.

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  - **Note.** We have $J_1 = 0$. Also, $J_k^* = J_k$ for each $k \in [n]$.
- **Theorem 3.1.** The YJM elements $J_1, J_2, \ldots, J_n$ commute: We have $J_i J_j = J_j J_i$ for all i, j.
  - *Proof:* Easy computational exercise.

* **Theorem 3.2.** The minimal polynomial of J_k over $\mathbb{Q}$ divides

$$\prod_{i=-k+1}^{k-1} (X - i) = (X - k + 1)(X - k + 2) \cdots (X + k - 1).$$

(For $k \leq 3$, some factors here are redundant.)

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- *First proof:* Study the action of J_k on each Specht module (simple S_n-module). See, e.g., G. E. Murphy, *A New Construction of Young's Seminormal Representation ...*, 1981 for details.
- *Second proof (Igor Makhlin):* Some linear algebra does the trick. Induct on k using the facts that J_k and J_{k+1} are simultaneously diagonalizable over $\mathbb{C}$ (since they are symmetric as real matrices and commute) and satisfy $s_k J_{k+1} = J_k s_k + 1$, where $s_k := \text{cyc}_{k,k+1}$. See `https://mathoverflow.net/a/83493/` for details.

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- **Theorem 3.3.** Assume that $\mathbf{k}$ is a field of characteristic 0. Then, there exists a basis $(e_{S,T})$ of $\mathbf{k}[S_n]$ indexed by pairs of standard Young tableaux of the same (partition) shape called the *seminormal basis*. This basis has the property that

$$J_k e_{S,T} = c_S(k) \cdot e_{S,T},$$

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$$J_k e_{S,T} = c_S(k) \cdot e_{S,T},$$

where $c_S(k) = j - i$ if the number k lies in cell (i, j) of S.
- Moreover, each Specht module $\mathcal{S}^\lambda$ (= irreducible representation of S_n) is spanned by part of the seminormal basis, and thus we find the eigenvalues of J_k on that $\mathcal{S}^\lambda$.

The YJM elements: Eigenvalues

* **Theorem 3.2.** The minimal polynomial of J_k over $\mathbb{Q}$ divides

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- Thus, the eigenvalues of J_k are $-k+1, -k+2, \ldots, k-1$ (except for 0 when $k \leq 3$). Their multiplicities can be computed in terms of standard Young tableaux. Even better:
- The seminormal basis exists only for char $\mathbf{k} = 0$ (or, more generally, when $n!$ is invertible in $\mathbf{k}$).
  But Theorem 3.2 and the algebraic multiplicities transfer automatically to all rings $\mathbf{k}$.
- **Question.** Is there a self-contained algebraic/combinatorial proof of Theorem 3.2 without linear algebra or representation theory? (Asked on MathOverflow: `https://mathoverflow.net/questions/420318/` .)

- **Theorem 3.4.** For each $k \in \mathbb{N}$, we can evaluate the k-th elementary symmetric polynomial e_k at the YJM elements $J_1, J_2, \ldots, J_n$ to obtain

$$e_k(J_1, J_2, \ldots, J_n) = \sum_{\substack{\sigma \in S_n; \\ \sigma \text{ has exactly } n-k \text{ cycles}}} \sigma.$$

- *Proof:* Nice homework exercise (once stripped of the algebra).

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- *Proof:* Nice homework exercise (once stripped of the algebra).
- There are formulas for other symmetric polynomials applied to $J_1, J_2, \ldots, J_n$ (see Garsia/Egecioglu).
  There is also a general fact:

- **Theorem 3.5 (Murphy).**

$$\{f(J_1, J_2, \ldots, J_n) \mid f \in \mathbf{k}[X_1, X_2, \ldots, X_n] \text{ symmetric}\}$$
$$= (\text{center of the group algebra } \mathbf{k}[S_n]).$$

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- *Proof:* See any of:
  - Gadi Moran, *The center of $\mathbb{Z}[S_{n+1}]$ ...*, 1992.
  - G. E. Murphy, *The Idempotents of the Symmetric Group ...*, 1983, Theorem 1.9 (for the case $\mathbf{k} = \mathbb{Z}$, but the general case easily follows).
  - Ceccherini-Silberstein/Scarabotti/Tolli, *Representation Theory of the Symmetric Groups*, 2010, Theorem 4.4.5 (for the case $\mathbf{k} = \mathbb{Q}$, but the proof is easily adapted to all $\mathbf{k}$).
    This book also has more on the $J_1, J_2, \ldots, J_n$ (but mind the errata).

The card shuffling point of view

- Permutations are often visualized as shuffled decks of cards: Imagine a deck of cards labeled $1, 2, \ldots, n$.
  A permutation $\sigma \in S_n$ corresponds to the *state* in which the cards are arranged $\sigma(1), \sigma(2), \ldots, \sigma(n)$ from top to bottom.

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  A permutation $\sigma \in S_n$ corresponds to the *state* in which the cards are arranged $\sigma(1), \sigma(2), \ldots, \sigma(n)$ from top to bottom.
- A *random state* is an element $\sum\limits_{\sigma \in S_n} a_\sigma \sigma$ of $\mathbb{R}[S_n]$ whose coefficients $a_\sigma \in \mathbb{R}$ are nonnegative and add up to 1. This is interpreted as a distribution on the $n!$ possible states, where a_σ is the probability for the deck to be in state σ.

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- We drop the "add up to 1" condition, and only require that $\sum\limits_{\sigma \in S_n} a_\sigma > 0$. The probabilities must then be divided by $\sum\limits_{\sigma \in S_n} a_\sigma$.

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- We drop the "add up to 1" condition, and only require that $\sum\limits_{\sigma \in S_n} a_\sigma > 0$. The probabilities must then be divided by $\sum\limits_{\sigma \in S_n} a_\sigma$.
- For instance, $1 + \operatorname{cyc}_{1,2,3}$ corresponds to the random state in which the deck is sorted as $1, 2, 3$ with probability $\dfrac{1}{2}$ and sorted as $2, 3, 1$ with probability $\dfrac{1}{2}$.

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- An $\mathbb{R}$-vector space endomorphism of $\mathbb{R}[S_n]$, such as $L(a)$ or $R(a)$ for some $a \in \mathbb{R}[S_n]$, acts as a *(random) shuffle*, i.e., a transformation of random states. This is just the standard way how Markov chains are constructed from transition matrices.

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- An $\mathbb{R}$-vector space endomorphism of $\mathbb{R}[S_n]$, such as $L(a)$ or $R(a)$ for some $a \in \mathbb{R}[S_n]$, acts as a *(random) shuffle*, i.e., a transformation of random states. This is just the standard way how Markov chains are constructed from transition matrices.
- For example, if $k > 1$, then the right multiplication $R(J_k)$ by the YJM element J_k corresponds to swapping the k-th card with some card above it (chosen uniformly at random).

- Permutations are often visualized as shuffled decks of cards: Imagine a deck of cards labeled $1, 2, \ldots, n$.
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- For example, if $k > 1$, then the right multiplication $R(J_k)$ by the YJM element J_k corresponds to swapping the k-th card with some card above it (chosen uniformly at random).
- Transposing such a matrix means time-reversing the random shuffle.

- Another family of elements of $\mathbf{k}[S_n]$ are the *k-bottom-to-random shuffles*

$$\mathcal{B}_{n,k} := \sum_{\substack{\sigma \in S_n; \\ \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n-k)}} \sigma$$

defined for all $k \in \{0, 1, \ldots, n\}$. Thus,

$$\mathcal{B}_{n,n} = \mathcal{B}_{n,n-1} = \sum_{\sigma \in S_n} \sigma;$$

$$\mathcal{B}_{n,1} = \sum_{i=1}^{n} \mathrm{cyc}_{n,n-1,\ldots,i};$$

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- As a random shuffle, $\mathcal{B}_{n,k}$ (to be precise, $R\left(\mathcal{B}_{n,k}\right)$) takes the bottom k cards and moves them to random positions. Its antipode $\mathcal{B}_{n,k}^*$ takes k random cards and moves them to the bottom positions.

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- $\mathcal{B}_n := \mathcal{B}_{n,1}$ is known as the *bottom-to-random shuffle* or the *Tsetlin library*.

- **Theorem 5.1 (Diaconis, Fill, Pitman).** We have

$$\mathcal{B}_{n,k+1} = (\mathcal{B}_n - k)\,\mathcal{B}_{n,k} \qquad \text{for each } k \in \{0, 1, \ldots, n-1\}.$$

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- These are not hard to prove in this order. See https://mathoverflow.net/questions/308536 for the details.

- More can be said: in particular, the multiplicities of the eigenvalues $0, 1, \ldots, n-2, n$ of $R(\mathcal{B}_n)$ over $\mathbb{Q}$ are known.

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  of $\mathcal{B}_{n,k}$ are known as the *k-random-to-bottom shuffles* and have the same properties (since S is an algebra anti-automorphism).
- Moreover, there are *top-to-random* and *random-to-top* shuffles defined in the same way but with renaming $1, 2, \ldots, n$ as $n, n-1, \ldots, 1$. They are just images of the $\mathcal{B}_{n,k}$ and $\mathcal{B}^*_{n,k}$ under the automorphism $a \mapsto w_0 a w_0^{-1}$ of $\mathbf{k}[S_n]$, where w_0 is the permutation with one-line notation $(n, n-1, \ldots, 1)$. Thus, top vs. bottom is mainly a matter of notation.

- Main references:
  - Nolan R. Wallach, *Lie Algebra Cohomology and Holomorphic Continuation of Generalized Jacquet Integrals*, 1988, Appendix.
  - Persi Diaconis, James Allen Fill and Jim Pitman, *Analysis of Top to Random Shuffles*, 1992.

Chapter 2

Random-to-random shuffles

References:

- Victor Reiner, Franco Saliola, Volkmar Welker, *Spectra of Symmetrized Shuffling Operators*, arXiv:1102.2460.
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- Here is a further family. For each $k \in \{0, 1, \ldots, n\}$, we let

$$\mathcal{R}_{n,k} := \sum_{\sigma \in S_n} \operatorname{noninv}_{n-k}(\sigma) \cdot \sigma,$$

where $\operatorname{noninv}_{n-k}(\sigma)$ denotes the number of $(n-k)$-element subsets of $[n]$ on which σ is increasing. This is called the *k-random-to-random shuffle*.

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- **Example:** Writing permutations in one-line notation,

$$\begin{aligned}\mathcal{R}_{4,2} = 6[1,2,3,4] &+ 5[1,2,4,3] + 5[1,3,2,4] + 4[1,3,4,2] \\ &+ 4[1,4,2,3] + 3[1,4,3,2] + 5[2,1,3,4] + 4[2,1,4,3] \\ &+ 4[2,3,1,4] + 3[2,3,4,1] + 3[2,4,1,3] + 2[2,4,3,1] \\ &+ 4[3,1,2,4] + 3[3,1,4,2] + 3[3,2,1,4] + 2[3,2,4,1] \\ &+ 2[3,4,1,2] + [3,4,2,1] + 3[4,1,2,3] + 2[4,1,3,2] \\ &+ 2[4,2,1,3] + [4,2,3,1] + [4,3,1,2].\end{aligned}$$

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- **Note:** $\mathcal{R}_{n,0} = \operatorname{id}$ and $\mathcal{R}_{n,n-1} = n \sum_{\sigma \in S_n} \sigma$ and $\mathcal{R}_{n,n} = \sum_{\sigma \in S_n} \sigma$.

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- The card-shuffling interpretation of $\mathcal{R}_{n,k}$ is "pick any k cards from the deck and move them to k randomly chosen positions".

* **Theorem 6.1 (Reiner, Saliola, Welker).** The $n+1$ elements $\mathcal{R}_{n,0}, \mathcal{R}_{n,1}, \ldots, \mathcal{R}_{n,n}$ commute (but are not polynomials in $\mathcal{R}_{n,1}$ in general).

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- **Theorem 6.2 (Dieker, Saliola, Lafrenière).** The minimal polynomial of each $\mathcal{R}_{n,k}$ over $\mathbb{Q}$ is a product of $X-i$'s for distinct integers i. For example, the one of $\mathcal{R}_{n,1}$ divides

$$\prod_{i=0}^{n^2} (X-i)\,.$$

The exact factors can be given in terms of certain statistics on Young diagrams.

- Main references: the “classics”
  - Victor Reiner, Franco Saliola, Volkmar Welker, *Spectra of Symmetrized Shuffling Operators*, arXiv:1102.2460.
  - A.B. Dieker, F.V. Saliola, *Spectral analysis of random-to-random Markov chains*, 2018.
  - Nadia Lafrenière, *Valeurs propres des opérateurs de mélanges symétrisés*, thesis, 2019.

and the two recent preprints

  - Ilani Axelrod-Freed, Sarah Brauner, Judy Hsin-Hui Chiang, Patricia Commins, Veronica Lang, *Spectrum of random-to-random shuffling in the Hecke algebra*, arXiv:2407.08644.
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- The “classical” proofs are complicated, technical and long. In this talk, I will outline some parts of the two recent preprints, including a simpler proof of Theorem 6.1 and most of Theorem 6.2. (The full proof of Theorem 6.2 is still long and hard.)

- The first step is a formula that is easy to prove combinatorially:

* **Proposition 6.3.** For each $k \in \{0, 1, \ldots, n\}$, we have

$$\mathcal{R}_{n,k} = \frac{1}{k!} \cdot \mathcal{B}^*_{n,k} \, \mathcal{B}_{n,k}.$$

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$$\mathcal{R}_{n,k} = \frac{1}{k!} \cdot \mathcal{B}^*_{n,k} \, \mathcal{B}_{n,k}.$$

- However, the $\mathcal{B}_{n,k}$ do not commute with the $\mathcal{B}^*_{n,k}$, so this is not by itself an answer.

The Hecke algebra, 1

- Let $q \in \mathbf{k}$ be a parameter.
  The n-th *Hecke algebra* (or *Iwahori–Hecke algebra*) is a q-deformation of the group algebra $\mathbf{k}[S_n]$.
  It has generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$\begin{aligned}
T_i^2 &= (q-1)\, T_i + q && \text{for all } i \in [n-1]\,; \\
T_i T_j &= T_j T_i && \text{whenever } |i-j| > 1; \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for all } i \in [n-2]\,.
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  We call this algebra $\mathcal{H}_n$.
- For $q = 1$, this is the group algebra $\mathbf{k}[S_n]$ (and the generator T_i is the simple transposition $s_i = \operatorname{cyc}_{i,i+1}$).
- For general q, it still is a free $\mathbf{k}$-module of rank $n!$, with a basis $(T_w)_{w \in S_n}$ indexed by permutations $w \in S_n$. The basis vectors are defined by $T_w := T_{i_1} T_{i_2} \cdots T_{i_k}$, where $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for w. For $q = 1$, this T_w is just w.

- Much of the theory of $\mathbf{k}[S_n]$ exists in a subtler form for $\mathcal{H}_n$. Sometimes, the added difficulty brings the best proofs to light.

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Main change: The random-to-random shuffle must now be **defined** as

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  Noninversions no longer work!
  But we will stick to the $q = 1$ case in this talk.

* **Theorem 8.1 (Brauner–Commins–G.–Saliola 2025, based on Axelrod-Freed–Brauner–Chiang–Commins–Lang 2024).** For any $1 \le k \le n$, we have

$$\mathcal{B}_n \, \mathcal{R}_{n,k} = \underbrace{\left(\mathcal{R}_{n-1,k} + \left((n+1-k) + J_n\right) \mathcal{R}_{n-1,k-1}\right)}_{=:\mathcal{W}_{n,k}} \mathcal{B}_n.$$

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- The proof takes about 5 pages, relying on some more elementary computations from prior work (ca. 10–15 pages in total).
- This recursion does not actually compute $\mathcal{R}_{n,k}$. But it says enough about $\mathcal{R}_{n,k}$ to carry our proofs.
- Note also that $\mathcal{R}_{n,k} \in \mathcal{B}_n^* \mathbf{k}[S_n]$ by its definition (when $k \geq 1$). This makes the recursion so useful.

Commutativity of random-to-random

- Theorem 8.1 leads fairly easily to a proof of commutativity (Theorem 6.1).
  Indeed, inducting on n, we observe that the $\mathcal{W}_{n,k}$s all commute by the induction hypothesis (and the easy fact that J_n commutes with everything in $\mathbf{k}[S_{n-1}]$). Thus, using $\mathcal{B}_n \mathcal{R}_{n,k} = \mathcal{W}_{n,k} \mathcal{B}_n$, we find

  $$\begin{aligned}\mathcal{B}_n \mathcal{R}_{n,i} \mathcal{R}_{n,j} &= \mathcal{W}_{n,i} \mathcal{B}_n \mathcal{R}_{n,j} = \mathcal{W}_{n,i} \mathcal{W}_{n,j} \mathcal{B}_n \\ &= \mathcal{W}_{n,j} \mathcal{W}_{n,i} \mathcal{B}_n = \mathcal{W}_{n,j} \mathcal{B}_n \mathcal{R}_{n,i} = \mathcal{B}_n \mathcal{R}_{n,j} \mathcal{R}_{n,i}.\end{aligned}$$

  Remains to get rid of the $\mathcal{B}_n$ factor at the front. Recall that all $\mathcal{R}_{n,i}$ (except for the trivial $\mathcal{R}_{n,0}$) lie in $\mathcal{B}_n^* \mathbf{k}[S_n]$. But we can WLOG assume that $\mathbf{k} = \mathbb{Q}$, and then the equality $\mathcal{B}_n \mathcal{B}_n^* a = 0$ entails $\mathcal{B}_n^* a = 0$ (positivity trick! cf. linear algebra: $\operatorname{Ker}(A^T A) = \operatorname{Ker} A$ for a real matrix A).
- Alternatively, the trick can be avoided (see `arXiv:2503.17580`).

The approach to eigenvalues, 1

- Now to Theorem 6.2:
  The eigenvalues of $\mathcal{R}_{n,k}$ are **nonnegative reals**, since $\mathcal{R}_{n,k}$ is represented by a positive semidefinite symmetric matrix (Proposition 6.3).
  But why are they **integers**?

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- An element a of a $\mathbf{k}$-algebra A is said to be *split* (over $\mathbf{k}$) if there exist some scalars $u_1, u_2, \ldots, u_n \in \mathbf{k}$ (not necessarily distinct) such that $\prod_{i=1}^{n} (a - u_i) = 0$.

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- $*$ When $\mathbf{k}$ is an integral domain and A is a free $\mathbf{k}$-module of finite rank, this is the same as saying that $R(a)$ has all eigenvalues in $\mathbf{k}$.
- In particular, for $\mathbf{k} = \mathbb{Z}$ and $A = \mathbf{k}[S_n]$, this means that all eigenvalues of $R(a)$ are $\in \mathbb{Z}$. This is what we want to show for $a = \mathcal{R}_{n,k}$.

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- We have a theory of "split elements" that can help answer such questions in general. Here is an outline:

* An element a of a $\mathbf{k}$-algebra A is said to be *split* (over $\mathbf{k}$) if there exist some scalars $u_1, u_2, \ldots, u_n \in \mathbf{k}$ (not necessarily distinct) such that $\prod_{i=1}^{n} (a - u_i) = 0$.

* When $\mathbf{k}$ is an integral domain and A is a free $\mathbf{k}$-module of finite rank, this is the same as saying that $R(a)$ has all eigenvalues in $\mathbf{k}$.

- In particular, for $\mathbf{k} = \mathbb{Z}$ and $A = \mathbf{k}[S_n]$, this means that all eigenvalues of $R(a)$ are $\in \mathbb{Z}$. This is what we want to show for $a = \mathcal{R}_{n,k}$.
- So we must show that $\mathcal{R}_{n,k}$ is split over $\mathbb{Z}$.

General theory of split elements, 1

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- **Theorem 9.3.** If b, c, f are elements of A such that f is split and such that $bc = fb$ and $c \in Ab$, then c is split.
- Theorem 9.3 is tailored to our use:

$bc = fb$	$c \in Ab$
$\mathcal{B}_n \mathcal{R}_{n,k} = \mathcal{W}_{n,k} \mathcal{B}_n$	$\mathcal{R}_{n,k} \in \mathbf{k}[S_n]\, \mathcal{B}_n$

.

The splitness of $\mathcal{W}_{n,k}$ follows from the splitness of the commuting elements J_n, $\mathcal{R}_{n-1,k-1}$ and $\mathcal{R}_{n-1,k}$ (induction!) by Corollary 9.2. We need the splitness of the YJM elements, which was proved (e.g.) by Murphy.

- Theorem 9.3 looks baroque, but in fact it easily decomposes into two particular cases:
  Corollary 9.4. If ba is split, then ab is also split.
  Corollary 9.5. If a is split and $b^2 = ab$, then b is split.
  (Both times, $a, b \in A$ are arbitrary.)

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Formulas for eigenvalues, 1

- The splitness theory proves easily that all eigenvalues of $\mathcal{R}_{n,k}$ are integers, but it does not compute them explicitly. Indeed, it produces "phantom eigenvalues" which do not actually appear.
- With a lot more work (Specht modules, seminormal basis, Pieri rule, etc.), we have been able to compute the eigenvalues with their multiplicities fully.
- I only have time to state the main result.

Formulas for eigenvalues, 2

- **Theorem 10.1.** Let $n, k \geq 0$. The eigenvalues of $R(\mathcal{R}_{n,k})$ on $\mathbf{k}[S_n]$ are the elements

$$\mathcal{E}_{\lambda\setminus\mu}(k) := \sum_{j<(\ell_1<\ell_2<\cdots<\ell_k)\leq n} \prod_{m=1}^{k} \left(\ell_m + 1 - m + \mathsf{c}_{\mathfrak{t}^{\lambda\setminus\mu}}(\ell_m)\right)$$

for all horizontal strips $\lambda \setminus \mu$ that satisfy $\lambda \vdash n$ and $d^{\mu} \neq 0$. Here,

  - d^{μ} denotes the number of *desarrangement tableaux* of shape μ (that is, standard tableaux of shape μ whose smallest non-descent is even);
  - j is the size of μ;
  - $\mathfrak{t}^{\lambda\setminus\mu}$ is the skew tableau of shape $\lambda \setminus \mu$ obtained by filling in the boxes of $\lambda \setminus \mu$ with $j+1, j+2, \ldots, n$ from top to bottom;
  - $\mathsf{c}_{\mathfrak{t}^{\lambda\setminus\mu}}(p) = y - x$ if the cell of $\mathfrak{t}^{\lambda\setminus\mu}$ containing the entry p is (x, y).

Moreover, the multiplicity of each such eigenvalue $\mathcal{E}_{\lambda\setminus\mu}(k)$ is $d^{\mu} f^{\lambda}$, where f^{λ} is the number of standard tableaux of shape λ (unless there are collisions).

- We have explicit formulas for specific shapes and strips:

$$\mathcal{E}_{(n)\setminus\varnothing}(k) = k!\binom{n}{k}^2;$$
$$\mathcal{E}_{(n-1,1)\setminus(j,1)}(k) = k!\binom{n-j-1}{k}\binom{n+j}{k} \qquad \text{for all } j \in [n-1].$$

But there is no such nice formula for $\mathcal{E}_{(4,1,1)\setminus(1,1)}(1)$.

- **Question:** Any nicer formulas for the eigenvalues $\mathcal{E}_{\lambda\backslash\mu}(k)$?
- **Question (Reiner):** What is the dimension of the subalgebra of $\mathbb{Q}[S_n]$ generated by $\mathcal{R}_{n,0}, \mathcal{R}_{n,1}, \ldots, \mathcal{R}_{n,n}$?

n	1	2	3	4	5	6	7	8	9	10	11	12
dim (subalgebra)	1	2	4	7	15	30	54	95	159	257	400	613

(sequence not in the OEIS as of 2025-10-06).

The same numbers hold for the q-deformation!

CHAPTER 3

Somewhere-to-below shuffles

References:

- Darij Grinberg, Nadia Lafrenière, *The one-sided cycle shuffles in the symmetric group algebra*, Algebraic Combinatorics (2024), arXiv:2212.06274.
- Darij Grinberg, *Commutator nilpotency for somewhere-to-below shuffles*, arXiv:2309.05340.
- Darij Grinberg, *The representation theory of somewhere-to-below shuffles*, arXiv:2508.00752.

Somewhere-to-below shuffles: introduction

- Now to something completely different...

- Now to something completely different...
- In 2021, Nadia Lafrenière defined the *somewhere-to-below shuffles* $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ by setting

$$\mathbf{t}_\ell := \mathrm{cyc}_\ell + \mathrm{cyc}_{\ell,\ell+1} + \mathrm{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \mathrm{cyc}_{\ell,\ell+1,\ldots,n} \in \mathbf{k}[S_n]$$

for each $\ell \in [n]$.
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- Now to something completely different...
- In 2021, Nadia Lafrenière defined the *somewhere-to-below shuffles* $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ by setting

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for each $\ell \in [n]$.
- Note: $\mathbf{t}_n = \mathrm{id}$.
- As a card shuffle, $\mathbf{t}_\ell$ takes the ℓ-th card from the top and moves it further down the deck.
- $\mathbf{t}_1$ is called the *top-to-random shuffle*. Upon renaming $1, 2, \ldots, n$ as $n, n-1, \ldots, 1$, it becomes $\mathcal{B}_{n,1}$. (So it is conjugate to $\mathcal{B}_{n,1}$ by w_0.)

- $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ do not commute for $n \geq 3$. For $n = 3$, we have

$$[\mathbf{t}_1, \mathbf{t}_2] = \mathrm{cyc}_{1,2} + \mathrm{cyc}_{1,2,3} - \mathrm{cyc}_{1,3,2} - \mathrm{cyc}_{1,3} .$$

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- However, they come pretty close to commuting!

* **Theorem 20.1 (Lafreniere, G., 2022).** There exists a basis of the $\mathbf{k}$-module $\mathbf{k}[S_n]$ in which all of the endomorphisms $R(\mathbf{t}_1), R(\mathbf{t}_2), \ldots, R(\mathbf{t}_n)$ are represented by upper-triangular matrices.

- This basis is not hard to define, but I haven't seen it before.
- For each $w \in S_n$, we let

$$\operatorname{Des} w := \{i \in [n-1] \mid w(i) > w(i+1)\}.$$

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This is called the *descent set* of w.
- For each $i \in [n-1]$, we let $s_i := \operatorname{cyc}_{i,i+1}$.
- For each $I \subseteq [n-1]$, we let

$$G(I) := (\text{the subgroup of } S_n \text{ generated by the } s_i \text{ for } i \in I).$$

This is called a *Young parabolic subgroup* of S_n.

- For each $w \in S_n$, we let

$$\mathbf{a}_w := \sum_{\sigma \in G(\operatorname{Des} w)} w\sigma \in \mathbf{k}[S_n].$$

In other words, $\mathbf{a}_w$ is obtained by breaking up the word w into maximal decreasing factors and re-sorting each factor arbitrarily (without mixing different factors).

The descent-destroying basis, 2

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- The family $\left(\mathbf{a}_w\right)_{w \in S_n}$ is a basis of $\mathbf{k}\left[S_n\right]$ (by triangularity).

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* The family $(\mathbf{a}_w)_{w \in S_n}$ is a basis of $\mathbf{k}[S_n]$ (by triangularity).
* For instance, for $n = 3$, we have

$$\mathbf{a}_{[123]} = [123];$$
$$\mathbf{a}_{[132]} = [132] + [123];$$
$$\mathbf{a}_{[213]} = [213] + [123];$$
$$\mathbf{a}_{[231]} = [231] + [213];$$
$$\mathbf{a}_{[312]} = [312] + [132];$$
$$\mathbf{a}_{[321]} = [321] + [312] + [231] + [213] + [132] + [123].$$

* **Theorem 14.1 (Lafrenière, G.).** For any $w \in S_n$ and $\ell \in [n]$, we have

$$\mathbf{a}_w \mathbf{t}_\ell = \mu_{w,\ell} \mathbf{a}_w + \sum_{\substack{v \in S_n; \\ v \prec w}} \lambda_{w,\ell,v} \mathbf{a}_v$$

for some nonnegative integer $\mu_{w,\ell}$, some integers $\lambda_{w,\ell,v}$ and a certain partial order $\prec$ on S_n.

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for some nonnegative integer $\mu_{w,\ell}$, some integers $\lambda_{w,\ell,v}$ and a certain partial order $\prec$ on S_n.
Thus, the endomorphisms $R(\mathbf{t}_1), R(\mathbf{t}_2), \ldots, R(\mathbf{t}_n)$ are upper-triangular with respect to the basis $(\mathbf{a}_w)_{w \in S_n}$.

- **Example:** For $n = 4$, we have

$$\mathbf{a}_{[4312]}\mathbf{t}_2 = \mathbf{a}_{[4312]} + \underbrace{\mathbf{a}_{[4321]} - \mathbf{a}_{[4231]} - \mathbf{a}_{[3241]} - \mathbf{a}_{[2143]}}_{\text{subscripts are } \prec [4312]} .$$

- **Example:** For $n = 4$, we have

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- **Example:** For $n = 3$, the endomorphism $R(\mathbf{t}_1)$ is represented by the matrix

	$\mathbf{a}_{[321]}$	$\mathbf{a}_{[231]}$	$\mathbf{a}_{[132]}$	$\mathbf{a}_{[213]}$	$\mathbf{a}_{[312]}$	$\mathbf{a}_{[123]}$
$\mathbf{a}_{[321]}$	3	1	1		1	
$\mathbf{a}_{[231]}$				1	-1	1
$\mathbf{a}_{[132]}$				1		
$\mathbf{a}_{[213]}$				1		
$\mathbf{a}_{[312]}$					1	
$\mathbf{a}_{[123]}$						1

(empty cells = zero entries). For instance, the last column means $\mathbf{a}_{[123]}\mathbf{t}_1 = \mathbf{a}_{[123]} + \mathbf{a}_{[231]}$.

- **Corollary 14.2.** The eigenvalues of the endomorphisms $R(\mathbf{t}_1), R(\mathbf{t}_2), \ldots, R(\mathbf{t}_n)$ and of all their linear combinations

$$R(\lambda_1 \mathbf{t}_1 + \lambda_2 \mathbf{t}_2 + \cdots + \lambda_n \mathbf{t}_n)$$

  are integers as long as $\lambda_1, \lambda_2, \ldots, \lambda_n$ are.

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- How many different eigenvalues do they have?
- $R(\mathbf{t}_1) \cong R(\mathcal{B}_{n,1})$ has only n eigenvalues: $0, 1, \ldots, n-2, n$, as we have seen before. The other $R(\mathbf{t}_\ell)$'s have even fewer.

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- How many different eigenvalues do they have?
- $R(\mathbf{t}_1) \cong R(\mathcal{B}_{n,1})$ has only n eigenvalues: $0, 1, \ldots, n-2, n$, as we have seen before. The other $R(\mathbf{t}_\ell)$'s have even fewer.
- But their linear combinations $R(\lambda_1 \mathbf{t}_1 + \lambda_2 \mathbf{t}_2 + \cdots + \lambda_n \mathbf{t}_n)$ can have many more. How many?

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* **Theorem 15.1 (combinatorial interpretation of Fibonacci numbers, folklore).** The number of lacunar subsets of $[n-1]$ is the *Fibonacci number* f_{n+1}.
(Recall: $f_0 = 0, \qquad f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2}.$)

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- **Theorem 15.2.** When $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ are generic, the endomorphism $R(\lambda_1 \mathbf{t}_1 + \lambda_2 \mathbf{t}_2 + \cdots + \lambda_n \mathbf{t}_n)$ is diagonalizable and has f_{n+1} distinct eigenvalues.

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  - Note that $f_{n+1} \ll n!$.

* We prove this by finding a filtration

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n]$$

of the $\mathbf{k}$-module $\mathbf{k}[S_n]$ such that each $R(\mathbf{t}_\ell)$ acts as a **scalar** on each of its quotients F_i / F_{i-1}. In matrix terms, this means bringing $R(\mathbf{t}_\ell)$ to a block-triangular form, with the diagonal blocks being "scalar times I" matrices.

- It is only natural that the quotients should correspond to the lacunar subsets of $[n-1]$.
- Let us approach the construction of this filtration.

The Fibonacci filtration, 2

- For each $I \subseteq [n]$, we set

$$\mathrm{sum}\, I := \sum_{i \in I} i$$

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and

$$\widehat{I} := \{0\} \cup I \cup \{n+1\} \qquad \text{(“enclosure” of } I\text{)}$$

and

$$I' := [n-1] \setminus (I \cup (I-1)) \qquad \text{(“non-shadow” of } I\text{)}$$

and

$$F(I) := \left\{ \mathbf{q} \in \mathbf{k}[S_n] \ \mid \ \mathbf{q} s_i = \mathbf{q} \text{ for all } i \in I' \right\} \subseteq \mathbf{k}[S_n].$$

In probabilistic terms, $F(I)$ consists of those random states of the deck that do not change if we swap the i-th and $(i+1)$-st cards from the top as long as neither i nor $i+1$ is in I. To put it informally: $F(I)$ consists of those random states that are “fully shuffled” between any two consecutive $\widehat{I}$-positions.

The Fibonacci filtration, 3

- **Example:** If $n = 11$ and $I = \{3, 6, 7\}$, then

$$\widehat{I} := \{0\} \cup I \cup \{n+1\} = \{0, 3, 6, 7, 12\}$$

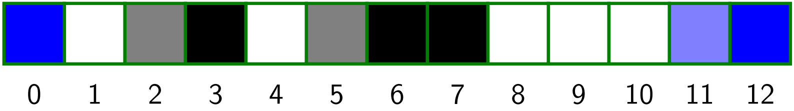
and

$$I' := [n-1] \setminus (I \cup (I-1)) = \{1, 4, 8, 9, 10\}$$

and

$$F(I) = \{\mathbf{q} \in \mathbf{k}[S_{11}] \mid \mathbf{q}s_1 = \mathbf{q}s_4 = \mathbf{q}s_8 = \mathbf{q}s_9 = \mathbf{q}s_{10} = \mathbf{q}\}.$$

Illustrating this:



(black $= I$; grey $= I - 1$; blue $= \widehat{I} \setminus I$;
lightblue $= n$; white $= I'$).

* For any $\ell \in [n]$, we let $m_{I,\ell}$ be the distance from ℓ to the next-higher element of $\widehat{I}$. In other words,

$$m_{I,\ell} := \left(\text{smallest element of } \widehat{I} \text{ that is } \geq \ell\right) - \ell \in \{0, 1, \ldots, n\}.$$

In our above example,

$$(m_{I,1},\ m_{I,2},\ \ldots,\ m_{I,11}) = (2,\ 1,\ 0,\ 2,\ 1,\ 0,\ 0,\ 4,\ 3,\ 2,\ 1).$$

	2	1	0	2	1	0	0	4	3	2	1	0
0	1	2	3	4	5	6	7	8	9	10	11	12

- We note that, for any $\ell \in [n]$, we have the equivalence

$$m_{I,\ell} = 0 \quad \Longleftrightarrow \quad \ell \in \widehat{I} \quad \Longleftrightarrow \quad \ell \in I.$$

* **Crucial Lemma 16.1.** Let $I \subseteq [n]$ and $\ell \in [n]$. Then,

$$\mathbf{q}\mathbf{t}_\ell \in m_{I,\ell}\mathbf{q} + \underbrace{\sum_{\substack{J \subseteq [n]; \\ \operatorname{sum} J < \operatorname{sum} I}} F(J)}_{\substack{\text{Think of these as} \\ \text{"lower-order terms"}}} \qquad \text{for each } \mathbf{q} \in F(I).$$

- *Proof:* Expand $\mathbf{q}\mathbf{t}_\ell$ by the definition of $\mathbf{t}_\ell$, and break up the resulting sum into smaller bunches using the interval decomposition

$$[\ell, n] = [\ell, i_k - 1] \sqcup [i_k, i_{k+1} - 1] \sqcup [i_{k+1}, i_{k+2} - 1] \sqcup \cdots \sqcup [i_p, n]$$

(where $i_k < i_{k+1} < \cdots < i_p$ are the elements of I larger or equal to ℓ). The $[\ell, i_k - 1]$ bunch gives the $m_{I,\ell}\mathbf{q}$ term; the others live in appropriate $F(J)$'s.
See arXiv:2212.06274 for the details.

The Fibonacci filtration, 6

- $*$ Thus, we obtain a filtration of $\mathbf{k}[S_n]$ if we label the subsets I of $[n]$ in the order of increasing sum I and add up the respective $F(I)$s.
  On each subquotient of this filtration, $\mathbf{t}_\ell$ acts as a scalar $m_{I,\ell}$.
- Unfortunately, this filtration has 2^n, not f_{n+1} terms.

- ∗ Thus, we obtain a filtration of $\mathbf{k}[S_n]$ if we label the subsets I of $[n]$ in the order of increasing sum I and add up the respective $F(I)$s.
  On each subquotient of this filtration, $\mathbf{t}_\ell$ acts as a scalar $m_{I,\ell}$.
- Unfortunately, this filtration has 2^n, not f_{n+1} terms.
- ∗ Fortunately, that's because many of its terms are redundant. The ones that aren't correspond precisely to the I's that are lacunar subsets of $[n-1]$:
- **Lemma 16.2.** Let $k \in \mathbb{N}$. Then,

$$\sum_{\substack{J \subseteq [n]; \\ \operatorname{sum} J < k}} F(J) = \sum_{\substack{J \subseteq [n-1] \text{ is lacunar;} \\ \operatorname{sum} J < k}} F(J).$$

- *Proof:* If $J \subseteq [n]$ contains n or fails to be lacunar, then $F(J)$ is a submodule of some $F(K)$ with $\operatorname{sum} K < \operatorname{sum} J$. (Exercise!)

- Now, we let $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ be the f_{n+1} lacunar subsets of $[n-1]$, listed in such an order that
$$\operatorname{sum}(Q_1) \leq \operatorname{sum}(Q_2) \leq \cdots \leq \operatorname{sum}\left(Q_{f_{n+1}}\right).$$
Then, for each $i \in [0, f_{n+1}]$, define a $\mathbf{k}$-submodule
$$F_i := F(Q_1) + F(Q_2) + \cdots + F(Q_i) \qquad \text{of } \mathbf{k}[S_n]$$
(so that $F_0 = 0$). The resulting filtration
$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n]$$
(which we call the *Fibonacci filtration* of $\mathbf{k}[S_n]$) satisfies the properties we need:

- **Theorem 16.3.** For each $i \in [f_{n+1}]$ and $\ell \in [n]$, we have
$$F_i \cdot (\mathbf{t}_\ell - m_{Q_i,\ell}) \subseteq F_{i-1}$$
(so that $R(\mathbf{t}_\ell)$ acts on F_i / F_{i-1} as multiplication by $m_{Q_i,\ell}$).
- *Proof:* Lemma 16.1 + Lemma 16.2.

- **Theorem 16.3.** For each $i \in [f_{n+1}]$ and $\ell \in [n]$, we have
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- *Proof:* Lemma 16.1 + Lemma 16.2.
- **Lemma 16.4.** The quotients F_i / F_{i-1} are nontrivial for all $i \in [f_{n+1}]$.
- *Proof:* See below.

- **Theorem 16.3.** For each $i \in [f_{n+1}]$ and $\ell \in [n]$, we have
$$F_i \cdot (\mathbf{t}_\ell - m_{Q_i,\ell}) \subseteq F_{i-1}$$
(so that $R(\mathbf{t}_\ell)$ acts on F_i / F_{i-1} as multiplication by $m_{Q_i,\ell}$).
- *Proof:* Lemma 16.1 + Lemma 16.2.
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- *Proof:* See below.
- **Corollary 16.5.** Let $\mathbf{k}$ be a field, and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$. Then, the eigenvalues of $R(\lambda_1 \mathbf{t}_1 + \lambda_2 \mathbf{t}_2 + \cdots + \lambda_n \mathbf{t}_n)$ are the linear combinations
$$\lambda_1 m_{I,1} + \lambda_2 m_{I,2} + \cdots + \lambda_n m_{I,n} \qquad \text{for } I \subseteq [n-1] \text{ lacunar.}$$

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- Theorem 15.2 easily follows by some linear algebra.
- More generally, this holds not just for linear combinations $\lambda_1 \mathbf{t}_1 + \lambda_2 \mathbf{t}_2 + \cdots + \lambda_n \mathbf{t}_n$ but for any noncommutative polynomials in $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$.

- The descent-destroying basis $(\mathbf{a}_w)_{w \in S_n}$ is compatible with our filtration:

* **Theorem 17.1.** For each $I \subseteq [n]$, the family $(\mathbf{a}_w)_{w \in S_n;\ I' \subseteq \operatorname{Des} w}$ is a basis of the $\mathbf{k}$-module $F(I)$.

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* If $w \in S_n$ is any permutation, then the *Q-index* of w is defined to be the **smallest** $i \in [f_{n+1}]$ such that $Q_i' \subseteq \operatorname{Des} w$. We call this Q-index $\operatorname{Qind} w$.

- **Proposition 17.2.** Let $w \in S_n$ and $i \in [f_{n+1}]$. Then, $\operatorname{Qind} w = i$ if and only if $Q_i' \subseteq \operatorname{Des} w \subseteq [n-1] \setminus Q_i$.

- **Note:** The numbering $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ of the lacunar subsets of $[n-1]$ is not unique; we just picked one. The Q-index $i = \operatorname{Qind} w$ of a $w \in S_n$ depends on this numbering. However, the corresponding lacunar set Q_i does not, since Proposition 17.2 determines it canonically (it is the unique lacunar $L \subseteq [n-1]$ satisfying $L' \subseteq \operatorname{Des} w \subseteq [n-1] \setminus L$). Thus, think of this set Q_i as the "real" index of w. We just found i easier to work with.
  ("Morally", the Fibonacci filtration should be indexed by a poset; then you need not choose any numbering.)

- **Theorem 17.3.** For each $i \in [0, f_{n+1}]$, the $\mathbf{k}$-module F_i is free with basis $(\mathbf{a}_w)_{w \in S_n;\ \text{Qind}\, w \leq i}$.

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Back to the basis, 3

* **Theorem 17.3.** For each $i \in [0, f_{n+1}]$, the $\mathbf{k}$-module F_i is free with basis $(\mathbf{a}_w)_{w \in S_n;\ \text{Qind}\, w \leq i}$.
* **Corollary 17.4.** For each $i \in [f_{n+1}]$, the $\mathbf{k}$-module F_i / F_{i-1} is free with basis $(\overline{\mathbf{a}_w})_{w \in S_n;\ \text{Qind}\, w = i}$.
  - This yields Lemma 9.4 and also leads to Theorem 7.1, made precise as follows:
* **Theorem 17.5 (Lafrenière, G.).** For any $w \in S_n$ and $\ell \in [n]$, we have

$$\mathbf{a}_w \mathbf{t}_\ell = \mu_{w,\ell} \mathbf{a}_w + \sum_{\substack{v \in S_n; \\ \text{Qind}\, v < \text{Qind}\, w}} \lambda_{w,\ell,v} \mathbf{a}_v$$

for some nonnegative integer $\mu_{w,\ell}$ and some integers $\lambda_{w,\ell,v}$. Thus, the endomorphisms $R(\mathbf{t}_1), R(\mathbf{t}_2), \ldots, R(\mathbf{t}_n)$ are upper-triangular with respect to the basis $(\mathbf{a}_w)_{w \in S_n}$ as long as the permutations $w \in S_n$ are ordered by increasing Q-index.

- In Corollary 9.5, we found the eigenvalues of the endomorphism $R(\lambda_1 \mathbf{t}_1 + \lambda_2 \mathbf{t}_2 + \cdots + \lambda_n \mathbf{t}_n)$. With Corollary 17.4, we can also find their algebraic multiplicities. To state a formula for them, we need a definition:

- In Corollary 9.5, we found the eigenvalues of the endomorphism $R(\lambda_1 \mathbf{t}_1 + \lambda_2 \mathbf{t}_2 + \cdots + \lambda_n \mathbf{t}_n)$. With Corollary 17.4, we can also find their algebraic multiplicities. To state a formula for them, we need a definition:

- For each $i \in [f_{n+1}]$, we set

$$\delta_i := (\text{the number of all } w \in S_n \text{ satisfying } \operatorname{Qind} w = i).$$

- **Corollary 18.1 (informal version).** Assume that $\mathbf{k}$ is a field. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$. Then, the endomorphism $R(\lambda_1 \mathbf{t}_1 + \lambda_2 \mathbf{t}_2 + \cdots + \lambda_n \mathbf{t}_n)$ has eigenvalues

$$\lambda_I := \lambda_1 m_{I,1} + \lambda_2 m_{I,2} + \cdots + \lambda_n m_{I,n} \qquad \text{for all lacunar } I \subseteq [n-1]$$

with respective multiplicities δ_i,

where $i \in [f_{n+1}]$ is such that $I = Q_i$.

(If some λ_I happen to coincide, then their algebraic multiplicities must be added together.)

- Can we compute the δ_i explicitly? Yes!

Theorem 18.2. Let $i \in [f_{n+1}]$. Then:

(a) Write the set Q_i in the form $Q_i = \{i_1 < i_2 < \cdots < i_p\}$, and set $i_0 = 1$ and $i_{p+1} = n+1$. Let $j_k = i_k - i_{k-1}$ for each $k \in [p+1]$. Then,

$$\delta_i = \underbrace{\binom{n}{j_1, j_2, \ldots, j_{p+1}}}_{\text{multinomial coefficient}} \cdot \prod_{k=2}^{p+1} (j_k - 1).$$

- Can we compute the δ_i explicitly? Yes!

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$$\delta_i = \underbrace{\binom{n}{j_1, j_2, \ldots, j_{p+1}}}_{\substack{\text{multinomial} \\ \text{coefficient}}} \cdot \prod_{k=2}^{p+1} (j_k - 1).$$

  (b) We have $\delta_i \mid n!$.

- **Note.** This reminds of the hook-length formula for standard tableaux, but is much simpler.

- Most of what we said about the somewhere-to-below shuffles $\mathbf{t}_\ell$ can be extended to their antipodes $\mathbf{t}_\ell^*$ (the "*below-to-somewhere shuffles*"). For instance:
- **Theorem 19.1.** There exists a basis of the $\mathbf{k}$-module $\mathbf{k}[S_n]$ in which all of the endomorphisms $R(\mathbf{t}_1^*), R(S\mathbf{t}_2^*), \ldots, R(\mathbf{t}_n^*)$ are represented by upper-triangular matrices.
- We can also use left instead of right multiplication:
- **Theorem 19.2.** There exists a basis of the $\mathbf{k}$-module $\mathbf{k}[S_n]$ in which all of the endomorphisms $L(\mathbf{t}_1), L(\mathbf{t}_2), \ldots, L(\mathbf{t}_n)$ are represented by upper-triangular matrices.
- These follow from Theorem 14.1 using dual bases and transpose matrices. No new combinatorics required!

- The simultaneous trigonalizability of the endomorphisms $R(\mathbf{t}_1), R(\mathbf{t}_2), \ldots, R(\mathbf{t}_n)$ yields that their pairwise commutators are nilpotent. Hence, the pairwise commutators $[\mathbf{t}_i, \mathbf{t}_j]$ are also nilpotent.
- **Question.** How small an exponent works in $[\mathbf{t}_i, \mathbf{t}_j]^* = 0$?
- **Theorem 20.1.** We have $[\mathbf{t}_i, \mathbf{t}_j]^{j-i+1} = 0$ for any $1 \leq i \leq j \leq n$.
- **Theorem 20.2.** We have $[\mathbf{t}_i, \mathbf{t}_j]^{\lceil (n-j)/2 \rceil + 1} = 0$ for any $i, j \in [n]$.
- Depending on i and j, one of the exponents is better than the other.

  Conjecture. The better one is optimal! (Checked for all $n \leq 12$.)

- Stronger results hold, replacing powers by products.
- Several other curious facts hold: For example,

$$\mathbf{t}_{i+1}\mathbf{t}_i = (\mathbf{t}_i - 1)\,\mathbf{t}_i \qquad \text{and} \qquad \mathbf{t}_{i+2}\,(\mathbf{t}_i - 1) = (\mathbf{t}_i - 1)\,(\mathbf{t}_{i+1} - 1)$$

and

$$\mathbf{t}_{n-1}\,[\mathbf{t}_i, \mathbf{t}_{n-1}] = 0 \qquad \text{and} \qquad [\mathbf{t}_i, \mathbf{t}_{n-1}]\,[\mathbf{t}_j, \mathbf{t}_{n-1}] = 0$$

for all i and j.

- All this is completely elementary but surprisingly hard to prove (dozens of pages of manipulations with sums and cycles). The proofs can be found in `arXiv:2309.05340`.
- What is "really" going on? No idea...

- Two natural questions:
  1. The $F(I)$ and the F_i are left ideals of $\mathbf{k}[S_n]$; how do they decompose into Specht modules?
  2. How do $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ act on a given Specht module?

- Two natural questions:
  1. The $F(I)$ and the F_i are left ideals of $\mathbf{k}[S_n]$; how do they decompose into Specht modules?
  2. How do $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ act on a given Specht module?
- We can answer these.
- The answer uses symmetric functions, specifically:
- Let s_λ be the Schur function for a partition λ.
- Let $h_m = s_{(m)}$ be the m-th complete homogeneous symmetric function for each $m \geq 0$.
- Let $z_m = s_{(m-1,1)} = h_{m-1}h_1 - h_m$ for each $m > 1$.

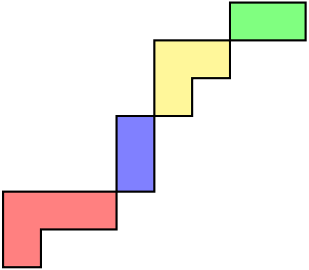
Representations, 2

- For each lacunar subset I of $[n-1]$, we define a symmetric function

$$\mathbf{z}_I := h_{i_1-1} \prod_{j=2}^{k} z_{i_j - i_{j-1}} \qquad \text{(over } \mathbb{Z}\text{)},$$

where $i_1, i_2, \ldots, i_k$ are the elements of $I \cup \{n+1\}$ in increasing order (so that $i_k = n+1$ and $I = \{i_1 < i_2 < \cdots < i_{k-1}\}$). This is a skew Schur function corresponding to a disjoint union of hooks: e.g., if $n = 11$ and $I = \{3, 6, 8\}$, then it is

$$\begin{array}{c} i_1 - 1 \\ i_2 - i_1 - 1 \\ 1 \\ i_3 - i_2 - 1 \\ 1 \\ i_4 - i_3 - 1 \\ 1 \end{array}$$



- For each lacunar $I \subseteq [n-1]$ and each partition λ of n, we let c_λ^I be the coefficient of s_λ in the Schur expansion of z_I.
  This is a Littlewood–Richardson coefficient (since z_I is a skew Schur function), thus $\in \mathbb{N}$.

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  This is a Littlewood–Richardson coefficient (since z_I is a skew Schur function), thus $\in \mathbb{N}$.
- **Theorem 21.1.** Let ν be a partition. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$. Then, the shuffle $\lambda_1 \mathbf{t}_1 + \lambda_2 \mathbf{t}_2 + \cdots + \lambda_n \mathbf{t}_n$ acts on the Specht module $\mathcal{S}^\nu$ as a linear map with eigenvalues

$$\lambda_1 m_{I,1} + \lambda_2 m_{I,2} + \cdots + \lambda_n m_{I,n} \qquad \text{for all lacunar } I \subseteq [n-1] \text{ satisfying } c_\nu^I \neq 0,$$

  and the multiplicity of each such eigenvalue is c_ν^I in the generic case.
  If all these linear combinations are distinct, then this linear map is diagonalizable.

- **Theorem 21.2 (lazy version).** Let $\mathbf{k}$ be a field of characteristic 0. Let $i \in [f_{n+1}]$. As a representation of S_n, the quotient module F_i / F_{i-1} has Frobenius characteristic z_{Q_i}.
- **Theorem 21.2 (careful version, true in every characteristic).** Let $i \in [f_{n+1}]$. Consider the lacunar subset Q_i of $[n-1]$. Let $i_1, i_2, \ldots, i_k$ be the elements of $Q_i \cup \{n+1\}$ in increasing order. Then, as representations of S_n, we have

$$F_i / F_{i-1} \cong \mathcal{H}_{i_1 - 1} * \mathcal{Z}_{i_2 - i_1} * \mathcal{Z}_{i_3 - i_2} * \cdots * \mathcal{Z}_{i_k - i_{k-1}},$$

  where $*$ means induction product (that is, $U * V = \operatorname{Ind}_{S_i \times S_j}^{S_{i+j}} (U \otimes V)$), and where $\mathcal{H}_m$ is the trivial 1-dimensional representation of S_m, whereas $\mathcal{Z}_m$ is the reflection representation of S_m (that is, $\mathbf{k}^m$ modulo the span of $(1, 1, \ldots, 1)$).
- Proofs appear in arXiv:2508.00752.

- **Question.** What can be said about the $\mathbf{k}$-subalgebra $\mathbf{k}[\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n]$ of $\mathbf{k}[S_n]$? Note:

n	1	2	3	4	5	6	7	8
$\dim(\mathbb{Q}[\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n])$	1	2	4	9	23	66	212	761

  (this sequence is not in the OEIS as of 2025-10-08).

- **Question.** Do the results about commutators and representations generalize to the Hecke algebra? (The Fibonacci filtration and descent-destroying basis definitely do. Proofs forthcoming...)

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