

Schur functions: Theme and variations*I. G. Macdonald*

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Actes 28^e Séminaire Lotharingien, p. 5-39.**Errata and addenda by Darij Grinberg**

I will refer to the results appearing in the paper “Schur functions: Theme and variations” by the numbers under which they appear in this paper. Page numbering goes from 5 to 39.

I have read pages ... of the paper so far.

The list below contains both actual corrections and what I believe to be clarifications and pertinent comments. I have not tried to separate the former from the latter, as I suspect that the precise boundary is in the eyes of the beholder.

B. Errata and addenda

I shall use the following notations:

- If p and q are two integers such that $p \leq q + 1$, then $[p, q]$ shall denote the set of all integers m such that $p \leq m \leq q$. We call this set an *integer interval*. It has size $q - p + 1$ (so it is empty if $p = q + 1$).
- For a given $r \in \mathbb{N}$, we shall denote the integer interval $[1, r] = \{1, 2, \dots, r\}$ by $[r]$.
- If $A = (a_{i,j})_{i \in S, j \in T}$ is an arbitrary matrix (where the sets S and T may be finite or infinite), and if $P = \{p_1 < p_2 < \dots < p_\alpha\}$ is a finite subset of S , and if $Q = \{q_1 < q_2 < \dots < q_\beta\}$ is a finite subset of T , then $\text{sub}_P^Q A$ shall denote the submatrix $(a_{p_i, q_j})_{i \in [a], j \in [b]}$ of A . For instance, if $A = (a_{i,j})_{i,j \in [4]}$ is a 4×4 -matrix, then $\text{sub}_{\{2,3\}}^{\{2,4\}} A = \begin{pmatrix} a_{2,2} & a_{2,4} \\ a_{3,2} & a_{3,4} \end{pmatrix}$.

Now, the actual corrections:

1. **page 15, line 2:** “for each $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^n$ ” should be “for each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ ”.
2. **page 15, after (6.3):** For a detailed proof of the fact that the quotient

$$s_\lambda(x | a) = A_{\lambda+\delta}(x | a) / A_\delta(x | a)$$

is a symmetric polynomial in x_1, \dots, x_n with coefficients in R , see [Grinbe18, Corollary 9.14]. (Apply this corollary to $X_i = x_i$ and $P_j(T) = (T | a)^{\lambda_j + n - j}$.)

3. **page 16, proof of (6.6):** This argument only shows that $\mathbf{E}(x | a) \cdot \mathbf{H}(x | a) = I_{\mathbb{Z}}$ (where $I_{\mathbb{Z}}$ is the identity matrix with rows and columns indexed by all integers). In order to prove that the two matrices $\mathbf{E}(x | a)$ and $\mathbf{H}(x | a)$ are inverse to each other (i.e., in order to prove (6.6)), it must also be shown that $\mathbf{H}(x | a) \cdot \mathbf{E}(x | a) = I_{\mathbb{Z}}$.

Fortunately, there is a simple shortcut for this: Let $UT_{\mathbb{Z}}$ be the set of all upper unitriangular matrices with rows and columns indexed by all integers (and with entries in a given base ring, which in our case is the polynomial ring over \mathbb{Z} in the variables x_i for $i \in [n]$ and a_j for $j \in \mathbb{Z}$). This set $UT_{\mathbb{Z}}$ is closed under matrix multiplication, thus is a monoid. Moreover, each matrix $A \in UT_{\mathbb{Z}}$ can be written as $I + M$ for some strictly upper-triangular matrix M , and thus has an inverse $A^{-1} = (I + M)^{-1}$, which can be computed by the formula $(I + M)^{-1} = I - M + M^2 - M^3 + M^4 \pm \dots$ (this infinite sum makes sense, since the nonzero entries of each power M^i start no earlier than i steps above the main diagonal¹). This inverse $A^{-1} = I - M + M^2 - M^3 + M^4 \pm \dots$ again belongs to $UT_{\mathbb{Z}}$. Hence, each element of the monoid $UT_{\mathbb{Z}}$ has an inverse. Thus, $UT_{\mathbb{Z}}$ is a group with respect to matrix multiplication. Since both matrices $\mathbf{E}(x | a)$ and $\mathbf{H}(x | a)$ belong to this group $UT_{\mathbb{Z}}$, we can thus conclude $\mathbf{E}(x | a) = \mathbf{H}(x | a)^{-1}$ from $\mathbf{E}(x | a) \cdot \mathbf{H}(x | a) = I_{\mathbb{Z}}$.

4. **page 17, proof of (6.7):** It is worth saying that all the matrices that appear in the proof of the first of the formulas (6.7) are understood to be $n \times n$ -matrices.

5. **page 17, proof of (6.7):** Let me give a proof of the second of the formulas (6.7) along with the more general formula (6.9). We will need some notations and some lemmas.

Let $UT_{\mathbb{Z}}$ be the set of all upper unitriangular matrices with rows and columns indexed by all integers (and with entries in a given commutative ring). For any $m \in \mathbb{N}$, we let UT_m be the set of all upper unitriangular $m \times m$ -matrices (again, with entries in our given commutative ring). Both of these sets $UT_{\mathbb{Z}}$ and UT_m are groups (under matrix multiplication). The following fact is easy:

Lemma B.1. Let $T = [p, q]$ be an integer interval of size $m = q - p + 1$ (so that $q = p + m - 1$). Then, the map

$$\begin{aligned} UT_{\mathbb{Z}} &\rightarrow UT_m, \\ A &\mapsto \text{sub}_T^T A \end{aligned} \tag{1}$$

¹That is: For any $u, v \in \mathbb{Z}$ and any $i \in \mathbb{N}$, the (u, v) -th entry of M^i is 0 whenever $v - u < i$, and therefore the (u, v) -th entry of the infinite sum $I - M + M^2 - M^3 + M^4 \pm \dots$ is only affected by the first $v - u + 1$ addends of this sum.

is a group morphism.

Proof of Lemma B.1. It is easy to see that this map is well-defined (i.e., that $\text{sub}_T^T A \in \text{UT}_m$ for each $A \in \text{UT}_{\mathbb{Z}}$). (Indeed, more generally, any principal submatrix of an upper unitriangular matrix is again upper unitriangular.) It is clear that $\text{sub}_T^T(I_{\mathbb{Z}}) = I_m$. It remains to show that $\text{sub}_T^T(AB) = \text{sub}_T^T A \cdot \text{sub}_T^T B$ for all $A, B \in \text{UT}_{\mathbb{Z}}$.

So let $A, B \in \text{UT}_{\mathbb{Z}}$ be arbitrary. Write these matrices A and B as $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ and $B = (b_{i,j})_{i,j \in \mathbb{Z}}$. Since $T = [p, q] = [p, p+m-1]$ (because $q = p+m-1$), we thus have

$$\text{sub}_T^T A = (a_{p+i-1, p+j-1})_{i,j \in [m]} \quad \text{and} \quad \text{sub}_T^T B = (b_{p+i-1, p+j-1})_{i,j \in [m]}$$

and therefore

$$\begin{aligned} \text{sub}_T^T A \cdot \text{sub}_T^T B &= (a_{p+i-1, p+j-1})_{i,j \in [m]} \cdot (b_{p+i-1, p+j-1})_{i,j \in [m]} \\ &= \left(\sum_{k=1}^m a_{p+i-1, p+k-1} b_{p+k-1, p+j-1} \right)_{i,j \in [m]} \end{aligned} \tag{2}$$

(by the definition of matrix multiplication).

The matrix A is upper-triangular (since $A \in \text{UT}_{\mathbb{Z}}$), so we have

$$a_{i,k} = 0 \quad \text{for all } i > k. \tag{3}$$

Likewise,

$$b_{k,j} = 0 \quad \text{for all } k > j. \tag{4}$$

Thus, we can easily see that if $i, j \in T$, then

$$a_{i,k} b_{k,j} = 0 \quad \text{for all integers } k \notin T. \tag{5}$$

(*Proof:* Let k be an integer such that $k \notin T$. Thus, $k \notin T = [p, q]$. Hence, either $k < p$ or $k > q$. In the former case, we have $k < p \leq i$ (since $i \in T = [p, q]$) and therefore $i > k$, so that $a_{i,k} = 0$ (by (3)), whence $\underbrace{a_{i,k}}_{=0} b_{k,j} = 0$. In the latter case, we have $k > q \geq j$ (since $j \in T = [p, q]$) entails $j \leq q$ and therefore $b_{k,j} = 0$ (by (4)), whence $a_{i,k} \underbrace{b_{k,j}}_{=0} = 0$. Hence,

we have proved $a_{i,k} b_{k,j} = 0$ in both cases. Thus, (5) is proved.)

Hence, for any $i, j \in T$, the (i, j) -th entry of the matrix AB is

$$\begin{aligned} \sum_{k \in \mathbb{Z}} a_{i,k} b_{k,j} &\quad \text{(by the definition of } AB\text{)} \\ &= \sum_{k \in T} a_{i,k} b_{k,j} \quad \left(\begin{array}{l} \text{since (5) shows that} \\ \text{any addend } a_{i,k} b_{k,j} \text{ equals 0 unless } k \in T, \\ \text{and thus we can restrict the sum to} \\ \text{only range over the } k \in T \end{array} \right). \end{aligned}$$

Therefore, from $T = [p, q]$, we obtain

$$\text{sub}_T^T(AB) = \left(\sum_{k \in T} a_{p+i-1,k} b_{k,p+j-1} \right)_{i,j \in [m]}. \quad (6)$$

However, each $i, j \in [m]$ satisfy

$$\begin{aligned} & \sum_{k=1}^m a_{p+i-1,p+k-1} b_{p+k-1,p+j-1} \\ = & \underbrace{\sum_{k=p}^{p+m-1}}_{\substack{= \sum \\ k \in T \\ (\text{since } T=[p,p+m-1])}} a_{p+i-1,k} b_{k,p+j-1} \\ & \quad (\text{here, we have substituted } k \text{ for } p+k-1 \text{ in the sum}) \\ = & \sum_{k \in T} a_{p+i-1,k} b_{k,p+j-1}. \end{aligned}$$

Thus, the right hand sides of the equalities (2) and (6) are equal. Hence, their left hand sides are also equal. In other words, $\text{sub}_T^T(AB) = \text{sub}_T^T A \cdot \text{sub}_T^T B$. This completes the proof of Lemma B.1. ■

Lemma B.2. Let A be an invertible $m \times m$ -matrix. Let P and Q be two subsets of $[m]$ such that $|P| = |Q|$. Let $\tilde{P} := [m] \setminus P$ and $\tilde{Q} := [m] \setminus Q$ be their complements. Let $\sum P$ be the sum of all elements of P , and let $\sum Q$ be the sum of all elements of Q . Then,

$$\det(\text{sub}_P^Q A) = (-1)^{\sum P + \sum Q} \det A \cdot \det(\text{sub}_{\tilde{Q}}^{\tilde{P}}(A^{-1})).$$

Lemma B.2 is [Grinbe20, Exercise 6.56] (with slightly different notations: $\text{sub}_P^Q A$ is denoted $\text{sub}_{w(P)}^{w(Q)} A$ there). Alternatively, it can be easily derived from [LLPT95, (APP.1.5.2)] (since the adjugate matrix $\text{adj } A$ of A , which is denoted by ${}^t A$ in [LLPT95], is known to equal $(\det A) \cdot A^{-1}$). ■

Lemma B.3. Let B be an infinite matrix in $\text{UT}_{\mathbb{Z}}$. Let $T = [p, q]$ be an integer interval. Let U and V be two subsets of T such that $|U| = |V|$. Then,

$$\det(\text{sub}_U^V B) = (-1)^{\sum U + \sum V} \det(\text{sub}_{T \setminus V}^{T \setminus U}(B^{-1})).$$

(Note that B^{-1} exists: Indeed, the matrix B belongs to the group $\text{UT}_{\mathbb{Z}}$ and thus has an inverse.)

Proof of Lemma B.3. Let m be the size $q - p + 1$ of the interval $T = [p, q]$. Then, $q = m + p - 1$. Furthermore, Lemma B.1 says that the map (1) is a group morphism. Hence, $(\text{sub}_T^T B)^{-1} = \text{sub}_T^T (B^{-1})$. Moreover, $\text{sub}_T^T B \in \text{UT}_m$ (since the map (1) has target UT_m), so that the matrix $\text{sub}_T^T B$ is upper unitriangular. Thus, $\det(\text{sub}_T^T B) = 1$ (since the determinant of a triangular matrix is the product of its diagonal entries, and therefore the determinant of a unitriangular matrix is 1).

Let A be the submatrix $\text{sub}_T^T B$ of B . Then,

$$A^{-1} = (\text{sub}_T^T B)^{-1} = \text{sub}_T^T (B^{-1}) \quad \text{and} \quad (7)$$

$$\det A = \det(\text{sub}_T^T B) = 1. \quad (8)$$

For any subset S of \mathbb{Z} and any integer z , we let $S + z$ denote the set $\{s + z \mid s \in S\}$. Visually speaking, this is simply the set S shifted by z units to the right along the number line. Clearly, $|S + z| = |S|$ and $(S + z) + (-z) = S$. Moreover, any two subsets S and T of \mathbb{Z} and any integer z satisfy

$$(S \setminus T) + z = (S + z) \setminus (T + z), \quad (9)$$

since the operation of adding z to each integer is a bijection.

We have

$$\begin{aligned} T &= [p, q] \\ &= [1 + p - 1, m + p - 1] \quad (\text{since } p = 1 + p - 1 \text{ and } q = m + p - 1) \\ &= [1, m] + (p - 1) \\ &= [m] + (p - 1) \quad (\text{since } [1, m] = m). \end{aligned} \quad (10)$$

Therefore, the map $i \mapsto i + (p - 1)$ is a bijection from $[m]$ to T . This bijection induces a bijection $J \mapsto J + (p - 1)$ from the set of all subsets of $[m]$ to the set of all subsets of T . Hence, any subset S of T has the form $S = S' + (p - 1)$ for a unique subset $S' \subseteq [m]$. In particular, the subsets U and V of T thus have the forms $U = P + (p - 1)$ and $V = Q + (p - 1)$ for unique subsets $P, Q \subseteq [m]$. Consider these P, Q . Clearly, $|U| = |P + (p - 1)| = |P|$ and $|V| = |Q + (p - 1)| = |Q|$, so that $|P| = |U| = |V| = |Q|$.

Let $\tilde{P} := [m] \setminus P$ and $\tilde{Q} := [m] \setminus Q$ be the complements of P and Q within $[m]$. Let $\sum P$ be the sum of all elements of P , and let $\sum Q$ be the sum of all elements of Q . From $U = P + (p - 1)$, we obtain

$$\sum U = \sum (P + (p - 1)) = \sum P + |P| \cdot (p - 1)$$

(since the elements of $P + (p - 1)$ are simply the $|P|$ elements of P with $p - 1$ added to each). Likewise, $\sum V = \sum Q + |Q| \cdot (p - 1)$. Adding these

two equalities together, we find

$$\begin{aligned}
\sum U + \sum V &= (\sum P + |P| \cdot (p-1)) + (\sum Q + |Q| \cdot (p-1)) \\
&= \sum P + \sum Q + \underbrace{(|P| + |Q|)}_{\substack{=|Q|+|Q| \\ (\text{since } |P|=|Q|)}} \cdot (p-1) \\
&= \sum P + \sum Q + \underbrace{(|Q| + |Q|)}_{\substack{=2|Q| \equiv 0 \pmod{2}}} \cdot (p-1) \\
&\equiv \sum P + \sum Q \pmod{2}.
\end{aligned}$$

Hence,

$$(-1)^{\sum U + \sum V} = (-1)^{\sum P + \sum Q}. \quad (11)$$

Next, we recall that $A = \text{sub}_T^T B = \text{sub}_{[p,q]}^{[p,q]} B$ (since $T = [p, q]$). Hence, for all $i, j \in [m]$, the (i, j) -th entry of A is the $(i + p - 1, j + p - 1)$ -th entry of B . Consequently, for any $X, Y \subseteq [m]$, we have

$$\text{sub}_X^Y A = \text{sub}_{X+(p-1)}^{Y+(p-1)} B.$$

Applying this to $X = P$ and $Y = Q$, we obtain

$$\text{sub}_P^Q A = \text{sub}_{P+(p-1)}^{Q+(p-1)} B = \text{sub}_U^V B \quad (12)$$

(since $P + (p-1) = U$ and $Q + (p-1) = V$).

Furthermore, from $\tilde{P} = [m] \setminus P$, we obtain

$$\begin{aligned}
\tilde{P} + (p-1) &= ([m] \setminus P) + (p-1) \\
&= \underbrace{([m] + (p-1))}_{=T} \setminus \underbrace{(P + (p-1))}_{=U} \quad (\text{by (9)}) \\
&= T \setminus U.
\end{aligned}$$

Similarly, $\tilde{Q} + (p-1) = T \setminus V$.

However, (7) says that $A^{-1} = \text{sub}_T^T (B^{-1}) = \text{sub}_{[p,q]}^{[p,q]} (B^{-1})$ (since $T = [p, q]$). Thus, for all $i, j \in [m]$, the (i, j) -th entry of A^{-1} is the $(i + p - 1, j + p - 1)$ -th entry of B^{-1} . Consequently, for any $X, Y \subseteq [m]$, we have

$$\text{sub}_X^Y (A^{-1}) = \text{sub}_{X+(p-1)}^{Y+(p-1)} (B^{-1}).$$

Applying this to $X = \tilde{Q}$ and $Y = \tilde{P}$, we obtain

$$\text{sub}_{\tilde{Q}}^{\tilde{P}} (A^{-1}) = \text{sub}_{\tilde{Q}+(p-1)}^{\tilde{P}+(p-1)} (B^{-1}) = \text{sub}_{T \setminus V}^{T \setminus U} (B^{-1}) \quad (13)$$

(since $\tilde{P} + (p-1) = T \setminus U$ and $\tilde{Q} + (p-1) = T \setminus V$).

Now, Lemma B.2 yields

$$\begin{aligned} \det(\text{sub}_P^Q A) &= \underbrace{(-1)^{\sum P + \sum Q}}_{=(-1)^{\sum U + \sum V} \text{ (by (11))}} \underbrace{\det A}_{=1 \text{ (by (8))}} \cdot \det(\text{sub}_{\tilde{Q}}^{\tilde{P}}(A^{-1})) \\ &= (-1)^{\sum U + \sum V} \cdot \det(\text{sub}_{\tilde{Q}}^{\tilde{P}}(A^{-1})). \end{aligned}$$

In view of (12) and (13), we can rewrite this as

$$\det(\text{sub}_U^V B) = (-1)^{\sum U + \sum V} \det(\text{sub}_{T \setminus V}^{T \setminus U}(B^{-1})).$$

Thus, Lemma B.3 is proved. ■

Lemma B.4. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)$ be a partition, and let $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_p)$ be its conjugate partition. (The entries λ_i and λ'_j are allowed to be 0.) For each $i \in [q]$, let us set $\alpha_i := \lambda_i - i$. For each $j \in [q]$, let us set $\beta_j := \lambda'_j - j$ and $\eta_j := -1 - \beta_j$. Then, the two sets $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$ and $\{\eta_1, \eta_2, \dots, \eta_p\}$ are disjoint, and their union is the integer interval $[-q, p-1]$.

Lemma B.4 is Proposition 3.18 (f) in the detailed version of the paper [Grinbe19] (this detailed version is downloadable from the arXiv as an ancillary file). (Note that the μ in the paper corresponds to our λ' , and that the conditions $p \geq \lambda_1$ and $q \geq \mu_1$ in the paper follow from our assumptions $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_p)$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)$, respectively.) ■

Lemma B.5. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)$ be a partition, and let $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_p)$ be its conjugate partition. (The entries λ_i and λ'_j are allowed to be 0.)

Let $\mu = (\mu_1, \mu_2, \dots, \mu_q)$ be a partition, and let $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_p)$ be its conjugate partition. (The entries μ_i and μ'_j are allowed to be 0.)

Let $B = (b_{i,j})_{i,j \in \mathbb{Z}} \in \text{UT}_{\mathbb{Z}}$ be an upper unitriangular matrix. Let $(c_{i,j})_{i,j \in \mathbb{Z}} = B^{-1}$ be its inverse matrix. Then,

$$\begin{aligned} \det(b_{\mu_i - i, \lambda_j - j})_{i,j \in [q]} \\ = (-1)^{(\lambda_1 + \lambda_2 + \dots + \lambda_q) + (\mu_1 + \mu_2 + \dots + \mu_q)} \det(c_{i - \lambda'_i - 1, j - \mu'_j - 1})_{i,j \in [p]}. \end{aligned}$$

Proof of Lemma B.5. Let T be the integer interval $[-q, p-1]$. For each $i \in [q]$, let us set $\alpha_i := \lambda_i - i$ and $\gamma_i := \mu_i - i$. For each $j \in [p]$, let us set $\beta_j := \lambda'_j - j$ and $\eta_j := -1 - \beta_j$ and $\delta_j := \mu'_j - j$ and $\omega_j := -1 - \delta_j$.

Then, Lemma B.4 says that the two sets $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$ and $\{\eta_1, \eta_2, \dots, \eta_p\}$ are disjoint, and their union is the integer interval $[-q, p-1]$. Therefore, the set $\{\eta_1, \eta_2, \dots, \eta_p\}$ is the complement of the set $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$ in the interval $[-q, p-1]$. In other words,

$$\begin{aligned} \{\eta_1, \eta_2, \dots, \eta_p\} &= \underbrace{[-q, p-1]}_{=T} \setminus \{\alpha_1, \alpha_2, \dots, \alpha_q\} \\ &= T \setminus \{\alpha_1, \alpha_2, \dots, \alpha_q\}. \end{aligned} \quad (14)$$

The same argument (applied to μ , μ' , γ_i , δ_j and ω_j instead of λ , λ' , α_i , β_j and η_j) yields

$$\{\omega_1, \omega_2, \dots, \omega_p\} = T \setminus \{\gamma_1, \gamma_2, \dots, \gamma_q\}. \quad (15)$$

Since λ is a partition, we have $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q$ and thus $\lambda_1 - 1 > \lambda_2 - 2 > \dots > \lambda_q - q$. In other words, $\alpha_1 > \alpha_2 > \dots > \alpha_q$ (since $\alpha_i = \lambda_i - i$ for each i). Hence,

$$\{\alpha_1, \alpha_2, \dots, \alpha_q\} = \{\alpha_q < \alpha_{q-1} < \dots < \alpha_1\}. \quad (16)$$

Similarly,

$$\{\gamma_1, \gamma_2, \dots, \gamma_q\} = \{\gamma_q < \gamma_{q-1} < \dots < \gamma_1\}. \quad (17)$$

These two equalities show that both sets $\{\gamma_1, \gamma_2, \dots, \gamma_q\}$ and $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$ have size q , so that they have the same size. In other words, $|\{\gamma_1, \gamma_2, \dots, \gamma_q\}| = |\{\alpha_1, \alpha_2, \dots, \alpha_q\}|$.

Furthermore, recall that the two sets $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$ and $\{\eta_1, \eta_2, \dots, \eta_p\}$ are disjoint, and their union is the integer interval $[-q, p-1]$. Hence, in particular, $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$ is a subset of $[-q, p-1] = T$. Similarly, $\{\gamma_1, \gamma_2, \dots, \gamma_q\}$ is a subset of T as well.

Thus, Lemma B.3 (applied to $-q, p-1, \{\gamma_1, \gamma_2, \dots, \gamma_q\}$ and $\{\alpha_1, \alpha_2, \dots, \alpha_q\}$ instead of p, q, U and V) yields

$$\begin{aligned} &\det \left(\text{sub}_{\{\gamma_1, \gamma_2, \dots, \gamma_q\}}^{\{\alpha_1, \alpha_2, \dots, \alpha_q\}} B \right) \\ &= (-1)^{\sum \{\gamma_1, \gamma_2, \dots, \gamma_q\} + \sum \{\alpha_1, \alpha_2, \dots, \alpha_q\}} \\ &\quad \det \left(\text{sub}_{T \setminus \{\alpha_1, \alpha_2, \dots, \alpha_q\}}^{T \setminus \{\gamma_1, \gamma_2, \dots, \gamma_q\}} (B^{-1}) \right). \end{aligned} \quad (18)$$

In view of (16) and (17), we have

$$\text{sub}_{\{\gamma_1, \gamma_2, \dots, \gamma_q\}}^{\{\alpha_1, \alpha_2, \dots, \alpha_q\}} B = \text{sub}_{\{\gamma_q < \gamma_{q-1} < \dots < \gamma_1\}}^{\{\alpha_q < \alpha_{q-1} < \dots < \alpha_1\}} B = \left(b_{\gamma_{q+1-i}, \alpha_{q+1-j}} \right)_{i,j \in [q]}$$

(since $B = (b_{i,j})_{i,j \in \mathbb{Z}}$). Hence,

$$\begin{aligned} \det \left(\text{sub}_{\{\gamma_1, \gamma_2, \dots, \gamma_q\}}^{\{\alpha_1, \alpha_2, \dots, \alpha_q\}} B \right) &= \det \left(b_{\gamma_{q+1-i}, \alpha_{q+1-j}} \right)_{i,j \in [q]} \\ &= \det \left(b_{\gamma_i, \alpha_j} \right)_{i,j \in [q]} \end{aligned} \quad (19)$$

(here, we have substituted $q+1-i$ and $q+1-j$ for i and j in the matrix, which effectively rotates the matrix by 180° ; this rotation does not change the determinant, because it is a composition of a row permutation and a column permutation with the same sign).

On the other hand, let us recall that $\{\eta_1, \eta_2, \dots, \eta_p\} = T \setminus \{\alpha_1, \alpha_2, \dots, \alpha_q\}$. But λ' is a partition; thus, $\lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_p$ and therefore $\lambda'_1 - 1 > \lambda'_2 - 2 > \dots > \lambda'_p - p$. In other words, $\beta_1 > \beta_2 > \dots > \beta_p$ (since $\beta_j = \lambda'_j - j$ for each j). Hence, $-1 - \beta_1 < -1 - \beta_2 < \dots < -1 - \beta_p$. In other words, $\eta_1 < \eta_2 < \dots < \eta_p$ (since $\eta_j = -1 - \beta_j$ for each j). Thus, $\{\eta_1, \eta_2, \dots, \eta_p\} = \{\eta_1 < \eta_2 < \dots < \eta_p\}$. Comparing this with (14), we obtain

$$T \setminus \{\alpha_1, \alpha_2, \dots, \alpha_q\} = \{\eta_1 < \eta_2 < \dots < \eta_p\}.$$

The same argument (applied to $\mu, \mu', \gamma_i, \delta_j$ and ω_j instead of $\lambda, \lambda', \alpha_i, \beta_j$ and η_j) yields

$$T \setminus \{\gamma_1, \gamma_2, \dots, \gamma_q\} = \{\omega_1 < \omega_2 < \dots < \omega_p\}.$$

In view of these two equalities, we have

$$\text{sub}_{T \setminus \{\alpha_1, \alpha_2, \dots, \alpha_q\}}^{T \setminus \{\gamma_1, \gamma_2, \dots, \gamma_q\}} (B^{-1}) = \text{sub}_{\{\eta_1 < \eta_2 < \dots < \eta_p\}}^{\{\omega_1 < \omega_2 < \dots < \omega_p\}} (B^{-1}) = (c_{\eta_i, \omega_j})_{i,j \in [p]}$$

(since $B^{-1} = (c_{i,j})_{i,j \in \mathbb{Z}}$). Thus,

$$\det \left(\text{sub}_{T \setminus \{\alpha_1, \alpha_2, \dots, \alpha_q\}}^{T \setminus \{\gamma_1, \gamma_2, \dots, \gamma_q\}} (B^{-1}) \right) = \det \left(c_{\eta_i, \omega_j} \right)_{i,j \in [p]}. \quad (20)$$

Furthermore, $\alpha_1 > \alpha_2 > \dots > \alpha_q$ shows that the numbers $\alpha_1, \alpha_2, \dots, \alpha_q$ are distinct; hence,

$$\begin{aligned} \sum \{\alpha_1, \alpha_2, \dots, \alpha_q\} &= \alpha_1 + \alpha_2 + \dots + \alpha_q \\ &= (\lambda_1 - 1) + (\lambda_2 - 2) + \dots + (\lambda_q - q) \\ &\quad (\text{since } \alpha_i = \lambda_i - i \text{ for all } i) \\ &= (\lambda_1 + \lambda_2 + \dots + \lambda_q) - (1 + 2 + \dots + q). \end{aligned}$$

Similarly,

$$\sum \{\gamma_1, \gamma_2, \dots, \gamma_q\} = (\mu_1 + \mu_2 + \dots + \mu_q) - (1 + 2 + \dots + q).$$

Adding these two equalities together, we find

$$\begin{aligned}
& \sum \{\alpha_1, \alpha_2, \dots, \alpha_q\} + \sum \{\gamma_1, \gamma_2, \dots, \gamma_q\} \\
&= (\lambda_1 + \lambda_2 + \dots + \lambda_q) - (1 + 2 + \dots + q) \\
&\quad + (\mu_1 + \mu_2 + \dots + \mu_q) - (1 + 2 + \dots + q) \\
&= (\lambda_1 + \lambda_2 + \dots + \lambda_q) + (\mu_1 + \mu_2 + \dots + \mu_q) - \underbrace{2(1 + 2 + \dots + q)}_{\equiv 0 \pmod{2}} \\
&\equiv (\lambda_1 + \lambda_2 + \dots + \lambda_q) + (\mu_1 + \mu_2 + \dots + \mu_q) \pmod{2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& (-1)^{\sum \{\alpha_1, \alpha_2, \dots, \alpha_q\} + \sum \{\gamma_1, \gamma_2, \dots, \gamma_q\}} \\
&= (-1)^{(\lambda_1 + \lambda_2 + \dots + \lambda_q) + (\mu_1 + \mu_2 + \dots + \mu_q)}.
\end{aligned} \tag{21}$$

Finally, using (19), we can rewrite (18) as

$$\begin{aligned}
& \det \left(b_{\gamma_i, \alpha_j} \right)_{i, j \in [q]} \\
&= \underbrace{(-1)^{\sum \{\gamma_1, \gamma_2, \dots, \gamma_q\} + \sum \{\alpha_1, \alpha_2, \dots, \alpha_q\}}}_{=(-1)^{\sum \{\alpha_1, \alpha_2, \dots, \alpha_q\} + \sum \{\gamma_1, \gamma_2, \dots, \gamma_q\}}} \underbrace{\det \left(\text{sub}_{T \setminus \{\alpha_1, \alpha_2, \dots, \alpha_q\}}^{T \setminus \{\gamma_1, \gamma_2, \dots, \gamma_q\}} (B^{-1}) \right)}_{=\det \left(c_{\eta_i, \omega_j} \right)_{i, j \in [p]}} \\
&\quad \stackrel{(\text{by (21)})}{=} (-1)^{(\lambda_1 + \lambda_2 + \dots + \lambda_q) + (\mu_1 + \mu_2 + \dots + \mu_q)} \stackrel{(\text{by (20)})}{=} \det \left(c_{\eta_i, \omega_j} \right)_{i, j \in [p]}.
\end{aligned}$$

In view of $\gamma_i = \mu_i - i$ and $\alpha_j = \lambda_j - j$ and $\eta_i = -1 - \underbrace{\beta_i}_{=\lambda'_i - i} = -1 - (\lambda'_i - i)$ and $\omega_j = -1 - \underbrace{\delta_j}_{=\mu'_j - j} = -1 - (\mu'_j - j) = j - \mu'_j - 1$, we can rewrite this as

$$\begin{aligned}
& \det \left(b_{\mu_i - i, \lambda_j - j} \right)_{i, j \in [q]} \\
&= (-1)^{(\lambda_1 + \lambda_2 + \dots + \lambda_q) + (\mu_1 + \mu_2 + \dots + \mu_q)} \det \left(c_{i - \lambda'_i - 1, j - \mu'_j - 1} \right)_{i, j \in [p]}.
\end{aligned}$$

This proves Lemma B.5. ■

Lemma B.6. Let $(u_{i,j})_{i, j \in [p]}$ be a $p \times p$ -matrix. Furthermore, let $\alpha_1, \alpha_2, \dots, \alpha_p, \beta_1, \beta_2, \dots, \beta_p$ be any $2p$ scalars. Then,

$$\det \left(\alpha_i \beta_j u_{i,j} \right)_{i, j \in [p]} = \left(\prod_{i=1}^p \alpha_i \right) \left(\prod_{j=1}^p \beta_j \right) \cdot \det \left(u_{i,j} \right)_{i, j \in [p]}.$$

Proof of Lemma B.6. The matrix $(\alpha_i \beta_j u_{i,j})_{i,j \in [p]}$ is obtained from the matrix $(u_{i,j})_{i,j \in [p]}$ by

- multiplying the i -th row by α_i for each $i \in [p]$, and then
- multiplying the j -th column by β_j for each $j \in [p]$.

Each of these operations multiplies the determinant of the matrix by the corresponding factor α_i or β_j . Thus, in total, the determinant gets multiplied by $\left(\prod_{i=1}^p \alpha_i\right) \left(\prod_{j=1}^p \beta_j\right)$. This proves Lemma B.6. ■

Now, we can prove (6.9):

Proof of (6.9): Write the partitions λ , λ' , μ and μ' in the forms

$$\begin{aligned} \lambda &= (\lambda_1, \lambda_2, \dots, \lambda_q), & \lambda' &= (\lambda'_1, \lambda'_2, \dots, \lambda'_p), \\ \mu &= (\mu_1, \mu_2, \dots, \mu_q), & \mu' &= (\mu'_1, \mu'_2, \dots, \mu'_p) \end{aligned}$$

for some $p, q \in \mathbb{N}$ (where, of course, the entries λ_i , λ'_j , μ_i and μ'_j are allowed to be 0).

Recall that

$$\begin{aligned} \mathbf{H}(x \mid a) &= \left(h_{j-i}(x \mid \tau^{i+1}a) \right)_{i,j \in \mathbb{Z}} \quad \text{and} \\ \mathbf{E}(x \mid a) &= \left((-1)^{j-i} e_{j-i}(x \mid \tau^j a) \right)_{i,j \in \mathbb{Z}} \end{aligned}$$

are two upper unitriangular matrices in $\text{UT}_{\mathbb{Z}}$ (since $e_0(x \mid a) = 0$ and $e_k(x \mid a) = 0$ whenever $k < 0$). From (6.6), we know that the matrix $\mathbf{E}(x \mid a)$ is the inverse matrix of $\mathbf{H}(x \mid a)$. Hence, Lemma B.5 (applied to $B = \mathbf{H}(x \mid a)$ and $b_{i,j} = h_{j-i}(x \mid \tau^{i+1}a)$ and $c_{i,j} = (-1)^{j-i} e_{j-i}(x \mid \tau^j a)$) yields

$$\begin{aligned} &\det \left(h_{(\lambda_j - j) - (\mu_i - i)}(x \mid \tau^{\mu_i - i + 1}a) \right)_{i,j \in [q]} \\ &= (-1)^{(\lambda_1 + \lambda_2 + \dots + \lambda_q) + (\mu_1 + \mu_2 + \dots + \mu_q)} \\ &\quad \det \left((-1)^{(j - \mu'_j - 1) - (i - \lambda'_i - 1)} e_{(j - \mu'_j - 1) - (i - \lambda'_i - 1)}(x \mid \tau^{j - \mu'_j - 1}a) \right)_{i,j \in [p]}. \end{aligned}$$

In view of $(\lambda_j - j) - (\mu_i - i) = \lambda_j - \mu_i - j + i$ and $(j - \mu'_j - 1) - (i - \lambda'_i - 1) =$

$\lambda'_i - \mu'_j - i + j$, we can rewrite this as

$$\begin{aligned} & \det \left(h_{\lambda_j - \mu_i - j + i} \left(x \mid \tau^{\mu_i - i + 1} a \right) \right)_{i,j \in [q]} \\ &= (-1)^{(\lambda_1 + \lambda_2 + \dots + \lambda_q) + (\mu_1 + \mu_2 + \dots + \mu_q)} \\ & \quad \det \left((-1)^{\lambda'_i - \mu'_j - i + j} e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{j - \mu'_j - 1} a \right) \right)_{i,j \in [p]}. \end{aligned}$$

In view of

$$\det \left(h_{\lambda_j - \mu_i - j + i} \left(x \mid \tau^{\mu_i - i + 1} a \right) \right)_{i,j \in [q]} = \det \left(h_{\lambda_i - \mu_j - i + j} \left(x \mid \tau^{\mu_j - j + 1} a \right) \right)_{i,j \in [q]}$$

(since the determinant of a matrix does not change when we transpose it) and

$$\begin{aligned} & \det \left(\underbrace{(-1)^{\lambda'_i - \mu'_j - i + j} e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{j - \mu'_j - 1} a \right)}_{= (-1)^{\lambda'_i - i} (-1)^{\mu'_j - j}} \right)_{i,j \in [p]} \\ &= \det \left((-1)^{\lambda'_i - i} (-1)^{\mu'_j - j} e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{j - \mu'_j - 1} a \right) \right)_{i,j \in [p]} \\ &= \left(\prod_{i=1}^p (-1)^{\lambda'_i - i} \right) \left(\prod_{j=1}^p (-1)^{\mu'_j - j} \right) \cdot \det \left(e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{j - \mu'_j - 1} a \right) \right)_{i,j \in [p]} \end{aligned}$$

(by Lemma B.6, applied to $\alpha_i = (-1)^{\lambda'_i - i}$ and $\beta_j = (-1)^{\mu'_j - j}$ and $u_{i,j} = e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{j - \mu'_j - 1} a \right)$), we can rewrite this as

$$\begin{aligned} & \det \left(h_{\lambda_i - \mu_j - i + j} \left(x \mid \tau^{\mu_j - j + 1} a \right) \right)_{i,j \in [q]} \\ &= (-1)^{(\lambda_1 + \lambda_2 + \dots + \lambda_q) + (\mu_1 + \mu_2 + \dots + \mu_q)} \\ & \quad \left(\prod_{i=1}^p (-1)^{\lambda'_i - i} \right) \left(\prod_{j=1}^p (-1)^{\mu'_j - j} \right) \cdot \det \left(e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{j - \mu'_j - 1} a \right) \right)_{i,j \in [p]} \\ &= (-1)^{(\lambda_1 + \lambda_2 + \dots + \lambda_q) + (\mu_1 + \mu_2 + \dots + \mu_q)} \left(\prod_{i=1}^p (-1)^{\lambda'_i - i} \right) \left(\prod_{j=1}^p (-1)^{\mu'_j - j} \right) \\ & \quad \cdot \det \left(e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{j - \mu'_j - 1} a \right) \right)_{i,j \in [p]}. \end{aligned} \tag{22}$$

However,

$$\begin{aligned}
 & (-1)^{(\lambda_1 + \lambda_2 + \cdots + \lambda_q) + (\mu_1 + \mu_2 + \cdots + \mu_q)} \left(\prod_{i=1}^p (-1)^{\lambda'_i - i} \right) \left(\prod_{j=1}^p (-1)^{\mu'_j - j} \right) \\
 & = (-1)^{(\lambda_1 + \lambda_2 + \cdots + \lambda_q) + (\mu_1 + \mu_2 + \cdots + \mu_q) + \sum_{i=1}^p (\lambda'_i - i) + \sum_{j=1}^p (\mu'_j - j)} \\
 & = 1,
 \end{aligned}$$

since

$$\begin{aligned}
 & \underbrace{(\lambda_1 + \lambda_2 + \cdots + \lambda_q)}_{=|\lambda|} + \underbrace{(\mu_1 + \mu_2 + \cdots + \mu_q)}_{=|\mu|} + \underbrace{\sum_{i=1}^p (\lambda'_i - i)}_{= \sum_{i=1}^p \lambda'_i - \sum_{i=1}^p i} + \underbrace{\sum_{j=1}^p (\mu'_j - j)}_{= \sum_{j=1}^p \mu'_j - \sum_{j=1}^p j} \\
 & = |\lambda| + |\mu| + \underbrace{\sum_{i=1}^p \lambda'_i}_{=|\lambda'|=|\lambda|} - \underbrace{\sum_{i=1}^p i}_{=1+2+\cdots+p} + \underbrace{\sum_{j=1}^p \mu'_j}_{=|\mu'|=|\mu|} - \underbrace{\sum_{j=1}^p j}_{=1+2+\cdots+p} \\
 & = |\lambda| + |\mu| + |\lambda| + (1 + 2 + \cdots + p) + |\mu| + (1 + 2 + \cdots + p) \\
 & = 2(|\lambda| + |\mu| + (1 + 2 + \cdots + p)) \quad \text{is even.}
 \end{aligned}$$

Thus, we can rewrite (22) as

$$\begin{aligned}
 & \det \left(h_{\lambda_i - \mu_j - i + j} \left(x \mid \tau^{\mu_j - j + 1} a \right) \right)_{i,j \in [q]} \\
 & = 1 \cdot \det \left(e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{j - \mu'_j - 1} a \right) \right)_{i,j \in [p]} \\
 & = \det \left(e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{j - \mu'_j - 1} a \right) \right)_{i,j \in [p]} \\
 & = \det \left(e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{-\mu'_j + j - 1} a \right) \right)_{i,j \in [p]} \quad \left(\text{since } j - \mu'_j - 1 = -\mu'_j + j - 1 \right).
 \end{aligned}$$

In view of (6.8), this can be rewritten as

$$s_{\lambda/\mu}(x \mid a) = \det \left(e_{\lambda'_i - \mu'_j - i + j} \left(x \mid \tau^{-\mu'_j + j - 1} a \right) \right)_{i,j \in [p]}.$$

This proves (6.9). ■

Applying (6.9) to $\mu = \emptyset$, we obtain the second equality in (6.7).

6. **page 18, (6.9):** See the previous bullet point for a proof of (6.9).
7. **page 18, (6.10):** Let me prove (6.10) here. We will need a simple lemma about determinants.

If p and q are two integers such that $p \leq q + 1$, then $[p, q]$ shall denote the set of all integers m such that $p \leq m \leq q$. We call this set an *integer interval*. It has size $q - p + 1$ (so it is empty if $p = q + 1$).

For a given $r \in \mathbb{N}$, we shall denote the integer interval $[1, r] = \{1, 2, \dots, r\}$ by $[r]$.

Our lemma says the following:

Lemma B.7. Let $(u_{i,j})_{i,j \in [p]}$ be a $p \times p$ -matrix for some $p \in \mathbb{N}$. Let U and V be two subsets of $[p]$ satisfying $|U| + |V| > p$. Assume that

$$u_{i,j} = 0 \quad \text{for all } i \in U \text{ and } j \in V. \quad (23)$$

Then, $\det(u_{i,j})_{i,j \in [p]} = 0$.

Proof of Lemma B.7. By the definition of a determinant, we have

$$\det(u_{i,j})_{i,j \in [p]} = \sum_{\sigma \in S_p} \text{sign } \sigma \cdot \prod_{i=1}^p u_{i,\sigma(i)} \quad (24)$$

(where S_p is the symmetric group of all permutations of $[p]$). Now we shall show that each $\sigma \in S_p$ satisfies

$$\prod_{i=1}^p u_{i,\sigma(i)} = 0. \quad (25)$$

(*Proof:* Let $\sigma \in S_p$. Then, $|\sigma(U)| = |U| > p - |V|$ (since $|U| + |V| > p$), so that $\sigma(U) \not\subseteq [p] \setminus V$ (since $\sigma(U) \subseteq [p] \setminus V$ would entail $|\sigma(U)| \leq |[p] \setminus V| = p - |V|$, contradicting $|\sigma(U)| > p - |V|$). In other words, there exists some $j \in \sigma(U)$ such that $j \notin [p] \setminus V$. Consider this j . From $j \in \sigma(U)$, we obtain $j = \sigma(k)$ for some $k \in U$. Consider this k . Now, $k \in U$ and $j \in V$ (since $j \in [p]$ but $j \notin [p] \setminus V$). Hence, (23) (applied to $i = k$) yields $u_{k,j} = 0$. In other words, $u_{k,\sigma(k)} = 0$ (since $j = \sigma(k)$). Thus, one of the factors of the product $\prod_{i=1}^p u_{i,\sigma(i)}$ is 0 (namely, the k -th factor). Hence, this whole product is 0. This proves (25).)

Now, (24) becomes

$$\det(u_{i,j})_{i,j \in [p]} = \sum_{\sigma \in S_p} \text{sign } \sigma \cdot \underbrace{\prod_{i=1}^p u_{i,\sigma(i)}}_{\substack{=0 \\ (\text{by (24)})}} = 0.$$

This proves Lemma B.7. ■

We can now prove (6.10):

Proof of (6.10). Assume that we don't have $0 \leq \lambda'_i - \mu'_i \leq n$ for all i . We must prove that $s_{\lambda/\mu}(x | a) = 0$.

Write the partitions λ' and μ' as $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_p)$ and $\mu' = (\mu'_1, \mu'_2, \dots, \mu'_p)$.

Then, all $i > p$ satisfy $\lambda'_i = 0$ and $\mu'_i = 0$ and therefore $\lambda'_i - \mu'_i = 0 - 0 = 0$.

Note also that $\lambda'_1 \geq \lambda'_2 \geq \lambda'_3 \geq \dots$ (since λ' is a partition) and $\mu'_1 \geq \mu'_2 \geq \mu'_3 \geq \dots$ (similarly). From (6.9), we obtain

$$s_{\lambda/\mu}(x | a) = \det \left(e_{\lambda'_i - \mu'_j - i + j} (x | \tau^{-\mu'_j + j - 1} a) \right)_{i,j \in [p]}. \quad (26)$$

We have assumed that we don't have $0 \leq \lambda'_i - \mu'_i \leq n$ for all i . In other words, there exists some $i \geq 1$ such that we don't have $0 \leq \lambda'_i - \mu'_i \leq n$. Consider this i , and denote it by k . Thus, we don't have $0 \leq \lambda'_k - \mu'_k \leq n$. Hence, we have either $\lambda'_k - \mu'_k < 0$ or $\lambda'_k - \mu'_k > n$. We are thus in one of the following two cases:

Case 1: We have $\lambda'_k - \mu'_k < 0$.

Case 2: We have $\lambda'_k - \mu'_k > n$.

Consider Case 1 first. In this case, we have $\lambda'_k - \mu'_k < 0$. Hence, we cannot have $k > p$ (since all $i > p$ satisfy $\lambda'_i - \mu'_i = 0$, which would yield $\lambda'_k - \mu'_k = 0$ if we had $k > p$). Thus, $k \leq p$, so that $k \in [p]$.

Now, let $i \in [k, p]$ and $j \in [k]$. Then, $i \geq k$ (since $i \in [k, p]$), so that $k \leq i$ and thus $\lambda'_k \geq \lambda'_i$ (since $\lambda'_1 \geq \lambda'_2 \geq \lambda'_3 \geq \dots$). Thus, $\lambda'_i \leq \lambda'_k$. Furthermore, $j \leq k$ (since $j \in [k]$), so that $\mu'_j \geq \mu'_k$. Thus,

$$\underbrace{\lambda'_i}_{\leq \lambda'_k} - \underbrace{\mu'_j}_{\geq \mu'_k} - \underbrace{i}_{\geq k} + \underbrace{j}_{\leq k} \leq \lambda'_k - \mu'_k - k + k = \lambda'_k - \mu'_k < 0,$$

and therefore $e_{\lambda'_i - \mu'_j - i + j} (x | \tau^{-\mu'_j + j - 1} a) = 0$ (since $e_\ell (x | \tau^{-\mu'_j + j - 1} a) = 0$ for any $\ell < 0$).

Forget that we fixed i and j . We thus have shown that

$$e_{\lambda'_i - \mu'_j - i + j} (x | \tau^{-\mu'_j + j - 1} a) = 0 \quad \text{for all } i \in [k, p] \text{ and } j \in [k].$$

Hence, Lemma B.7 (applied to $u_{i,j} = e_{\lambda'_i - \mu'_j - i + j} (x | \tau^{-\mu'_j + j - 1} a)$) yields that

$$\det \left(e_{\lambda'_i - \mu'_j - i + j} (x | \tau^{-\mu'_j + j - 1} a) \right)_{i,j \in [p]} = 0$$

(since $\underbrace{[k, p]}_{=p-k+1} + \underbrace{[k]}_{=k} = (p - k + 1) + k = p + 1 > p$). Hence, (26) rewrites

as $s_{\lambda/\mu}(x | a) = 0$. This proves (6.10) in Case 1.

Let us now consider Case 2. In this case, we have $\lambda'_k - \mu'_k > n$. Hence, we cannot have $k > p$ (since all $i > p$ satisfy $\lambda'_i - \mu'_i = 0$, which would yield $\lambda'_k - \mu'_k = 0 \leq n$ if we had $k > p$). Thus, $k \leq p$, so that $k \in [p]$.

Now, let $i \in [k]$ and $j \in [k, p]$. Then, $j \geq k$ (since $j \in [k, p]$), so that $k \leq j$ and thus $\mu'_k \geq \mu'_j$ (since $\mu'_1 \geq \mu'_2 \geq \mu'_3 \geq \dots$). Hence, $\mu'_j \leq \mu'_k$. Furthermore, $i \leq k$ (since $i \in [k]$), so that $\lambda'_i \geq \lambda'_k$. Thus,

$$\underbrace{\lambda'_i}_{\geq \lambda'_k} - \underbrace{\mu'_j}_{\leq \mu'_k} - \underbrace{i}_{\leq k} + \underbrace{j}_{\geq k} \geq \lambda'_k - \mu'_k - k + k = \lambda'_k - \mu'_k > n,$$

and therefore $e_{\lambda'_i - \mu'_j - i + j} (x \mid \tau^{-\mu'_j + j - 1} a) = 0$ (since $e_\ell (x \mid \tau^{-\mu'_j + j - 1} a) = 0$ for any $\ell > n$).

Forget that we fixed i and j . We thus have shown that

$$e_{\lambda'_i - \mu'_j - i + j} (x \mid \tau^{-\mu'_j + j - 1} a) = 0 \quad \text{for all } i \in [k] \text{ and } j \in [k, p].$$

Hence, Lemma B.7 (applied to $u_{i,j} = e_{\lambda'_i - \mu'_j - i + j} (x \mid \tau^{-\mu'_j + j - 1} a)$) yields that

$$\det \left(e_{\lambda'_i - \mu'_j - i + j} (x \mid \tau^{-\mu'_j + j - 1} a) \right)_{i,j \in [p]} = 0$$

(since $\underbrace{|[k]|}_{=k} + \underbrace{|[k, p]|}_{=p-k+1} = k + (p - k + 1) = p + 1 > p$). Hence, (26) rewrites

as $s_{\lambda/\mu} (x \mid a) = 0$. This proves (6.10) in Case 2.

We have now proved (6.10) in both Cases 1 and 2; this completes the proof of (6.10). ■

8. **page 19, proof of (6.12):** In the last displayed equation of the proof, replace “ $e_{j-k} (y \mid \tau^{n+j} a)$ ” by “ $e_{j-i} (y \mid \tau^{n+j} a)$ ”.

9. **page 19, proof of (6.13):** Let me explain in some more detail how the equality

$$\bigwedge^r \mathbf{H}(x, y \mid a) = \bigwedge^r \mathbf{H}(x \mid a) \cdot \bigwedge^r \mathbf{H}(y \mid \tau^n a)$$

yields the claim (6.13).

Indeed, a well-known corollary of the Cauchy–Binet theorem (specifically, [Grinbe20, Corollary 7.182], or rather its version for infinite matrices²)

²The version for infinite matrices is proved in the same way as the version for finite matrices, as long as (formal) convergence is taken care of (and that is easy when the matrices in question are upper-triangular).

yields

$$\begin{aligned} & \det \left(\text{sub}_{\{\mu_1-1, \mu_2-2, \dots, \mu_r-r\}}^{\{\lambda_1-1, \lambda_2-2, \dots, \lambda_r-r\}} (\mathbf{H}(x \mid a) \cdot \mathbf{H}(y \mid \tau^n a)) \right) \\ &= \sum_{g_1 < g_2 < \dots < g_r} \det \left(\text{sub}_{\{\mu_1-1, \mu_2-2, \dots, \mu_r-r\}}^{\{g_1, g_2, \dots, g_r\}} (\mathbf{H}(x \mid a)) \right) \\ & \quad \cdot \det \left(\text{sub}_{\{g_1, g_2, \dots, g_r\}}^{\{\lambda_1-1, \lambda_2-2, \dots, \lambda_r-r\}} (\mathbf{H}(y \mid \tau^n a)) \right), \end{aligned}$$

where the sum ranges over all strictly increasing r -tuples $(g_1 < g_2 < \dots < g_r)$ of integers. Using (6.12)(ii), we can rewrite this as

$$\begin{aligned} & \det \left(\text{sub}_{\{\mu_1-1, \mu_2-2, \dots, \mu_r-r\}}^{\{\lambda_1-1, \lambda_2-2, \dots, \lambda_r-r\}} (\mathbf{H}(xy \mid a)) \right) \\ &= \sum_{g_1 < g_2 < \dots < g_r} \det \left(\text{sub}_{\{\mu_1-1, \mu_2-2, \dots, \mu_r-r\}}^{\{g_1, g_2, \dots, g_r\}} (\mathbf{H}(x \mid a)) \right) \\ & \quad \cdot \det \left(\text{sub}_{\{g_1, g_2, \dots, g_r\}}^{\{\lambda_1-1, \lambda_2-2, \dots, \lambda_r-r\}} (\mathbf{H}(y \mid \tau^n a)) \right) \\ &= \sum_{\substack{S \text{ is an } r\text{-element} \\ \text{set of integers}}} \det \left(\text{sub}_S^S (\mathbf{H}(x \mid a)) \right) \\ & \quad \cdot \det \left(\text{sub}_S^{\{\lambda_1-1, \lambda_2-2, \dots, \lambda_r-r\}} (\mathbf{H}(y \mid \tau^n a)) \right) \\ &= \sum_{\substack{\nu \text{ is a partition} \\ \text{of length } \leq r}} \det \left(\text{sub}_{\{\mu_1-1, \mu_2-2, \dots, \mu_r-r\}}^{\{\nu_1-1, \nu_2-2, \dots, \nu_r-r\}} (\mathbf{H}(x \mid a)) \right) \\ & \quad \cdot \det \left(\text{sub}_{\{\nu_1-1, \nu_2-2, \dots, \nu_r-r\}}^{\{\lambda_1-1, \lambda_2-2, \dots, \lambda_r-r\}} (\mathbf{H}(y \mid \tau^n a)) \right) \end{aligned} \tag{27}$$

(here, we have reindexed the sum, since each r -element set S of integers can be written uniquely in the form $\{\nu_1-1, \nu_2-2, \dots, \nu_r-r\}$ for a partition ν of length $\leq r$, and conversely, any set of the latter form is an r -element set of integers).

However, in the second sentence of the proof of (6.13), it was said that

$$s_{\lambda/\mu}(x, y \mid a) = \det \left(\text{sub}_{\{\mu_1-1, \mu_2-2, \dots, \mu_r-r\}}^{\{\lambda_1-1, \lambda_2-2, \dots, \lambda_r-r\}} (\mathbf{H}(xy \mid a)) \right).$$

Similarly, for any partition ν of length $\leq n$, we have

$$s_{\nu/\mu}(x \mid a) = \det \left(\text{sub}_{\{\mu_1-1, \mu_2-2, \dots, \mu_r-r\}}^{\{\nu_1-1, \nu_2-2, \dots, \nu_r-r\}} (\mathbf{H}(x \mid a)) \right)$$

and

$$s_{\lambda/\nu}(y \mid \tau^n a) = \det \left(\text{sub}_{\{\nu_1-1, \nu_2-2, \dots, \nu_r-r\}}^{\{\lambda_1-1, \lambda_2-2, \dots, \lambda_r-r\}} (\mathbf{H}(y \mid \tau^n a)) \right).$$

In view of these three equalities, we can rewrite (27) as

$$\begin{aligned} s_{\lambda/\mu}(x, y \mid a) &= \sum_{\substack{\nu \text{ is a partition} \\ \text{of length } \leq r}} s_{\nu/\mu}(x \mid a) s_{\lambda/\nu}(y \mid \tau^n a) \\ &= \sum_{\substack{\nu \text{ is a partition}}} s_{\nu/\mu}(x \mid a) s_{\lambda/\nu}(y \mid \tau^n a) \end{aligned}$$

(here, we have removed the condition "of length $\leq r$ " from the sum; this does not change the sum, since all newly introduced addends are zero³). This proves (6.13).

10. **page 19, the paragraph containing (6.14):** Replace "Let $x^{(i)}, \dots, x^{(n)}$ be" by "Let $x^{(1)}, \dots, x^{(n)}$ be".
11. **page 19, the paragraph containing (6.14):** Replace "where $x^{(i)} = (x_1^{(1)}, \dots, x_{r_i}^{(i)})$ " by "where $x^{(i)} = (x_1^{(i)}, \dots, x_{r_i}^{(i)})$ ".
12. **page 19, (6.14):** On the left hand side of (6.14), replace " $x^{(i)}, \dots, x^{(n)}$ " by " $x^{(1)}, \dots, x^{(n)}$ ".
13. **page 19, last paragraph:** Let me explain why

$$s_{\lambda/\mu}(x \mid a) = \prod_{i \geq 1} h_{\lambda_i - \mu_i}(x \mid \tau^{\mu_i - i + 1} a) \quad (28)$$

when $\lambda - \mu$ is a horizontal strip:

Assume that $\lambda - \mu$ is a horizontal strip. Write the partitions λ and μ as $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_q)$, where $q \in \mathbb{N}$ is sufficiently large (namely, $q \geq \max(\ell(\lambda), \ell(\mu))$). Then, for any $i, j \in [q]$ satisfying $i > j$, we have

$$h_{\lambda_i - \mu_j - i + j}(x \mid \tau^{\mu_j - j + 1} a) = 0. \quad (29)$$

(*Proof:* Let $i, j \in [q]$ satisfy $i > j$. Thus, $j < i$, so that $\lambda_j \geq \lambda_i$ (since $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$) and $\mu_j \geq \mu_i$ (similarly). In other words, $\lambda_i \leq \lambda_j$ and $\mu_i \leq \mu_j$.

The skew partition $\lambda - \mu$ is a horizontal strip, i.e., contains no two cells in the same column. However, if we had $\mu_j < \lambda_i$, then the two distinct cells (i, λ_i) and (j, λ_i) would both belong to $\lambda - \mu$ (indeed, we would have $(i, \lambda_i) \in \lambda - \mu$ because of $\mu_i \leq \mu_j < \lambda_i \leq \lambda_i$, and we would have $(j, \lambda_i) \in \lambda - \mu$ because of $\mu_j < \lambda_i \leq \lambda_j$), which would contradict the preceding sentence (since these two cells clearly lie in the same column). Thus, we cannot have $\mu_j < \lambda_i$. In other words, we have $\mu_j \geq \lambda_i$. Hence, $\lambda_i -$

³*Proof.* Let ν is a partition of length $> r$. We must show that $s_{\nu/\mu}(x \mid a) s_{\lambda/\nu}(y \mid \tau^n a) = 0$.

Since ν has length $> r$, we have $\ell(\nu) > r$. Thus, $\nu'_1 = \ell(\nu) > r \geq \max(\ell(\lambda), \ell(\mu)) \geq \ell(\lambda) = \lambda'_1$ and therefore $0 > \lambda'_1 - \nu'_1$. Hence, we don't have $0 \leq \lambda'_1 - \nu'_1 \leq n$. Thus, we don't have $0 \leq \lambda'_i - \nu'_i \leq n$ for all i (since this inequality fails for $i = 1$). Thus, (6.10) (applied to $\nu, \tau^n a$ and y instead of μ, a and x) yields $s_{\lambda/\nu}(y \mid \tau^n a) = 0$. Therefore, $s_{\nu/\mu}(x \mid a) \underbrace{s_{\lambda/\nu}(y \mid \tau^n a)}_{=0} = 0$ as well.

$\underbrace{\mu_j}_{\geq \lambda_i} - \underbrace{i}_{>j} + j < \lambda_i - \lambda_i - j + j = 0$, and therefore $h_{\lambda_i - \mu_j - i + j}(x \mid \tau^{\mu_j - j + 1} a) = 0$ (since $h_k(x \mid \tau^{\mu_j - j + 1} a) = 0$ for any $k < 0$). This proves (29).)

Now, (29) shows that the matrix $(h_{\lambda_i - \mu_j - i + j}(x \mid \tau^{\mu_j - j + 1} a))_{i,j \in [q]}$ is upper-triangular. Hence, its determinant is the product of its diagonal entries. In other words,

$$\begin{aligned} & \det(h_{\lambda_i - \mu_j - i + j}(x \mid \tau^{\mu_j - j + 1} a))_{i,j \in [q]} \\ &= \prod_{i=1}^q \underbrace{h_{\lambda_i - \mu_i - i + i}(x \mid \tau^{\mu_i - i + 1} a)}_{\substack{= h_{\lambda_i - \mu_i}(x \mid \tau^{\mu_i - i + 1} a) \\ (\text{since } \lambda_i - \mu_i - i + i = \lambda_i - \mu_i)}} \\ &= \prod_{i=1}^q h_{\lambda_i - \mu_i}(x \mid \tau^{\mu_i - i + 1} a). \end{aligned}$$

In view of (6.8), we can rewrite this as

$$s_{\lambda/\mu}(x \mid a) = \prod_{i=1}^q h_{\lambda_i - \mu_i}(x \mid \tau^{\mu_i - i + 1} a). \quad (30)$$

This is a finite product, but we can extend it to an infinite product over all $i \geq 1$; this will not change the value of the product, since all the newly inserted factors $h_{\lambda_i - \mu_i}(x \mid \tau^{\mu_i - i + 1} a)$ for $i > q$ will equal 1 (because if $i > q$, then $\lambda_i = 0$ and $\mu_i = 0$ and therefore $\lambda_i - \mu_i = 0 - 0 = 0$, so that $h_{\lambda_i - \mu_i}(x \mid \tau^{\mu_i - i + 1} a) = h_0(x \mid \tau^{\mu_i - i + 1} a) = 1$). Hence, (30) can be rewritten as

$$s_{\lambda/\mu}(x \mid a) = \prod_{i \geq 1} h_{\lambda_i - \mu_i}(x \mid \tau^{\mu_i - i + 1} a).$$

This proves (28). ■

14. **page 19, last paragraph:** At the end of the last display on this page, there is a period. This period should be a comma.
15. **page 20, proof of (6.17):** Replace " $A_{\delta_{m+n}}(x, y)$ " by " $A_{\delta_{m+n}}(x, y \mid a)$ ". Likewise, replace " $A_{\delta_n}(x)$ " by " $A_{\delta_n}(x \mid a)$ ". Likewise, replace " $A_{\delta_m}(y)$ " by " $A_{\delta_m}(y \mid a)$ ".
16. **page 20, proof of (6.17):** The equality

$$A_{\delta_{m+n}}(x, y) = \sum_{\lambda \subset (m^n)} (-1)^{|\widehat{\lambda}|} A_{\lambda + \delta_n}(x) A_{\widehat{\lambda}' + \delta_m}(y)$$

should be

$$A_{\delta_{m+n}}(x, y \mid a) = \sum_{\lambda \subset (m^n)} (-1)^{|\widehat{\lambda}|} A_{\lambda + \delta_n}(x \mid a) A_{\widehat{\lambda}' + \delta_m}(y \mid a) \quad (31)$$

instead.

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