Notes on Schubert Polynomials
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Errata and addenda by Darij Grinberg

I will refer to the results appearing in the book "Notes on Schubert Polynomials" by the numbers under which they appear in this book. All page numbers are from the LaTeX-recompiled version of 19 February 2025, not from the original 1991 edition.

I have read pages 1–12, 23–45, 70–77 and 103–105 of the book so far.

The list below contains both actual corrections and what I believe to be clarifications and pertinent comments. I have not tried to separate the former from the latter, as I suspect that the precise boundary is in the eyes of the beholder.

B. Errata

1. **Page 1, line 1 of the main text:** "of degree n" $\to$ "of degree n". (That is, the "n" should be in mathmode. This typo does not appear in the original printed version.)

2. **Page 1, second displayed equation from the bottom:** The expression "$(w(1), \ldots, w(r+1), w(r), \ldots, w(n))$" could be misconstrued. Its correct interpretation is $(w(1), \ldots, w(r-1), w(r+1), w(r), w(r+2), \ldots, w(n))$ (that is, the result of swapping the r-th and $(r+1)$-st entries of the n-tuple $(w(1), w(2), \ldots, w(n))$).

3. **Page 2, line 1:** The spacing in "$\ell(w) = \operatorname{Card} I(w)$" is off. (This typo does not appear in the original printed version.)

4. **Page 2, proof of (1.4):** A whitespace is missing in "$\ell(w)$that". (This typo does not appear in the original printed version. I suggest fixing it throughout the book, possibly by a regex search for whatever caused it. My guess is that some linebreaks or whitespaces got hurt as the file was moved across operating systems.)

5. **Page 2, proof of (1.6) (iv):** It is not true that "$I(ww_0)$ is the complement of $I(w)$ in $I(w_0)$". Instead, $I(w_0w)$ is the complement of $I(w)$ in $I(w_0)$. Thus, the roles of w_0w and ww_0 in this proof should be swapped (except in the equality $ww_0 = (w(n), w(n-1), \ldots, w(1))$, which is correct but not relevant here).

6. **Page 3, before (1.7):** After "Such sequences are called *reduced words* for w", add ", and the corresponding expressions $w = s_{a_1} \ldots s_{a_p}$ are called *reduced expressions* for w". This notation is used several times below.

7. **Page 4, (1.9):** A whitespace is missing in "(p,q)such that".

8. **Page 4, proof of (1.9):** "it follows that $v = s_{a_1} \cdots \widehat{s}_{a_p} \cdots s_{a_{q-1}}$" should be "it follows that $v = s_{a_1} \cdots \widehat{s}_{a_p} \cdots s_{a_{q-1}} s_{a_q}$".

9. **Page 4, before (1.11):** "if and only if $e_{ij} = 0$" should be "if and only if $e_{ij}(w) = 0$".

10. **Page 4, proof of (1.11):** "we have $(i,j) = s_{a_p} \cdots s_{a_{r+1}} (a_r, a_{r+1})$" should be "we have $(i,j) = s_{a_p} \cdots s_{a_{r+1}} (a_r, a_r + 1)$". (NB: Only the last a_{r+1} should be replaced by $a_r + 1$.)

11. **Pages 5–6, proof of (1.15):** "it follows from (1.13) that $s_i v = s_i s_{a_1} \cdots \widehat{s}_{a_r} \cdots s_{a_p}$" should be "it follows from (1.13) that there is a reduced expression $s_i v = s_i s_{a_1} \cdots \widehat{s}_{a_r} \cdots s_{a_p}$". The reducedness of this expression is important, since it is (tacitly!) used later in showing that $v = s_{a_1} \cdots \widehat{s}_{a_r} \cdots s_{a_p}$ is a reduced expression.

12. **Page 6, proof of (1.16):** "should have $s_i v_j = v_j - 1$" should be "should have $s_i v_j = v_{j-1}$".

13. **Page 6, proof of (1.16):** In equality (1), replace the "$\neq$" sign by a "$\rightarrow$" sign.

14. **Page 6, proof of (1.16):** "This shows that (i) and (iii) are equivalent" should be "This shows that (i) and (ii) are equivalent".

15. **Page 7, proof of (1.16):** "by $(a) \Longleftrightarrow (b)$" should be "by (i)$\Longleftrightarrow$(ii)".

16. **Page 7, (1.17):** In condition (iii), replace "exits" by "exists".

17. **Page 7, proof of (1.17):** Add a whitespace after "(b)".

18. **Page 7, proof of (1.17):** In Case (b), replace "$p - 1 + q$" by "$q - 1 + p$". (Of course, the original "$p - 1 + q$" is equally valid, but less clear in its provenance.)

19. **Page 7, (1.19):** "then" should be "Then".

20. **Page 8, proof of (1.19):** Why is "$v' \leq w$ by the induction hypothesis"? This would require showing that $K(v') \leq K(w)$, but this is not done anywhere in this proof.

    What looks like a complete proof of (1.19) (I haven't checked) can be found in Proposition 3.2 of

    > Yufei Zhao, *On the Bruhat order of the symmetric group and its shellability*, `https://web.mit.edu/yufeiz/www/papers/bruhat.pdf` .

    An even stronger result is proved in Theorem 2.6.3 of

Anders Björner, Francesco Brenti, *Combinatorics of Coxeter groups*, Springer 2005.

(To be specific, we only need the direction (ii)$\Longrightarrow$(i) of this stronger result, and only in the weaker form in which "for all $k \in D_R(x)$" is replaced by "for all $k \in \{1, 2, \ldots, n\}$".)

21. **Page 8, after (1.20):** "either $i \geq w^{-1}$ or" should be "either $i \geq w^{-1}j$ or".

22. **Page 9, (1.21) (i):** Add a whitespace in "$D\left(w^{-1}\right)$is".

23. **Page 9, (1.22):** Remove the extraneous closing parentheses in "$\ell(ws_r) > \ell(w))$" and "$\ell(s_r w) > \ell(w))$".

24. **Page 10, (1.23) (ii):** Add a comma before "a_p" in "$(a_1, \ldots a_p)$".

25. **Page 10, proof of (1.23):** Replace "from (1.21)" by "from (1.22)".

26. **Page 11:** The notation λ' stands for the conjugate partition of λ. This should probably be said somewhere.

27. **Page 11, (1.27):** You are using the notation $D(\lambda)$ for the Young diagram $\left\{(i, j) \in \{1, 2, 3, \ldots\}^2 \mid j \leq \lambda_i\right\}$ of a partition λ. This should also be defined.

28. **Page 11, proof of (1.27):** In the condition "there exist $a, \beta, c, \delta \in [1, n]$ such that $a < c$, $\beta < \delta$ and $(a, \beta), (c, \delta)$ belong to $D(w)$, whilst $(a, \delta), (c, \beta)$ do not", the inequalities "$a < c$, $\beta < \delta$" cannot necessarily be guaranteed to hold at the same time. Instead, you can either have $a \neq c$ and $\beta \neq \delta$ (in this form, the criterion is symmetric in rows vs. columns, which yields the equivalence (i)$\Longleftrightarrow$(ii)), or you can ensure that $a \neq c$ and $\beta < \delta$ (this is used in the proof of (ii)$\Longleftrightarrow$(iii)).

29. **Page 11, proof of (1.27):** The proof of (v)$\Longrightarrow$(iv) here tacitly relies on the following lemma (which ensures that the "suitable permutations" turn $D(w)$ into the actual diagram $D(\lambda)$ rather than merely into a diagram that has the same numbers of cells in each row and in each column):

    Lemma. Let λ be a partition. Let U be a subset of $\{1, 2, 3, \ldots\}^2$ such that the number of elements in the i-th row of U is λ_i for each i, and such that the number of elements in the j-th column of U is λ'_j for each j. Then, $U = D(\lambda)$.

    This lemma is fairly well-known when restated in the language of symmetric functions as "the elementary symmetric function $e_{\lambda'}$ has leading monomial x^λ and x^λ-coefficient 1". In this form, it is very easy to prove (since the leading terms of polynomials multiply when we multiply the polynomials), but it is less obvious when stated in the above combinatorial form.

30. **Page 11:** "it it satisfies" should be "if it satisfies".

31. **Page 12, (1.28) (iii):** This claim is false in general (and its proof appears to suffer from a confusion between $I(w)$ and $D(w)$). A counterexample is $w = (2, 1, 4, 3) \in S_4$, in which case $\overline{w} = w$ but $\lambda(w)$ is not self-conjugate.

    However, what is true is that
    $$\lambda(\overline{w}) = \lambda\left(w^{-1}\right) \qquad \text{for each } w \in S_n.$$
    This is because each $i \in \{1, 2, \ldots, n\}$ satisfies $c_{n+1-i}(\overline{w}) = c_{w(i)}\left(w^{-1}\right)$, as can be easily verified by comparing the definitions of both sides. This (combined with (1.27)(v)) shows that the original claim $\lambda(\overline{w}) = \lambda(w)'$ is true if and only if w is vexillary.

32. **Page 12, proof of (1.30):** I don't understand why "Then we have $w(k) \leq \lambda_{i-1}$ for $1 \leq k \leq i-1$, and $w(k) = \lambda_{i-1}$ for some $k \leq i-1$".

    However, I see a different way to prove the implication (i)$\Longrightarrow$(iii). Namely, let us add two more equivalent conditions to (1.30):

    (iv) there exist no $a \geq 1$ and $c > a+1$ such that $w(a) < w(c) < w(a+1)$;

    (v) there exist no $c > b > a \geq 1$ such that $w(a) < w(c) < w(b)$ (in modern parlance, this is called "w is 132-avoiding").

    The implication (i)$\Longrightarrow$(iv) follows easily from a closer look at (1.24) (indeed, for a given $a \geq 1$, the existence of a $c > a+1$ satisfying $w(a) < w(c) < w(a+1)$ is equivalent to $c_a(w) < c_{a+1}(w)$). The implication (iv)$\Longrightarrow$(v) is fairly easy (fix a $c > 1$ such that some $a < b < c$ satisfy $w(a) < w(c) < w(b)$; then pick such a, b with minimum $b - a$, and argue that b must be $a+1$). The implication (v)$\Longrightarrow$(i) follows easily from the definition of $D(w)$. Combining the three implications, (i)$\Longrightarrow$(iii) follows.

33. **Page 13, (1.31):** After "Let $w \in S_\infty$", add "and $r \geq 0$".

34. **Page 13, proof of (1.31):** "By (1.15)" should be "By (1.24)".

35. **Page 13:** "sequence c" should be "sequence c" (mathmode).

36. **Page 16, (1.38):** Insert whitespace in "$[1, f_1]$;then".

37. **Page 18, large picture, last diagram on the second line:** This has a circle too much. Namely, the rightmost circle on the second line from the bottom should not be a circle but rather start a fat hook.

38. **Page 18, large picture, middle diagram on the bottom line:** The code here should be "30221", not "302201". (The "302201" on the bottom right, however, is correct.)

39. **Page 19:** Insert whitespace in "λ',by (1.27)".

40. **Page 20, footnote:** Indeed, $N_8 = 24553$. For more terms of the sequence, or rather for the numbers $V_n = n! - N_n$ of vexillary permutations, see the OEIS (https://oeis.org/A005802).

41. **Page 22, Added in proof:** A proof of (1) can now be found in: Julian West, *Generating trees and the Catalan and Schröder numbers*, Discrete Mathematics **146** (1995), pp. 247–262. (More precisely, combine his Corollary 3.5 with the well-known enumeration of 1234-avoiding permutations in terms of standard tableaux. See Exercise 7.22 (e) of Richard Stanley, *Enumerative Combinatorics, volume 2*, 2nd edition 2024.)

42. **Page 23:** "If f is a function" should probably be "If f is a rational function" (you don't want actual functions here, lest it have a hole at $x = y$).

43. **Page 23 onward:** The symbols "s_{xy}" and "t_{xy}" mean the same thing here. The only difference is that t_{xy} denotes the transposition in S_n (or S_∞) while s_{xy} denotes its action on rational functions; but this distinction is not made for other permutations (such as s_i).

44. **Page 23:** In the displayed equation "$\partial_{xy} f = (x-y)^{-1} \left(1 - s_{xy}\right)$", remove the "$f$".

45. **Page 26, proof of (2.10):** "in ∂_0 is" should be "in ∂_{w_0} is".

46. **Page 26, proof of (2.10):** The claim that "$c_{w_0} = \epsilon(w_0)\, a_\delta^{-1}$" behooves a more detailed justification. It is a particular case (for $w = w_0$) of the following lemma:

    Lemma (also mentioned on pages 28–29): For each $w \in S_n$, we can write ∂_w as

    $$\partial_w = \epsilon(w) \prod_{(i,j) \in I\left(w^{-1}\right)} \left(x_i - x_j\right)^{-1} \cdot w + \sum_{\substack{v < w \\ \text{(in Bruhat order)}}} d_v v$$

    for some rational functions d_v.

    This lemma can be proved by induction on $\ell(w)$. (The induction step proceeds from ws_r to w, using (1.17) to transform the sum as well as a fact that is similar to (1.2): If $w \in S_n$ and $1 \leq r \leq n-1$, then

    $$I(s_r w) = \begin{cases} I(w) \cup \left\{w^{-1}(r), w^{-1}(r+1)\right\}, & \text{if } w^{-1}(r) < w^{-1}(r+1); \\ I(w) - \left\{w^{-1}(r+1), w^{-1}(r)\right\}, & \text{if } w^{-1}(r) > w^{-1}(r+1). \end{cases}$$

    Note that this proof does not require the specific sequence (2.9); it just needs the fact that $I(w_0)$ is the set of all pairs (i,j) with $1 \leq i < j \leq n$.)

47. **Page 27, between (2.11) and (2.12):** You write: "Thus ∂_{w_0} is a Λ_n-linear mapping of P_n onto Λ_n". Strictly speaking, the Λ_n-linearity of ∂_{w_0} follows not from (2.11) but from (2.10).

48. **Page 27, (2.13):** It is worth saying that the $\alpha_1, \alpha_2, \ldots$ are supposed to be scalars.

49. **Page 27, proof of (2.13):** In the first display, I would replace "$s_{a_1} \cdots \partial_{a_r} \cdots s_{a_p}$" by "$s_{a_1} \cdots s_{a_{r-1}} \partial_{a_r} s_{a_{r+1}} \cdots s_{a_p}$" for clarity.

50. **Page 27, proof of (2.13):** Replace "where $wt = s_{a_p} \cdots \widehat{s}_{a_r} \cdots s_{a_p}$" by "where $wt = s_{a_1} \cdots \widehat{s}_{a_r} \cdots s_{a_p}$".

51. **Page 27, proof of (2.13):** Replace "(a_r, a_{r+1})" by "$(a_r, a_r + 1)$".

52. **Page 27, proof of (2.13):** The last step (rewriting the $\sum\limits_{r=1}^{p}$ sum as a sum over pairs $(i < j)$) relies on (1.7).

53. **Page 28, proof of (2.16) and (2.16′):** I think "From this and (2.10)" should be "From this and (2.9) and (2.2′)". More precisely, the preceding equality (applied many times) shows that

$$\begin{aligned}
&(\pi_1 \cdots \pi_{n-1})(\pi_1 \cdots \pi_{n-2}) \cdots (\pi_1\pi_2\pi_3)(\pi_1\pi_2)\,\pi_1 f \\
&= (\partial_1 \cdots \partial_{n-1})\left((x_1 \cdots x_{n-1})(\partial_1 \cdots \partial_{n-2})\left((x_1 \cdots x_{n-2}) \cdots\right.\right. \\
&\qquad\qquad (\partial_1\partial_2\partial_3)\left((x_1x_2x_3)(\partial_1\partial_2)\left((x_1x_2)\,\partial_1\,(x_1 f)\right)\right)\left.\left.\right)\right) \\
&= \underbrace{(\partial_1 \cdots \partial_{n-1})(\partial_1 \cdots \partial_{n-2}) \cdots (\partial_1\partial_2\partial_3)(\partial_1\partial_2)\,\partial_1}_{\substack{=\partial_{w_0} \\ \text{(by (2.9))}}} \\
&\qquad\qquad \left(\underbrace{(x_1 \cdots x_{n-1})(x_1 \cdots x_{n-2}) \cdots (x_1x_2x_3)(x_1x_2)\,x_1}_{=x^\delta} f\right) \\
&\qquad\qquad\qquad \left(\begin{array}{c}\text{here, we used (2.2′) to commute the } x\text{-products past} \\ \text{the } \partial\text{-products, since each } x\text{-product } x_1 \cdots x_k \\ \text{is symmetric in } x_1, x_2, \ldots, x_k \text{ and thus commutes with} \\ \text{any product of } \partial_i\text{'s for } i < k \text{ (by (2.2′))}\end{array}\right) \\
&= \partial_{w_0}\left(x^\delta f\right).
\end{aligned}$$

54. **Page 29:** When you say "It follows that the ∂_w are linearly independent over the field of rational functions $\mathbf{Q}_\infty$", you are using the fact that the (actions of the) permutations $w \in S_\infty$ are linearly independent over this field. This fact is a consequence of Dedekind's linear independence of characters (Theorem 12 in: Emil Artin, *Galois Theory*, Notre Dame Mathematical Lectures **2**, second edition, 1971; the theorem must be applied to G being the multiplicative group of $\mathbf{Q}_\infty$ and F being the field $\mathbf{Q}_\infty$).

55. **Page 29:** Add a period at the end of the first formula for $\partial_w \circ \mu$ (the one before "On expansion").

56. **Page 30, Example 3:** There is a lone period that somehow ended up on its own line in the align environment.

57. **Page 30, (2.18):** Replace "$\partial_{w/u} g$" by "$\partial_{v/u} g$" on the right hand side.

58. **Page 31, proof of (2.18):** Replace "$v\partial_{w/u}(f)$" by "$v\partial_{w/v}(f)$" (twice).

59. **Page 32:** There is no need to "work in an arbitrary λ-ring R". Everything in Chapter III makes sense in the following more general setting: Let R be a commutative ring, and let V be a subgroup of the additive group $(R, +, 0)$. For each $r \in \mathbb{Z}$, let $e_r : V \to R$ be a map such that all $X \in V$ satisfy $e_0(X) = 1$ and $e_1(X) = X$. Assume that $e_r(X) = 0$ for all $r < 0$ and $X \in V$. Assume furthermore that

$$e_r(X+Y) = \sum_{p+q=r} e_p(X) e_q(Y) \qquad \text{for all } X, Y \in V \text{ and } r \geq 0.$$

I shall refer to this setting as the "VR-setting".

The VR-setting is all that is needed in order to make the arguments of Chapter III work. Having $V = R$ (as in the case of a λ-ring), or having additional axioms for $e_r(XY)$ and $e_r(e_s(X))$ (as in the case of a λ-ring, at least according to the definition in Knutson's 1973 LNM book or in Berthelot's SGA 6 exposé) is entirely unnecessary here.

Combinatorialists often use the instance of the VR-setting when R is the ring of formal power series $\mathbf{Z}[[x_1, x_2, x_3, \ldots]]$, and V is the R (as an additive group); the maps $e_r : V \to R$ are given by the following rule: If $f \in V$ is any formal power series, then we can write f as $f = (\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 + \cdots) - (\mathfrak{n}_1 + \mathfrak{n}_2 + \mathfrak{n}_3 + \cdots)$ for two countable (possibly finite) sequences $(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots)$ and $(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \ldots)$ of monomials in the $x_1, x_2, x_3, \ldots$, and then we set

$$e_r(f) = \sum_{p+q=r} e_p(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots) \cdot (-1)^q h_q(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \ldots),$$

where $e_p(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots) = \sum\limits_{i_1 < i_2 < \cdots < i_p} \mathfrak{m}_{i_1} \mathfrak{m}_{i_2} \cdots \mathfrak{m}_{i_p}$ denotes the p-th elementary symmetric function in the $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots$, whereas $h_q(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \ldots) = \sum\limits_{i_1 \leq i_2 \leq \cdots \leq i_q} \mathfrak{n}_{i_1} \mathfrak{n}_{i_2} \cdots \mathfrak{n}_{i_q}$ denotes the q-th complete homogeneous symmetric function in the $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \ldots$.

Another instance of the VR-setting is obtained when both V and R are the polynomial ring $\mathbf{Z}[x_1, x_2, x_3, \ldots]$, and the maps e_r are defined as above. The same formula for $e_r(f)$ holds here, but now the sequences $(\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \ldots)$ and $(\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3, \ldots)$ are always finite. This instance is used in (3.9).

An even smaller instance of the VR-setting is when V and R are the ring of symmetric functions in $x_1, x_2, x_3, \ldots$; the maps e_r are still defined in the same way as above (but now the monomials no longer belong to V).

Returning to the general case of an arbitrary VR-setting (with V a group and R a ring and $e_r : V \to R$ being maps satisfying certain axioms), we define maps $h_r : V \to R$ for all $r \in \mathbb{Z}$ by setting

$$h_r(X) := (-1)^r e_r(-X) \qquad \text{for each } X \in V.$$

Then, it is very easy to see that

$$h_r(X+Y) = \sum_{p+q=r} h_p(X) h_q(Y) \qquad \text{for all } X, Y \in V \text{ and } r \geq 0.$$

It is also easy to see (by induction) that if $X \in V$ is any element satisfying $e_r(X) = 0$ for all $r > 1$, then $h_r(X) = X^r$ for all $r \geq 0$. This fact is used on page 35, in the proof of (3.5).

60. **Page 33, (3.3):** Add a comma before "X_n" in "$s_{\lambda/\mu}(X_1, \ldots X_n)$".

61. **Page 34, proof of (3.4):** Replace "$h_{i-j}(-Y_j)$" by "$h_{j-i}(-Y_j)$".

62. **Page 34, (3.5):** It is worth saying that "$(i,j) \in \lambda$" is shorthand for "$(i,j) \in D(\lambda)$".

63. **Page 35, proof of (3.5):** In "$\sum_{r\geq 0} (-1)^r e_r(Y_{\alpha_i}) x_i^{\alpha_i - r}$", the sum should range from $r = 0$ to α_i (not to ∞).

64. **Page 35, between (3.5′) and (3.5″):** "(and α is a partition λ)" should better be "(and α is a partition λ of length $\leq n$)".

65. **Page 35, (3.5″):** This equality requires $\ell(\lambda') \leq n$.

66. **Page 35, (3.6):** I would replace "Suppose that $i < j$" by "Suppose that $i \leq j$". This does not add any real generality, but makes the theorem easier to apply and also simplifies the induction proof.

67. **Page 35, proof of (3.6):** Replace "$rk(Y) \leq j - 1$" by "$rk(Y) \leq j - i$".

68. **Page 36, proof of (3.8):** In the second displayed equality, "$(1 \leq j \leq n)$" should be "$(1 \leq j \leq m)$".

69. **Page 36, proof of (3.8):** "the interval $[-m, n-1]$" should be "the interval $[-n, m-1]$".

70. **Page 37, proof of (3.8):** In the first display, replace "$-X_{\xi_i}$" by "$-X_{\xi_1}$".

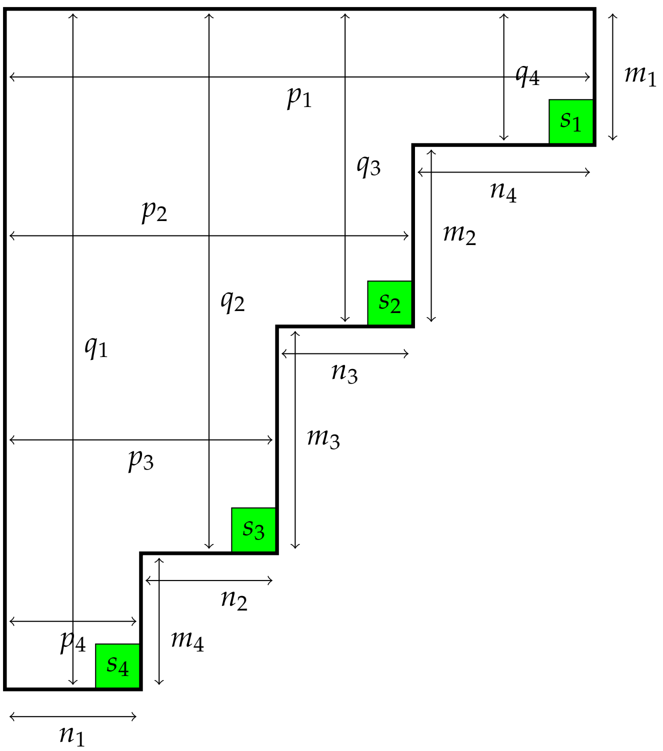
71. **Page 37, proof of (3.8):** Replace "$(h_{j-i}(X_{i+1}))_{-m\leq i,j\leq n-1}$" by "$(h_{j-i}(X_{i+1}))_{-n\leq i,j\leq m-1}$".

72. **Page 37, proof of (3.8):** In the last display, replace "$\left(-X_{\xi'_1}, \ldots, -X_{\xi'_m}\right)$" by "$\left(X_{-\xi'_1}, \ldots, X_{-\xi'_m}\right)$".

73. **Page 37, Remark:** "Hence (3.8) gives a duality theorem for the multi-Schur function $s_{\lambda/\mu}(Y_1, \ldots, Y_n)$ provided that $rk(Y_{i+1} - Y_i) \leq \lambda_i - \lambda_{i+1} + 1$ for $1 \leq i \leq n-1$" is perhaps an overstatement. It relies on being able to write each $Y_{i+1} - Y_i$ as a sum of $\lambda_i - \lambda_{i+1} + 1$ elements $X_m - X_{m-1}$ of rank ≤ 1 each; it is not clear to me how this can be ensured. There might be a splitting principle that allows for this, but even then, if the right hand side of (3.8) cannot be reexpressed in terms of the original Y_i's, then the usefulness of such a theorem is questionable.

74. **Page 37:** "becuase" should be "because".

75. **Pages 37–38, paragraph above (3.8′):** This becomes clearer when illustrated by a picture (showing the case $k = 4$, which however is pretty representative of the general case):



76. **Page 38, (3.8″):** Strictly speaking, this has only been proved under the assumption that each $Z_{i+1} - Z_i$ can be written as sum of at most $m_{i+1} + n_{k+1-i}$ elements of rank ≤ 1 each. This assumption is stronger than the given requirement $rk(Z_{i+1} - Z_i) \leq m_{i+1} + n_{k+1-i}$. From what I see, however, this stronger assumption is always satisfied when (3.8″) is applied in this book.

77. **Page 38, before (3.9):** "Let $X_i = x_1 + \cdots + x_i$ for each $i \geq 1$" should be "Let $X_i = x_1 + \cdots + x_i$ for each $i \geq 0$". (Otherwise, the "X_{i-1}" in (3.9) makes no sense.)

78. **Page 38:** It is worth warning the reader that the Schur function $s_\alpha (x_1, x_2, \ldots, x_n)$ used in Chapter II is not $s_\alpha (x_1, x_2, \ldots, x_n)$ in the notation of Chapter III (the latter is not even a symmetric polynomial!), but rather is $s_\alpha (X_n) = s_\alpha (x_1 + x_2 + \cdots + x_n)$ in the notation of Chapter III. (This is not obvious, but follows from the Jacobi–Trudi formula.)

79. **Page 39, (3.10):** This should also require $r_i > 0$ (otherwise, the operators make no sense).

80. **Page 39, proof of (3.10):** "By definition, we have $s_\alpha =$" should be "By definition, we have $s_\alpha (X_{r_1}, \ldots, X_{r_n}) =$".

81. **Page 39, proof of (3.8‴):** I think "equal to $(X_1 \cdots X_m)^n$" should be "equal to $(X_1 \cdots X_n)^m$".

82. **Page 39, proof of (3.11):** Replace "w_o" by "w_0" twice.

83. **Page 39, proof of (3.11):** The claim that "By (3.10), $\pi_1 \pi_2 \cdots \pi_{n-1}$ applied to $s_\alpha (X_1 + Z_1, \ldots, X_n + Z_n)$ will produce

$$s_\alpha (X_2 + Z_1, X_3 + Z_2, \ldots, X_n + Z_{n-1}, X_n + Z_n)$$

" is not as self-evident as it is being sold here. It does not follow from (3.10) directly, since there are no Z-variables in (3.10). Instead, this claim can be justified as follows: We have

$$s_\alpha (X_1 + Z_1, \ldots, X_n + Z_n) = \det \left(h_{\alpha_i - i + j} (X_i + Z_i) \right)_{1 \leq i, j \leq n}.$$

In the matrix on the right hand side, all entries except for those in the $(n-1)$-st row are symmetric in x_{n-1} and x_n, and thus can "pass through" the map π_{n-1} unchanged (this follows from the fact that $\pi_{n-1} (fg) = f \pi_{n-1} (g)$ whenever f is symmetric in x_{n-1} and x_n; this follows easily from (2.2′)), while the entries $h_{\alpha_{n-1} - (n-1) + j} (X_{n-1} + Z_i)$ in the $(n-1)$-st row are transformed by π_{n-1} into

$$\begin{aligned}
&\pi_{n-1} \underbrace{h_{\alpha_{n-1}-(n-1)+j}\left(X_{n-1}+Z_i\right)}_{=\sum_{p+q=\alpha_{n-1}-(n-1)+j} h_p(X_{n-1})h_q(Z_i)} \\
&= \sum_{p+q=\alpha_{n-1}-(n-1)+j} \underbrace{\pi_{n-1}\left(h_p\left(X_{n-1}\right)h_q\left(Z_i\right)\right)}_{\substack{=\pi_{n-1}\left(h_p(X_{n-1})\right)\cdot h_q(Z_i) \\ =h_p(X_n)\cdot h_q(Z_i) \\ \text{(since (3.9) yields } \pi_{n-1}\left(h_p(X_{n-1})\right)=h_p(X_n))}} \\
&= \sum_{p+q=\alpha_{n-1}-(n-1)+j} h_p\left(X_n\right)\cdot h_q\left(Z_i\right) \\
&= h_{\alpha_{n-1}-(n-1)+j}\left(X_n+Z_i\right).
\end{aligned}$$

Hence, applying π_{n-1} to the determinant of this matrix results in the same determinant, except that the X_{n-1}'s in the $(n-1)$-st row are turned into X_n's. That is, applying π_{n-1} to $s_\alpha\left(X_1+Z_1,\ldots,X_n+Z_n\right)$ turns the X_{n-1} in the second-to-last argument into an X_n. Subsequently applying π_{n-2} to the result then turns the X_{n-2} in the third-to-last argument into an X_{n-1} (for a similar reason). Subsequently applying π_{n-3} then turns the X_{n-3} in the fourth-to-last argument into an X_{n-2}. And so on, until we eventually conclude that $\pi_1\pi_2\cdots\pi_{n-1}$ applied to $s_\alpha\left(X_1+Z_1,\ldots,X_n+Z_n\right)$ results in each of the subscripts under $X_1,X_2,\ldots,X_{n-1}$ being increased by 1. That is, $\pi_1\pi_2\cdots\pi_{n-1}$ applied to $s_\alpha\left(X_1+Z_1,\ldots,X_n+Z_n\right)$ produces

$$s_\alpha\left(X_2+Z_1,X_3+Z_2,\ldots,X_n+Z_{n-1},X_n+Z_n\right).$$

The same reasoning is then applied again to conclude that $\pi_2\pi_3\cdots\pi_{n-1}$ applied to $s_\alpha\left(X_2+Z_1,X_3+Z_2,\ldots,X_n+Z_{n-1},X_n+Z_n\right)$ produces

$$s_\alpha\left(X_3+Z_1,X_4+Z_2,\ldots,X_n+Z_{n-2},X_n+Z_{n-1},X_n+Z_n\right),$$

and so on.

84. **Page 40, Remark:** This does indeed yield "independent proofs of (2.11) and (2.16′)" provided we know that the $s_\alpha\left(X_n\right)$ in the terminology of Chapter III (defined as $\det\left(h_{\alpha_i-i+j}\left(X_n\right)\right)_{1\le i,j\le n}$) agrees with the Schur polynomials $s_\alpha\left(x_1,x_2,\ldots,x_n\right)$ in the notations of Chapter II (defined, e.g., as the ratio of alternants $a_{\alpha+\delta}/a_\delta$, where $a_\beta := \det\left(x_i^{\beta_j}\right)_{1\le i,j\le n}$). The latter fact is known as the Jacobi–Trudi formula (or, rather, its generalization to indices in $\mathbf{Z}^n$, not just partitions). If we don't know this formula, then we instead obtain an independent proof of this formula.

85. **Page 40, Remark:** The words "(by linearity)" here are somewhat of an overstatement: By linearity, the formulas (2.11) and (2.16′) yield (2.10) and

(2.16) for polynomials only, not for all rational functions. However, the case of rational functions can easily be reduced to the case of polynomials using the (easy) fact that every rational function in $x_1, x_2, \ldots, x_n$ can be written as a ratio $\frac{f}{g}$ where f is a polynomial and g is a nonzero symmetric polynomial. (This is useful since symmetric polynomials "pass through" ∂_{w_0} and π_{w_0} undisturbed.)

86. **Page 40, (3.12):** The first displayed formula here could use some clarification. First, any pair $w = (u, v) \in S_m \times S_n$ acts on polynomials in the variables $x_1, \ldots, x_m, y_1, \ldots, y_n$ in the "obvious" way (i.e., by $w(x_i) = x_{u(i)}$ for all i and $w(y_j) = y_{v(j)}$ for all j). Second, "$f_\lambda(x, y) / D(x) D(y)$" is to be understood as "$f_\lambda(x, y) / (D(x) D(y))$".

87. **Page 40, proof of (3.12):** Replace "w_o" by "w_0".

88. **Page 40, proof of (3.12):** "and in view of (2.16)" could better be explained as "and in view of the fact that each $f \in P_n$ satisfies

$$\begin{aligned}
\pi_{w_0}(f) &= a_\delta^{-1} \sum_{w \in S_n} \epsilon(w)\, w\left(x^\delta f\right) \qquad \text{(by (2.16))} \\
&= \sum_{w \in S_n} \underbrace{\epsilon(w)\, a_\delta^{-1}}_{= w\left(a_\delta^{-1}\right)} w\left(x^\delta f\right) = \sum_{w \in S_n} \underbrace{w\left(a_\delta^{-1}\right) w\left(x^\delta f\right)}_{= w\left(a_\delta^{-1} x^\delta f\right)} \\
&= \sum_{w \in S_n} w \underbrace{\left(a_\delta^{-1} x^\delta f\right)}_{= f / \prod\limits_{1 \le i < j \le n} \left(1 - x_i^{-1} x_j\right)} = \sum_{w \in S_n} w\left(f / \prod_{1 \le i < j \le n} \left(1 - x_i^{-1} x_j\right)\right)
\end{aligned}$$

".

89. **Pages 40–41, proof of (3.12):** This proof appears to have a subtle flaw: Parts of it (specifically, the part where (3.11) is applied) only work when $n, m \ge r$.

    This flaw can be fixed with a bit of preparation. We define $\operatorname{zero}_{m+1} : P_{m+1} \to P_m$ to be the algebra morphism that substitutes 0 for x_{m+1} in a polynomial (i.e., it sends each $f(x_1, x_2, \ldots, x_{m+1}) \in P_{m+1}$ to $f(x_1, x_2, \ldots, x_m, 0) \in P_m$). We need two simple lemmas:

    (3.12'') Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{m+1}) \in \mathbf{N}^{m+1}$, and set $\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbf{N}^m$. Then,

$$\operatorname{zero}_{m+1}\left(s_\alpha(x_1, x_2, \ldots, x_{m+1})\right) = \begin{cases} s_{\alpha'}(x_1, x_2, \ldots, x_m), & \text{if } \alpha_{m+1} = 0; \\ 0, & \text{if } \alpha_{m+1} > 0. \end{cases}$$

To prove this, we recall the definition of Schur polynomials using alternants: For each $k \in \mathbf{N}$ and each $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbf{N}^k$, let $a_\beta := \det \left(x_i^{\beta_j} \right)_{1 \leq i,j \leq k}$ be the corresponding alternant. Also, let $\delta^{(k)}$ be the k-tuple $(k-1, k-2, \ldots, 1, 0)$ for each $k \in \mathbf{N}$. Then, the definition of Schur polynomials yields

$$s_\alpha = a_{\alpha+\delta^{(m+1)}} / a_{\delta^{(m+1)}} \qquad \text{and} \qquad s_{\alpha'} = a_{\alpha'+\delta^{(m)}} / a_{\delta^{(m)}}.$$

Now, as we apply the map $\operatorname{zero}_{m+1}$ (that is, substitute 0 for x_{m+1}), the denominator $a_{\delta^{(m+1)}} = \prod\limits_{1 \leq i < j \leq m+1} \left(x_i - x_j \right)$ of s_α becomes $x_1 x_2 \cdots x_m \cdot a_{\delta^{(m)}}$, whereas the numerator $a_{\alpha+\delta^{(m+1)}} = \det \left(x_i^{\alpha_j + (m+1-j)} \right)_{1 \leq i,j \leq m+1}$

- becomes 0 if $\alpha_{m+1} > 0$ (because in this case, the last row of the matrix $\left(x_i^{\alpha_j + (m+1-j)} \right)_{1 \leq i,j \leq m+1}$ becomes $(0, 0, \ldots, 0)$),
- and becomes $x_1 x_2 \cdots x_m a_{\alpha'}$ if $\alpha_{m+1} = 0$ (because in this case, the last row of the matrix $\left(x_i^{\alpha_j + (m+1-j)} \right)_{1 \leq i,j \leq m+1}$ becomes $(0, 0, \ldots, 0, 1)$, and so its determinant becomes

$$\begin{aligned} &\det \left(x_i^{\alpha_j + (m+1-j)} \right)_{1 \leq i,j \leq m} \qquad \text{(by Laplace expansion along the last row)} \\ &= \det \left(x_i \cdot x_i^{\alpha_j + (m-j)} \right)_{1 \leq i,j \leq m} = x_1 x_2 \cdots x_m \underbrace{\det \left(x_i^{\alpha_j + m - j} \right)_{1 \leq i,j \leq m}}_{= a_{\alpha'+\delta^{(m)}}} \\ &= x_1 x_2 \cdots x_m \cdot a_{\alpha'+\delta^{(m)}} \end{aligned}$$

).

Hence, the map $\operatorname{zero}_{m+1}$ takes $s_\alpha = a_{\alpha+\delta^{(m+1)}} / a_{\delta^{(m+1)}}$ to 0 if $\alpha_{m+1} > 0$, and to $\left(x_1 x_2 \cdots x_m \cdot a_{\alpha'+\delta^{(m)}} \right) / \left(x_1 x_2 \cdots x_m \cdot a_{\delta^{(m)}} \right) = a_{\alpha'+\delta^{(m)}} / a_{\delta^{(m)}} = s_{\alpha'}$ if $\alpha_{m+1} = 0$. This proves (3.12″).

(3.12‴) We have $\operatorname{zero}_{m+1} \circ \pi_{w_0^{(m+1)}} = \pi_{w_0^{(m)}}$ on P_m.

By linearity, it suffices to check this for each monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$; but in this case, it follows from (3.12′) and (2.16′). This proves (3.12‴).

Of course, (3.12‴) can be generalized to polynomials that involve several additional independent indeterminates $y_1, y_2, \ldots, y_n$ as well. The operator $\operatorname{zero}_{m+1}$ is supposed to leave these extra indeterminates undisturbed (i.e., treat them as constants).

Now, let us fix the proof of (3.12). It suffices to prove (3.12′), since all the reasoning before has been correct. Now, let us introduce a new variable x_{m+1} (replacing our convention $x_{m+1} = 0$), and write $f_\lambda \left(x^+, y \right) :=$

$\prod_{(i,j)\in\lambda}\left(x_i - y_j\right)$ (this looks the same as $f_\lambda\left(x,y\right)$, but keep in mind that x_{m+1} was 0 in the definition of $f_\lambda\left(x,y\right)$) and $\pi_{x^+} := \pi_{w_0^{(m+1)}}$ and $X_{m+1}^+ := x_1 + x_2 + \cdots + x_{m+1}$ (this is not X_{m+1} since X_{m+1} had $x_{m+1} = 0$). Clearly, $\operatorname{zero}_{m+1}\left(f_\lambda\left(x^+,y\right)\right) = f_\lambda\left(x,y\right)$ and $\operatorname{zero}_{m+1}\left(X_{m+1}^+\right) = X_{m+1} = X_m$. But (3.12‴) says that $\operatorname{zero}_{m+1}\circ\pi_{x^+} = \pi_x$ on P_m. Hence, $\operatorname{zero}_{m+1}\circ\pi_{x^+} = \pi_x$ holds also for polynomials in $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ (since the extra variables $y_1, y_2, \ldots, y_n$ "pass through" the π and $\operatorname{zero}_{m+1}$ operators unchanged). Hence,

$$\begin{aligned}
&\pi_y \underbrace{\pi_x}_{=\operatorname{zero}_{m+1}\circ\pi_{x^+}} f_\lambda\left(x,y\right) \\
&= \pi_y \operatorname{zero}_{m+1}\pi_{x^+} f_\lambda\left(x,y\right) \\
&= \operatorname{zero}_{m+1}\pi_y\pi_{x^+} f_\lambda\left(x,y\right) \qquad \left(\text{since } \pi_y \text{ and } \operatorname{zero}_{m+1} \text{ commute}\right) \\
&= \operatorname{zero}_{m+1}\left(\pi_y\pi_{x^+} f_\lambda\left(x^+,y\right)\right)
\end{aligned}$$

and furthermore

$$s_\lambda\left(X_m - Y_n\right) = \operatorname{zero}_{m+1}\left(s_\lambda\left(X_{m+1}^+ - Y_n\right)\right)$$

(since $\operatorname{zero}_{m+1}$ is an algebra morphism that takes X_{m+1}^+ to X_m and leaves Y_n unchanged). Therefore, in order to prove (3.12′), it suffices to prove the equality

$$s_\lambda\left(X_{m+1}^+ - Y_n\right) = \pi_y\pi_{x^+} f_\lambda\left(x^+,y\right),$$

since applying $\operatorname{zero}_{m+1}$ to this latter equality will then yield (3.12′). But this latter equality is nothing but the equality (3.12′) written down for $m+1$ instead of m. Hence, if we can prove (3.12′) for $m+1$ instead of m, then (3.12′) also holds for the original m.

Thus, we can replace m by $m+1$ in our proof of (3.12′). Iterating this argument, we can therefore replace m by any integer larger than m. In particular, we can WLOG assume that $m \geq \ell\left(\lambda\right)$ and $m \geq \ell\left(\lambda'\right)$. Likewise, we can WLOG assume that $n \geq \ell\left(\lambda\right)$ and $n \geq \ell\left(\lambda'\right)$. With these assumptions made, the proof in the text goes through.

90. **Page 40, proof of (3.12):** The derivation of (2) still needs a few more words. As we said above, we are working under the WLOG assumption that $m \geq \ell\left(\lambda\right)$ (among other things). That is, $m \geq r$. Hence, (1) becomes

$$f_\lambda\left(x,y\right) = s_\lambda\left(X_1 - Y_{\lambda_1}, \ldots, X_r - Y_{\lambda_r}\right) = s_\lambda\left(X_1 - Y_{\lambda_1}, \ldots, X_m - Y_{\lambda_m}\right)$$

by (3.3). Therefore,

$$\begin{aligned}
\pi_x f_\lambda\left(x,y\right) &= \pi_x s_\lambda\left(X_1 - Y_{\lambda_1}, \ldots, X_m - Y_{\lambda_m}\right) \\
&= s_\lambda\left(X_m - Y_{\lambda_1}, \ldots, X_m - Y_{\lambda_m}\right) \qquad \left(\text{by (3.11)}\right) \qquad (2') \\
&= s_\lambda\left(X_m - Y_{\lambda_1}, \ldots, X_m - Y_{\lambda_r}\right) \qquad \left(\text{by (3.3) again}\right).
\end{aligned}$$

This proves (2).

91. **Page 41, proof of (3.12):** I would replace "$rk\left(Y_{p_i} - Y_{p_{i+1}}\right) = p_i - p_{i+1}$" by "$rk\left(Y_{p_i} - Y_{p_{i+1}}\right) \leq p_i - p_{i+1}$".

92. **Page 41, proof of (3.12):** I don't understand the proof of (4) given here. But I have a simpler proof. More precisely, I will show not the equality (4), but rather the equality

$$\begin{aligned} & s_\lambda\left(X_m - Y_{\lambda_1}, \ldots, X_m - Y_{\lambda_m}\right) \\ & = (-1)^{|\lambda|} s_{\lambda'}\left(Y_1 - X_m, Y_2 - X_m, \ldots, Y_n - X_m\right), \end{aligned} \tag{4'}$$

which is equivalent to (4) by way of (3.3) but is even closer to what is actually needed for proving (3.12).

My proof of (4') relies on the duality theorem (3.8), not (3.8"). First, we need some combinatorics: For each $k \in \mathbf{Z}$, let $g_k \in \mathbf{N}$ denote the largest $i \geq 1$ such that $\lambda_i - i \geq k$ (or 0 if no such i exists). Visually speaking, this is the index of the bottommost row of $D(\lambda)$ that contains a cell in the k-th diagonal (i.e., in the main diagonal shifted by k upwards). It is easy to see that $g_{k-1} - g_k \in \{0, 1\}$ for each $k \in \mathbf{Z}$ (because if the diagram $D(\lambda)$ contains a cell (i, j) in the $(k-1)$-th diagonal, then it also contains the cell $(i-1, j)$ in the k-th diagonal, unless $i = 1$; and thus we must have $g_k \geq g_{k-1} - 1$; and a similar argument shows $g_{k-1} \geq g_k$). Now, set

$$Z_k := X_m - Y_{g_k} \qquad \text{for each } k \in \mathbf{Z}.$$

Then, $rk\left(Z_k - Z_{k-1}\right) \leq 1$ for all $k \in \mathbf{Z}$ (since $g_{k-1} - g_k \in \{0, 1\}$ for each $k \in \mathbf{Z}$). Therefore, (3.8) can be applied to Z_k instead of X_k. As a consequence, we obtain

$$s_\lambda\left(-Z_{\lambda_1 - 1}, \ldots, -Z_{\lambda_n - n}\right) = (-1)^{|\lambda|} s_{\lambda'}\left(Z_{1-\lambda_1'}, \ldots, Z_{m-\lambda_m'}\right). \tag{4''}$$

But each $1 \leq i \leq n$ satisfies $g_{\lambda_i - i} = i$ (since the cell (i, λ_i) of $D(\lambda)$ lies in the $(\lambda_i - i)$-th diagonal, but the next cell $(i+1, \lambda_i + 1)$ of this diagonal is already outside of $D(\lambda)$) and therefore

$$Z_{\lambda_i - i} = X_m - Y_{g_{\lambda_i - i}} = X_m - Y_i$$

and thus

$$-Z_{\lambda_i - i} = Y_i - X_m. \tag{5a}$$

Meanwhile, each $1 \leq j \leq m$ satisfies $g_{j-\lambda_j'} = \lambda_j'$ (since the cell $\left(\lambda_j', j\right)$ of $D(\lambda)$ lies in the $\left(j - \lambda_j'\right)$-th diagonal, but the next cell $\left(\lambda_j' + 1, j + 1\right)$ of this diagonal is already outside of $D(\lambda)$) and therefore

$$Z_{j-\lambda_j'} = X_m - Y_{g_{j-\lambda_j'}} = X_m - Y_{\lambda_j'}. \tag{5b}$$

In view of (5a) and (5b), we can rewrite (4″) as

$$\begin{aligned}
&s_\lambda \left(Y_1 - X_m, Y_2 - X_m, \ldots, Y_n - X_m\right) \\
&= (-1)^{|\lambda|} s_{\lambda'} \left(X_m - Y_{\lambda'_1}, X_m - Y_{\lambda'_2}, \ldots, X_m - Y_{\lambda'_m}\right).
\end{aligned}$$

In other words,

$$\begin{aligned}
&s_{\lambda'} \left(X_m - Y_{\lambda'_1}, X_m - Y_{\lambda'_2}, \ldots, X_m - Y_{\lambda'_m}\right) \\
&= (-1)^{|\lambda|} s_\lambda \left(Y_1 - X_m, Y_2 - X_m, \ldots, Y_n - X_m\right).
\end{aligned}$$

Substituting λ' for λ here (and recalling that $|\lambda'| = |\lambda|$ and $\lambda'' = \lambda$), we obtain precisely the desired equality (4′).

93. **Page 41, proof of (3.12):** The last equality sign might use some explanation: It is obtained by applying (3.8) to the elements $X_k := Y_n - X_m$ (for all $k \in \mathbf{Z}$).

94. **Page 42, proof of (4.2):** On the second-to-last line of the page, after "if $\ell(w) - \ell(v)$", add "$= \ell\left(wv^{-1}\right)$".

95. **Page 43, proof of (4.3) (ii):** Replace "$\max(\beta_i, \beta_{i+1}) \leq \max(\alpha_i, \alpha_{i+1}) - 1 \leq n - i - 1$" by "$\max(\beta_r, \beta_{r+1}) \leq \max(\alpha_r, \alpha_{r+1}) - 1 \leq n - r - 1$".

96. **Page 43, proof of (4.3) (ii):** The claim that $\mathfrak{S}_w$ is nonzero is not proved here. (But it follows easily from part (i): Indeed, $\ell(w_0) = \ell(w) + \ell\left(w^{-1}w_0\right)$, thus $\partial_{w_0} = \partial_w \partial_{w^{-1}w_0}$, and therefore (4.1) yields $\partial_w \mathfrak{S}_w = \underbrace{\partial_w \partial_{w^{-1}w_0}}_{=\partial_{w_0}} x^\delta = \partial_{w_0} x^\delta = \mathfrak{S}_1 = 1$ (by part (i)), and thus $\mathfrak{S}_w \neq 0$.)

97. **Page 43, proof of (4.3) (iv):** The claim that $\mathfrak{S}_w \notin P_{r-1}$ is not proved here. (But again, it is easy to see: If $\mathfrak{S}_w$ beloned to P_{r-1}, then $\mathfrak{S}_w$ would be symmetric in x_r, x_{r+1}, and therefore, by part (iii), we would have $w(r) < w(r+1)$.)

98. **Page 44, proof of (4.5):** "hence $\partial_2\partial_1\left(x_1^n x_2^{n-2} \cdots x_n\right)$" should be "hence $\partial_2\partial_1\left(x_1^n x_2^{n-1} \cdots x_n\right)$".

99. **Page 45, proof of (4.6):** In the display

$$\partial_i F(u, v) = \begin{cases} F(u, vs_i) & \text{if } \ell(vs_i) < \ell(v), \\ 0 & \text{otherwise,} \end{cases}$$

replace both "s_i"s by "s_{i-m}".

100. **Page 45:** It might be worth mentioning that (4.6) gives a second proof of (4.5), because the embedding $i : S_n \to S_m$ has the property that $i(w) = w \times 1_{m-n}$ for all $w \in S_n$ (and of course, we have $\mathfrak{S}_{1_n \times 1_{m-n}} = \mathfrak{S}_{1_n} = 1$).

101. **Page 45:** "premutations" should be "permutations".

102. **Page 49:** "indentically" should be "identically".

103. **Page 51, (4.16):** Replace "$\ell(v) = \ell(v) - 1$" by "$\ell(v) = \ell(w) - 1$".

104. **Page 51, (4.16):** After "and each w' in the sum precedes w in the reverse lexicographical ordering", add "and satisfies $\ell(w') = \ell(w)$".

105. **Page 51, (4.17):** Replace "S_w" by "$\mathfrak{S}_w$".

106. **Page 51:** In the last sentence of this page, "By (4.12) we can express" should be "By (4.13) we can express".

107. **Page 51, proof of (4.17):** "such that $\ell(v) - \ell(w)$" should be "such that $\ell(v) < \ell(w)$".

108. **Page 52, proof of (4.19):** Replace "$\rho(x_i)$" by "$\rho_m(x_i)$" twice.

109. **Page 54:** "let $\rho_n : P_\infty \mapsto P_n$" should be "let $\rho_n : P_\infty \to P_n$" (different kind of arrow).

110. **Page 59, proof of (4.35):** The "d_o" should probably be "d_0".

111. **Page 59, proof of (4.35):** The "diagram on p. 58" is not actually on p. 58 of the reedition. Instead it is on p. 57, or (to give a less fickle anchor for the reference) between (4.30) and (4.31).

112. **Page 61:** "aribtrarily" should be "arbitrarily".

113. **Page 70, proof of (5.1):** In the second displayed equation of the proof, "$e_r(x_1, \ldots, x_n)$" should be "$e_r(x_1, \ldots, x_{n-1})$".

114. **Page 70, proof of (5.1):** "follws" should be "follows".

115. **Page 71, (5.3):** The angular brackets here are misaligned with the equality sign. This is a LaTeX issue; use \left< instead of <.

116. **Page 71, proof of (5.3) (i):** "for $i \leq i$" should be "for $1 \leq i$".

117. **Page 72, proof of (5.5):** After "and then it is equal to $\epsilon(v)\left\langle w_0 \mathfrak{S}_{uv^{-1}}, x^\delta \right\rangle$", I would add "$= \epsilon(v)\,\partial_{w_0}\left(w_0\left(\mathfrak{S}_{uv^{-1}}\right) x^\delta\right)$" to make it clear what will happen next.

118. **Page 72, proof of (5.5):** After "Now $\mathfrak{S}_{uv^{-1}}$ is a linear combination of monomials x^α", add "with $\alpha \in \mathbf{N}^n$" (to avoid the alternative interpretation $\alpha \in \mathbf{N}^{n-1}$, which would lead to awkwardness further on).

119. **Page 72, proof of (5.5):** After "Now $\partial_{w_0} x^\beta = 0$ unless all components β_i of β are distinct", I would add "(by (2.11))".

120. **Page 72, proof of (5.6):** The expansions (1) and (2) exist because of (5.1′). This is perhaps worth pointing out.

121. **Page 73, proof of (5.7):** The set A_i is sometimes denoted by A (without subscript) here.

122. **Page 74, the line above (5.8):** In "$x = (x_1, \ldots x_n)$", add a comma before "x_n".

123. **Page 75:** The first displayed equation on this page,

$$T_w(y) = \langle \Delta(x,y), \; w_0 \mathfrak{S}_{ww_0}(-x) \rangle_x ,$$

could use a bit more justification. Namely, if $\alpha_u \in P_n(y)$ is a polynomial in $y_1, y_2, \ldots, y_n$ for each $u \in S_n$, and if $v \in \mathfrak{S}_n$ is any permutation, then

$$\begin{aligned} \left\langle \sum_{u \in S_n} \mathfrak{S}_u(x) \alpha_u, \; w_0 \mathfrak{S}_{vw_0}(-x) \right\rangle_x &= \sum_{u \in S_n} \alpha_u \left\langle \mathfrak{S}_u(x), \; w_0 \underbrace{\mathfrak{S}_{vw_0}(-x)}_{\substack{=\epsilon(vw_0)\mathfrak{S}_{vw_0}(x) \\ \text{(by homogeneity of } \mathfrak{S}_{vw_0})}} \right\rangle \\ &= \sum_{u \in S_n} \alpha_u \underbrace{\langle \mathfrak{S}_u(x), \; w_0 \epsilon(vw_0) \mathfrak{S}_{vw_0}(x) \rangle}_{\substack{=\epsilon(w_0)\langle w_0\mathfrak{S}_u(x), \; \epsilon(vw_0)\mathfrak{S}_{vw_0}(x)\rangle \\ \text{(by (5.3) (ii), since } w_0^{-1}=w_0)}} \\ &= \sum_{u \in S_n} \alpha_u \underbrace{\epsilon(w_0) \langle w_0 \mathfrak{S}_u(x), \; \epsilon(vw_0) \mathfrak{S}_{vw_0}(x) \rangle}_{=\epsilon(v)\langle w_0\mathfrak{S}_u(x), \; \mathfrak{S}_{vw_0}(x)\rangle} \\ &= \epsilon(v) \sum_{u \in S_n} \alpha_u \underbrace{\langle w_0 \mathfrak{S}_u(x), \; \mathfrak{S}_{vw_0}(x) \rangle}_{=\epsilon(v)\delta_{u,v}} \\ &= \sum_{u \in S_n} \alpha_u \delta_{u,v} = \alpha_v. \end{aligned}$$

Applying this to $\alpha_u = T_u(y)$ (so that $\sum_{u \in S_n} \mathfrak{S}_u(x) \alpha_u = \sum_{u \in S_n} \mathfrak{S}_u(x) T_u(y) = \Delta(x,y)$) and $v = w$, we conclude that

$$\langle \Delta(x,y), \; w_0 \mathfrak{S}_{ww_0}(-x) \rangle_x = T_w(y),$$

qed.

124. **Page 75, Remark:** "if wu is Grassmannian (with its only descent at r)" should be "if wu is Grassmannian (with its only descent at r, or with no descents at all)".

125. **Page 75, Remark:** "Grassmannian permutations v with descent at r (i.e. $v(i) < v(i+i)$ if $i \neq r$)" should be "Grassmannian permutations v with descent at r or no descent at all (i.e., satisfying $v(i) < v(i+1)$ for all $i \neq r$)".

126. **Page 75, Remark:** Let me explain how the "easily verified" equality

$$\partial_u \Delta(x,y) = \prod_{i=1}^{r} \prod_{j=1}^{s} (x_i - y_j)$$

is verified.

We begin by factoring the product $\Delta(x,y) = \prod_{i+j\leq n} (x_i - y_j)$ into three subproducts:

$$\begin{aligned} \Delta(x,y) &= ABC, \qquad \text{where} \\ A &= \prod_{\substack{i+j\leq n;\\ i\leq r \text{ and } j\leq s}} (x_i - y_j), \\ B &= \prod_{\substack{i+j\leq n;\\ i\leq r \text{ and } j>s}} (x_i - y_j), \\ C &= \prod_{\substack{i+j\leq n;\\ i>r \text{ and } j\leq s}} (x_i - y_j). \end{aligned}$$

(The reader might expect a fourth subproduct $\prod_{\substack{i+j\leq n;\\ i>r \text{ and } j>s}} (x_i - y_j)$, but this subproduct would be empty, since $i + j > r + s = n$ would contradict $i + j \leq n$.)

Note that

$$A = \prod_{\substack{i+j\leq n;\\ i\leq r \text{ and } j\leq s}} (x_i - y_j) = \prod_{i=1}^{r} \prod_{j=1}^{s} (x_i - y_j)$$

(since the conditions $i \leq r$ and $j \leq s$ entail $i + j \leq r + s = n$ and thus render the condition $i + j \leq n$ superfluous). This shows that the polynomial A is symmetric in $x_1, x_2, \ldots, x_r$ and also symmetric in $x_{r+1}, x_{r+2}, x_{r+3}, \ldots$ (since the latter variables don't appear in A at all). But the operator ∂_u is linear with respect to such polynomials (since u has a reduced expression built entirely of $s_1, s_2, \ldots, s_{r-1}, s_{r+1}, s_{r+2}, s_{r+3}, \ldots$, and thus ∂_u is built entirely of the operators $\partial_1, \partial_2, \ldots, \partial_{r-1}, \partial_{r+1}, \partial_{r+2}, \partial_{r+3}, \ldots$). Hence,

$$\partial_u (ABC) = A \partial_u (BC) .$$

Next, we observe that $\ell(u) = \ell\left(w_0^{(r)} \times w_0^{(s)}\right) = \ell\left(w_0^{(r)}\right) + \ell\left(w_0^{(s)}\right) = \binom{r}{2} + \binom{s}{2}$. But

$$\begin{aligned} B &= \prod_{\substack{i+j \leq n; \\ i \leq r \text{ and } j > s}} (x_i - y_j) \\ &= \underbrace{\prod_{\substack{i+j \leq n; \\ i \leq r \text{ and } j > s}} x_i}_{= x_1^{r-1} x_2^{r-2} \cdots x_r^0} + \left(\text{terms of degree } < \binom{r}{2} \text{ in the } x\text{-variables}\right) \\ &= x_1^{r-1} x_2^{r-2} \cdots x_r^0 + \left(\text{terms of degree } < \binom{r}{2} \text{ in the } x\text{-variables}\right). \end{aligned}$$

Similarly,

$$C = x_{r+1}^{s-1} x_{r+2}^{s-2} \cdots x_n^0 + \left(\text{terms of degree } < \binom{s}{2} \text{ in the } x\text{-variables}\right).$$

Multiplying these two equalities, we obtain

$$\begin{aligned} BC = {} & \left(x_1^{r-1} x_2^{r-2} \cdots x_r^0\right) \left(x_{r+1}^{s-1} x_{r+2}^{s-2} \cdots x_n^0\right) \\ & + \left(\text{terms of degree } < \binom{r}{2} + \binom{s}{2} \text{ in the } x\text{-variables}\right). \end{aligned}$$

Applying the map ∂_u to this equality, we obtain

$$\partial_u (BC) = \partial_u \left(\left(x_1^{r-1} x_2^{r-2} \cdots x_r^0\right) \left(x_{r+1}^{s-1} x_{r+2}^{s-2} \cdots x_n^0\right) \right),$$

since the operator ∂_u lowers degrees by $\ell(u) = \binom{r}{2} + \binom{s}{2}$ and thus annihilates all terms of degree $< \binom{r}{2} + \binom{s}{2}$ in the x-variables.

From $u = w_0^{(r)} \times w_0^{(s)}$, we easily obtain $\partial_u = \partial_{w_0^{(r)}} \partial'_{w_0^{(s)}}$, where $\partial'_{w_0^{(s)}}$ is like $\partial_{w_0^{(s)}}$ but acting on the variables $x_{r+1}, x_{r+2}, x_{r+3}, \ldots$ instead of $x_1, x_2, x_3, \ldots$.

Hence,

$$\begin{aligned}
\partial_u (BC) &= \partial_{w_0^{(r)}} \partial'_{w_0^{(s)}} (BC) = \partial_{w_0^{(r)}} \left(B \left(\partial'_{w_0^{(s)}} C \right) \right) \\
&\qquad \left(\begin{array}{c} \text{since } B \text{ does not involve} \\ \text{the variables } x_{r+1}, x_{r+2}, x_{r+3}, \ldots \\ \text{and thus can "pass through" } \partial'_{w_0^{(s)}} \end{array} \right) \\
&= \left(\partial_{w_0^{(r)}} B \right) \left(\partial'_{w_0^{(s)}} C \right) \\
&\qquad \left(\begin{array}{c} \text{since } \partial'_{w_0^{(s)}} C \text{ does not involve} \\ \text{the variables } x_1, x_2, \ldots, x_r \\ \text{and thus can "pass through" } \partial_{w_0^{(r)}} \end{array} \right).
\end{aligned}$$

Finally, recall that

$$B = x_1^{r-1} x_2^{r-2} \cdots x_r^0 + \left(\text{terms of degree } < \binom{r}{2} \text{ in the } x\text{-variables} \right).$$

Hence,

$$\partial_{w_0^{(r)}} B = \partial_{w_0^{(r)}} \left(x_1^{r-1} x_2^{r-2} \cdots x_r^0 \right)$$

(since the operator $\partial_{w_0^{(r)}}$ lowers degrees by $\ell\left(w_0^{(r)}\right) = \binom{r}{2}$ and thus annihilates all terms of degree $< \binom{r}{2}$ in the x-variables). Thus,

$$\begin{aligned}
\partial_{w_0^{(r)}} B &= \partial_{w_0^{(r)}} \left(x_1^{r-1} x_2^{r-2} \cdots x_r^0 \right) \\
&= \partial_{w_0^{(r)}} \left(x^{\delta^{(r)}} \right) \qquad \left(\text{where } \delta^{(r)} = (r-1, r-2, \ldots, 1, 0) \right) \\
&= s_{\delta^{(r)} - \delta^{(r)}} (x_1, x_2, \ldots, x_r) \qquad (\text{by (2.11)}) \\
&= s_{\varnothing} (x_1, x_2, \ldots, x_r) = 1.
\end{aligned}$$

Similarly,

$$\partial'_{w_0^{(s)}} C = 1$$

as well. Altogether, applying the map ∂_u to the equality $\Delta(x, y) = ABC$, we obtain

$$\begin{aligned}
\partial_u (\Delta(x, y)) &= \partial_u (ABC) = A \underbrace{\partial_u (BC)}_{= \left(\partial_{w_0^{(r)}} B \right) \left(\partial'_{w_0^{(s)}} C \right)} \\
&= A \underbrace{\left(\partial_{w_0^{(r)}} B \right)}_{=1} \underbrace{\left(\partial'_{w_0^{(s)}} C \right)}_{=1} = A = \prod_{i=1}^{r} \prod_{j=1}^{s} (x_i - y_j).
\end{aligned}$$

This proves the "easily verified" equality.

127. **Page 76:** The definition of the basis $(\mathfrak{S}^w)_{w\in S_n}$ as the "Λ_n-basis of P_n dual to the basis $(\mathfrak{S}_w)_{w\in S_n}$ relative to the scalar product (5.2)" tacitly relies on the fact that the scalar product (5.2) is nondegenerate. Arguably, this is an easy consequence of (5.5). Still, it would perhaps be easier to turn the argument around – i.e., to define the $\mathfrak{S}^w$ by (5.12) instead, and then show that they form a Λ_n-basis of P_n dual to the basis $(\mathfrak{S}_w)_{w\in S_n}$ relative to the scalar product (5.2).

128. **Page 77:** In the displayed equation "$\sum\limits_{\alpha} a_{u\alpha} b_{v\beta} = \delta_{uv}$", replace "$\beta$" by "$\alpha$".

129. **Pages 77–78, proof of (5.15):** I don't understand this proof (I don't know where the first equality comes from), but here is another:

We have
$$C(x,y) = \prod_{i<j}(y_i - x_j) = \sum_{w\in S_n} \mathfrak{S}_w(y)\,\mathfrak{S}^w(x) \tag{1}$$
(by (5.13), with x_i and y_i substituted for y_i and x_i). But the bases $(\mathfrak{S}^w)_{w\in S_n}$ and $(\mathfrak{S}_w)_{w\in S_n}$ of P_n are dual; thus, each $f \in P_n$ satisfies
$$\sum_{w\in S_n} \langle f,\ \mathfrak{S}^w\rangle\,\mathfrak{S}_w = f. \tag{2}$$
Furthermore, if $f \in H_n$, then all scalar products $\langle f,\ \mathfrak{S}^w\rangle$ belong to $\mathbf{Z}$ (because (4.11) shows that f is a linear combination $\sum\limits_{u\in S_n} q_u \mathfrak{S}_u$ with $q_u \in \mathbf{Z}$, and then we have $\langle f,\ \mathfrak{S}^w\rangle = q_w$ because $(\mathfrak{S}^w)_{w\in S_n}$ and $(\mathfrak{S}_w)_{w\in S_n}$ are dual bases). Thus, substituting the variables y_i for x_i in (2), we obtain
$$\sum_{w\in S_n} \langle f,\ \mathfrak{S}^w\rangle\,\mathfrak{S}_w(y) = f(y) \tag{3}$$
(for $f \in H_n$) However, from (1), we obtain (again for $f \in H_n$)
$$\begin{aligned}\langle f(x),\ C(x,y)\rangle_x &= \left\langle f(x),\ \sum_{w\in S_n} \mathfrak{S}_w(y)\,\mathfrak{S}^w(x)\right\rangle_x \\ &= \sum_{w\in S_n} \langle f,\ \mathfrak{S}^w\rangle\,\mathfrak{S}_w(y) = f(y) \qquad \text{(by (3))},\end{aligned}$$
which proves (5.15). ■

130. **Page 78:** I don't understand how the equalities (3) and (4) here are obtained. To apply (5.15′), we need $\partial_u v^{-1} C(x,z)$ to belong to H_n (allowing for the z_i to be treated as scalars), but I don't see why this should be true unless $v = w_0$.

131. **Page 79, proof of (5.20):** The first equality (which follows from (5.18)) is missing an "$\epsilon(w_0)$" factor. Or, alternatively, the "$\epsilon(w_0)$" factor in (5.18) should not be there (either fix works, but I don't know whether the factor is needed for further uses of (5.18)).

132. **Page 92:** "Recall the decomposition (4.17)" should be "Recall the decomposition (4.18)".

133. **Page 100, proof of (7.23):** "precedes $\lambda(i', j')$" should be "precedes (i', j')".

134. **Page 103:** Replace "$GL_n(k)$" by "$GL_n(K)$" twice.

135. **Page 104:** "it follows that that" $\rightarrow$ "it follows that".

136. **Page 104, proof of (A.2):** The symbol "$\supset$" here means "contains as a proper subset". This should be explained, since the symbol is otherwise used for just "contains as a subset".

137. **Page 105, proof of (A.2):** On the last line of this proof, replace "$\phi(b\mathbf{V})$" by "$\phi(b\mathbf{U})$".

138. **Page 106:** Replace "$\pi : F \mapsto P(E)$" by "$\pi : F \to P(E)$".

139. **Page 106:** The injectivity of the Plücker embedding π follows from the known theorems that:

    a) If two nonzero wedges $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ and $w_1 \wedge w_2 \wedge \cdots \wedge w_k$ in the k-th exterior power of a finite-dimensional vector space are equal, then $\operatorname{span}\{v_1, v_2, \ldots, v_k\} = \operatorname{span}\{w_1, w_2, \ldots, w_k\}$.

    b) If two nonzero pure tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ and $w_1 \otimes w_2 \otimes \cdots \otimes w_k$ in the tensor product of k finite-dimensional vector spaces are equal, then there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ in the base field such that $v_i = \lambda_i w_i$ for all $1 \leq i \leq k$.

140. **Page 107, proof of (A.5):** "Define $\xi, \ldots, \xi_n$" should be "Define $\xi_1, \ldots, \xi_n$".

141. **Page 109:** "theorm" should be "theorem".