A short survey on pre-Lie algebras
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`https://lmbp.uca.fr/~manchon/biblio/ESI-prelie2009.pdf`
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Errata and addenda by Darij Grinberg

I will refer to the results appearing in the paper "A short survey on pre-Lie algebras" by the numbers under which they appear in this paper (specifically, in its version of 18 December 2009). All page numbers appearing below are actual page numbers on the top headers of the pages, not the page numbers of the PDF file.

I have read only parts of the paper, so this list of errors is likely incomplete.

A. Errata

- **Page 3, §1.1:** In the whole Section 1.1, you appear to make the assumption that $\operatorname{char} k = 0$.

- **Page 3, §1.1:** The word "compatible" (in "with a compatible decreasing filtration") is confusing: It creates the impression that there is an additional "compatibility" condition on the filtration. I think it is really just an abbreviation for the requirement that "$A_p \rhd A_q \subset A_{p+q}$"; in this case, it is probably best to remove this word (since you already make the requirement that "$A_p \rhd A_q \subset A_{p+q}$" explicitly).

- **Page 3, §1.1:** In "$A = A_1 \supset A_2 \subset A_3 \supset \cdots$", replace the "$\subset$" sign by a "$\supset$" sign.

- **Page 3, §1.1:** Replace "reduces to $\{0\}$" by "is $\{0\}$".

- **Page 3, equation (9):** Replace "$\sum\limits_{i\geq 0} B_i L^i_{\Omega(a)} a$" by "$\sum\limits_{i\geq 0} \dfrac{B_i}{i!} L^i_{\Omega(a)} a$".

- **Page 5, equation (17):** It should be said here that the map L_a is extended to the fictitious unit $\mathbf{1}$ by $L_a \mathbf{1} := a$.

- **Page 6, §1.3:** Replace "$V = \bigoplus\limits_{n\geq 0}$" by "$V = \bigoplus\limits_{n\geq 1} V_n$". (The direct sum should start at $n = 1$ in order for the symmetric algebra $S(V)$ to have finite-dimensional homogeneous components, which in turn is necessary to ensure that its graded dual is a Hopf algebra.)

- **Page 6, §2.1:** In the definition of an augmented operad, replace "for any $a \in \mathcal{P}_k$, $b \in \mathcal{P}_l$, $a \in \mathcal{P}_m$" by "for any $a \in \mathcal{P}_k$, $b \in \mathcal{P}_l$, $c \in \mathcal{P}_m$".

- **Page 7, §2.1:** On the first line of page 7, replace "*free $\mathcal{O}$-algebra*" by "*free $\mathcal{P}$-algebra*".
- **Page 7, equation (27):** Replace ":" by ":=".
- **Page 8, §2.3:** "Here the notation $V < W$ means that $x < y$ for any vertex x of v and any vertex y of w such that x and y are comparable" is technically correct, but somewhat confusing because the sum in (31) has the condition $W < V$ rather than $V < W$. It would be better to replace this by "Here the notation $W < V$ means that $x < y$ for any vertex x of w and any vertex y of v such that x and y are comparable".
- **Page 8, §2.3:** In the last sentence of §2.3, replace "$\phi(t)$" by "$\Phi(t)$".
- **Page 10, equation (41):** Replace "$s \to_i v$" by "$s \to_v t$".
- **Page 11, Proposition 3.2:** Replace "$\alpha(\bullet = 1)$" by "$\alpha(\bullet) = 1$".
- **Page 11, after Proposition 3.2:** "The condition $\alpha(\bullet)$ is in fact dropped" should be "The condition $\alpha(\bullet) = 1$ is in fact dropped".

  In the same sentence "[4, Propsition 15]" should be "[4, Proposition 16]" (at least if we refer to arXiv version 4 of [4]; the referenced version 2 doesn't seem to include this proposition at all).
- **Page 12, reference [2]:** The correct pages are 521–534, not 512–534.
- **Page 13, reference [16]:** The arXiv ID is `arXiv:0910.2166`, not `arXiv:math/0910.2166`.
- **Page 13, reference [17]:** The page range for this paper is 295–316.

B. Addenda

B.1. Page 3, §1.1

In §1.1 on page 3, you write that "The application W is clearly a bijection". It took me a long time to see why this is true! I don't consider this argument simple enough as to be left to the reader. Let me give this argument in more detail. (This was partly written by GPT-5.5.)

First, I state a useful lemma:[1]

[1] Lemma B.1 is **not** concerned with the setting of §1.1; instead, it introduces a more general setting. In particular, the A in Lemma B.1 is not necessarily a Lie algebra, and not necessarily a vector space, but just an abelian group.

Lemma B.1. Let A be any abelian group (written additively). Let $A = A_1 \supset A_2 \supset A_3 \supset \cdots$ be any decreasing filtration of A (by subgroups) such that $\bigcap\limits_{j\geq 1} A_j = 0$. Let $W : A \to A$ be any map (not necessarily a group homomorphism). Assume that for every $n \geq 1$ and every $a \in A$ and $b \in A$, the following holds: If $a \equiv b \operatorname{mod} A_n$, then

$$W(a) - a \equiv W(b) - b \operatorname{mod} A_{n+1}. \tag{1}$$

Then:

(a) The map W is injective.

(b) Assume that A is complete with respect to the filtration $A = A_1 \supset A_2 \supset A_3 \supset \cdots$. Then, the map W is a bijection.

Proof of Lemma B.1. **(a)** Let a and b be two elements of A such that $W(a) = W(b)$. We shall show that

$$a - b \in A_m \qquad \text{for every } m \in \{1, 2, 3, \ldots\}. \tag{2}$$

Proof of (2). We shall prove (2) by induction on m:

Induction base: We have $a - b \in A = A_1$. In other words, (2) holds for $m = 1$. This completes the induction base.

Induction step: Let $M \in \{1, 2, 3, \ldots\}$ be such that (2) holds for $m = M$. We must show that (2) holds for $m = M + 1$ as well.

We have assumed that (2) holds for $m = M$. In other words, $a - b \in A_M$. In other words, $a \equiv b \operatorname{mod} A_M$. Hence, (1) (applied to $n = M$) yields $W(a) - a \equiv W(b) - b \operatorname{mod} A_{M+1}$. Hence,

$$a = \underbrace{W(a)}_{=W(b)} - \underbrace{(W(a) - a)}_{\equiv W(b) - b \operatorname{mod} A_{M+1}} \equiv W(b) - (W(b) - b) = b \operatorname{mod} A_{M+1}.$$

In other words, $a - b \in A_{M+1}$. That is, (2) holds for $m = M + 1$. This completes the induction step. Thus, (2) is proved. $\square$

Now, (2) shows that $a - b \in A_m$ for each $m \in \{1, 2, 3, \ldots\}$. Thus,

$$a - b \in \bigcap_{m \geq 1} A_m = \bigcap_{j \geq 1} A_j = 0.$$

That is, $a - b = 0$, so that $a = b$.

Forget that we fixed a and b. We thus have shown that if a and b are two elements of A such that $W(a) = W(b)$, then $a = b$. In other words, the map W is injective. This proves Lemma B.1 **(a)**.

(b) We assumed that A is complete. In other words, every Cauchy sequence in A has a limit. Here, a *Cauchy sequence* means an infinite sequence $(a_n)_{n \geq 1}$ of elements of A such that for each $N \geq 1$, there exists some $n \geq 1$ such that

$$\text{all integers } m_1, m_2 \geq n \text{ satisfy } a_{m_1} - a_{m_2} \in A_N. \tag{3}$$

Furthermore, a *limit* of such a sequence $(a_n)_{n \geq 1}$ means an element $x \in A$ such that for each $N \geq 1$, there exists some $n \geq 1$ such that

$$\text{all } m \geq n \text{ satisfy } x - a_m \in A_N. \tag{4}$$

Now, let $u \in A$. We shall construct an $x \in A$ satisfying $W(x) = u$. This will prove that W is surjective, and thus bijective (since Lemma B.1 **(a)** yields that W is injective).

We define a sequence $(a_n)_{n \geq 1}$ of elements of A recursively by

$$a_1 = u \qquad \text{and} \tag{5}$$

$$a_n = a_{n-1} - W(a_{n-1}) + u \qquad \text{for each } n \geq 2. \tag{6}$$

Then, we claim that

$$W(a_n) \equiv u \mod A_n \qquad \text{for each } n \geq 1. \tag{7}$$

Proof of (7). We shall prove (7) by induction on n:

Base case: For $n = 1$, the claim (7) is true, since $W(a_1) - u \in A = A_1$ and thus $W(a_1) \equiv u \mod A_1$.

Induction step: Let $M \in \{1, 2, 3, \ldots\}$ be such that (7) holds for $n = M$. We must show that (7) holds for $n = M + 1$ as well.

We have assumed that (7) holds for $n = M$. In other words, $W(a_M) \equiv u \mod A_M$. Hence, $u - W(a_M) \in A_M$. But (6) (applied to $n = M + 1$) yields

$$a_{M+1} = a_M - W(a_M) + u = a_M + (u - W(a_M)).$$

Thus, $a_{M+1} - a_M = u - W(a_M) \in A_M$. In other words, $a_{M+1} \equiv a_M \mod A_M$. Hence, (1) (applied to $n = M$, $a = a_{M+1}$ and $b = a_M$) yields

$$W(a_{M+1}) - a_{M+1} \equiv W(a_M) - a_M \mod A_{M+1}.$$

Therefore,

$$\begin{aligned} W(a_{M+1}) &= \underbrace{a_{M+1}}_{= a_M + (u - W(a_M))} + \underbrace{(W(a_{M+1}) - a_{M+1})}_{\equiv W(a_M) - a_M \mod A_{M+1}} \\ &\equiv (a_M + (u - W(a_M))) + (W(a_M) - a_M) \\ &= u \mod A_{M+1}. \end{aligned}$$

In other words, (7) holds for $n = M + 1$. This completes the induction step. Thus, (7) is proved. $\square$

Next, we claim that the sequence $(a_n)_{n\geq 1}$ is a Cauchy sequence.

Proof. Let $N \geq 1$. We must find some $n \geq 1$ such that (3) holds.

We claim that $n = N$ works. Indeed, let $m_1, m_2 \geq N$ be two integers. We must show that $a_{m_1} - a_{m_2} \in A_N$.

Since A_N is an abelian group, this is equivalent to showing that $a_{m_2} - a_{m_1} \in A_N$. Thus, by symmetry, we WLOG assume that $m_1 \geq m_2$. For each $i \geq N$, we have

$$a_{i+1} = a_i - W(a_i) + u \qquad \text{(by (6), applied to } n = i+1\text{)}$$

and thus

$$\begin{aligned} a_{i+1} - a_i = u - W(a_i) &\in A_i \qquad \text{(since (7) yields } W(a_i) \equiv u \operatorname{mod} A_i\text{)} \\ &\subset A_N \end{aligned} \tag{8}$$

(by $A_1 \supset A_2 \supset A_3 \supset \cdots$, since $i \geq N$). Hence, by the telescope principle,

$$\begin{aligned} a_{m_1} - a_{m_2} &= \sum_{i=m_2}^{m_1-1} \underbrace{(a_{i+1} - a_i)}_{\substack{\in A_N \\ \text{(by (8))}}} \qquad \text{(since } m_1 \geq m_2\text{)} \\ &\in \sum_{i=m_2}^{m_1-1} A_N \subset A_N, \end{aligned}$$

since A_N is a subgroup of A. Thus, (3) holds for $n = N$. This proves that $(a_n)_{n\geq 1}$ is a Cauchy sequence. $\square$

Since A is complete, the Cauchy sequence $(a_n)_{n\geq 1}$ has a limit. Let x be this limit. We shall show that $W(x) = u$.

Indeed, let $N \geq 1$. Since x is the limit of the sequence $(a_n)_{n\geq 1}$, there exists some $n \geq 1$ such that (4) holds. Consider this n. Set $m := \max\{n, N\}$. Then, $m \geq n$ and $m \geq N$. Hence, $x - a_m \in A_N$ (by (4)). From $A_1 \supset A_2 \supset A_3 \supset \cdots$, we obtain $A_{N+1} \subset A_N$ and $A_m \subset A_N$ (since $m \geq N$).

From $x - a_m \in A_N$, we obtain $x \equiv a_m \operatorname{mod} A_N$. Hence, (1) (applied to N, x and a_m instead of n, a and b) yields

$$W(x) - x \equiv W(a_m) - a_m \operatorname{mod} A_{N+1}.$$

Since $A_{N+1} \subset A_N$, this entails

$$W(x) - x \equiv W(a_m) - a_m \operatorname{mod} A_N.$$

Adding this congruence to the congruence $x \equiv a_m \operatorname{mod} A_N$, we obtain

$$W(x) \equiv W(a_m) \operatorname{mod} A_N.$$

On the other hand, (7) yields $W(a_m) \equiv u \operatorname{mod} A_m$, so that $W(a_m) \equiv u \operatorname{mod} A_N$ (since $A_m \subset A_N$). Thus,

$$W(x) \equiv W(a_m) \equiv u \operatorname{mod} A_N,$$

so that $W(x) - u \in A_N$.

Forget that we fixed N. We thus have proved that $W(x) - u \in A_N$ for each $N \geq 1$. Hence,

$$W(x) - u \in \bigcap_{N \geq 1} A_N = \bigcap_{j \geq 1} A_j = 0.$$

Thus, $W(x) - u = 0$, so that $W(x) = u$.

Now, forget that we fixed u. We have shown that for each $u \in A$, there exists some $x \in A$ satisfying $W(x) = u$. In other words, the map W is surjective. Combined with Lemma B.1 **(a)**, this shows that W is bijective. This proves Lemma B.1 **(b)**. □

We now apply Lemma B.1 to the setting of §1.1. Thus, A is a complete filtered left pre-Lie algebra over a field k of characteristic 0, with filtration

$$A = A_1 \supset A_2 \supset A_3 \supset \cdots$$

satisfying $A_p \rhd A_q \subset A_{p+q}$ and $\bigcap_{j \geq 1} A_j = 0$. We need to prove that the map

$$\begin{aligned} W : A &\to A \\ a &\mapsto e^{L_a}\mathbf{1} - \mathbf{1} \end{aligned}$$

is a bijection (where $\mathbf{1}$ is a fictitious unit outside of A, but we have $e^{L_a}\mathbf{1} - \mathbf{1} \in A$ nevertheless because the constant term of the power series e^x is 1).

Explicitly, each $a \in A$ satisfies $e^{L_a} = \sum_{r \geq 0} \frac{1}{r!} L_a^r$ and therefore

$$W(a) = e^{L_a}\mathbf{1} - \mathbf{1} = \sum_{r \geq 0} \frac{1}{r!} L_a^r \mathbf{1} - \mathbf{1} = \sum_{r \geq 1} \frac{1}{r!} L_a^r \mathbf{1}. \tag{9}$$

For each positive integer r, we shall denote the element

$$L_a^r \mathbf{1} = \underbrace{a \rhd (a \rhd (a \rhd (\cdots \rhd (a \rhd a))))}_{\text{with } r \text{ copies of } a} \in A$$

by $a_{[r]}$. Thus, the equality (9) can be rewritten as

$$W(a) = \sum_{r \geq 1} \frac{1}{r!} a_{[r]}.$$

(The sum converges because $a_{[r]} \in A_r$ for all $r \geq 1$, as can be easily checked by induction on r.) Hence, for each $a \in A$, we have

$$W(a) - a = \sum_{r \geq 1} \frac{1}{r!} a_{[r]} - a = \sum_{r \geq 2} \frac{1}{r!} a_{[r]} \tag{10}$$

(since $\frac{1}{1!} a_{[1]} = a_{[1]} = a$). Note that the definition of $a_{[r]}$ readily yields

$$a_{[r+1]} = L_a a_{[r]} = a \triangleright a_{[r]} \tag{11}$$

for each $r \geq 1$.

We shall now check that our map $W : A \to A$ satisfies the assumption (1) of Lemma B.1. So let $n \geq 1$ and $a, b \in A$ be such that $a \equiv b \operatorname{mod} A_n$. We must prove that

$$W(a) - a \equiv W(b) - b \operatorname{mod} A_{n+1}.$$

To do so, we first observe that

$$a_{[r]} - b_{[r]} \in A_{n+r-1} \quad \text{for every } r \geq 1. \tag{12}$$

Proof of (12). Indeed, (12) is proved by induction on r:

The *base case* $r = 1$ says that $a_{[1]} - b_{[1]} \in A_n$; in other words, it says that $a - b \in A_n$ (since $a_{[1]} = a$ and $b_{[1]} = b$). But this is just a restatement of our hypothesis $a \equiv b \operatorname{mod} A_n$. Hence the base case is complete.

For the *induction step*, assume that (12) holds for some $r \geq 1$. Then, (11) yields

$$\begin{aligned}
a_{[r+1]} - b_{[r+1]} &= a \triangleright a_{[r]} - b \triangleright b_{[r]} \\
&= \underbrace{a \triangleright a_{[r]} - b \triangleright a_{[r]}}_{=(a-b)\triangleright a_{[r]}} + \underbrace{b \triangleright a_{[r]} - b \triangleright b_{[r]}}_{=b\triangleright\left(a_{[r]}-b_{[r]}\right)} \\
&= \underbrace{(a-b)}_{\substack{\in A_n \\ (\text{since } a\equiv b \operatorname{mod} A_n)}} \triangleright \underbrace{a_{[r]}}_{\in A_r} + \underbrace{b}_{\in A = A_1} \triangleright \underbrace{\left(a_{[r]} - b_{[r]}\right)}_{\substack{\in A_{n+r-1} \\ (\text{by the induction hypothesis})}} \\
&\in \underbrace{A_n \triangleright A_r}_{\subset A_{n+r}} + \underbrace{A_1 \triangleright A_{n+r-1}}_{\subset A_{1+(n+r-1)} = A_{n+r}} \subset A_{n+r} + A_{n+r} = A_{n+r}.
\end{aligned}$$

This proves (12) for $r + 1$, and thus completes the induction. Hence, (12) is proved. □

Note that each $r \geq 2$ satisfies $n + r - 1 \geq n + 1$ and thus

$$A_{n+r-1} \subset A_{n+1} \tag{13}$$

(since $A_1 \supset A_2 \supset A_3 \supset \cdots$).

Now, using (10), we get

$$(W(a)-a)-(W(b)-b)=\sum_{r\geq 2}\frac{1}{r!}a_{[r]}-\sum_{r\geq 2}\frac{1}{r!}b_{[r]}=\sum_{r\geq 2}\frac{1}{r!}\underbrace{\left(a_{[r]}-b_{[r]}\right)}_{\substack{\in A_{n+r-1}\\ \text{(by (12))}}}$$
$$\in\sum_{r\geq 2}\frac{1}{r!}\underbrace{A_{n+r-1}}_{\substack{\subset A_{n+1}\\ \text{(by (13))}}}\subset\sum_{r\geq 2}\frac{1}{r!}A_{n+1}\subset A_{n+1}$$

(since A_{n+1} is closed with respect to the topology defined by our filtration). In other words,

$$W(a)-a\equiv W(b)-b \bmod A_{n+1}.$$

This proves the assumption (1) of Lemma B.1. Hence, Lemma B.1 **(b)** shows that W is bijective. This justifies the sentence "The application W is clearly a bijection".