The Littlewood-Richardson rule, and related combinatorics
Marc A. A. van Leeuwen
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Errata by Darij Grinberg - I

The following text is an annotation to Marc A. A. van Leeuwen's paper "The Littlewood-Richardson rule, and related combinatorics" in its version of 20 March 2012.

This annotation contains short comments as well as corrections of the occasional mistake (or what I believe to be mistakes).

Page 2, §1.1: "the the action" → "the action".

Page 2, §1.2: Here, you write: "The Schur polynomials of degree d form yet another $\mathbb{Z}$-basis of Λ_n parametrized by $\mathcal{P}_{d,n}$." The Λ_n here should be replaced by Λ_n^d.

Page 4, §1.2: In "the fact that $\{m_\lambda(n) \mid \lambda \in \mathcal{P}_{d,n}\}$ is a $\mathbb{Z}$-basis of Λ_n", the Λ_n again should be replaced by Λ_n^d.

Page 4, §1.3: You claim that "this representation is not unique in general (although in some cases it is, for instance when the sets of rows and columns meeting D are both initial intervals of $\mathbb{N}$)". This is not literally correct, since the representation is not unique when D is the empty skew diagram, whereas it is clear that the sets of rows and columns meeting the empty skew diagram are both initial intervals of $\mathbb{N}$. However, this is the only exception. In fact, something more general holds: If the set of rows meeting a skew diagram D is an initial interval of $\mathbb{N}$, and the set of columns meeting D contains 0, then the representation of D as a difference of two Young diagrams is unique. (Of course, the same holds with "rows" and "columns" switched. These two cases, however, don't cover all cases where the difference representation is unique; and in general, whether or not the representation is unique is not determined by the sets of rows and of columns meeting D.)

Page 6, Proposition 1.4.3: This proposition is left unproved here, and indeed the proof is not exactly deep – but I would not call it trivial either. I have spelled out the proof in detail in Exercise 2.9.20 (a) in:

Darij Grinberg, Victor Reiner, *Hopf algebras in combinatorics*, arXiv:1409.8356v7

(specifically, the claim of Proposition 1.4.3 is the equivalence of Assertions $\mathcal{F}^{(\kappa)}$ and $\mathcal{G}^{(\kappa)}$ in that exercise, since Assertion $\mathcal{G}^{(\kappa)}$ is saying that T is κ-dominant, whereas Assertion $\mathcal{F}^{(\kappa)}$ is saying that $\alpha \in \mathcal{P}$ throughout the test formulated in the proposition).

Page 8, §1.5: "another a valid reading order" should be "another valid reading order".

Page 9, §2.1: You haven't defined the relation "$\subseteq$" in "$(\mathcal{P}, \subseteq)$". (Of course, the definition is easy: If λ and μ are two partitions, then we set $\lambda \subseteq \mu$ if and only if every

$i \geq 0$ satisfies $\lambda_i \leq \mu_i$, where λ and μ have been written as $\lambda = (\lambda_0, \lambda_1, \lambda_2, ...)$ and $\mu = (\mu_0, \mu_1, \mu_2, ...)$, respectively. In other words, if λ and μ are two partitions, then $\lambda \subseteq \mu$ if and only if $Y(\lambda) \subseteq Y(\mu)$.)

Page 9, §2.2: On the first line of §2.2, remove the word "of" from the formulation "Among the skew shapes λ/μ of with $|\lambda/\mu| = 2$".

Page 10, §2.2: Shortly before Definition 2.2.1, you write:

"It follows that if values $\lambda^{[i,j]}$ are prescribed for indices $[i, j]$ traversing some "lattice path" going from $[k, m]$ to $[l, n]$ (a zig-zag path in which at each step either i or j increases by 1) by any skew standard tableau, then there is a unique way to extend these values to tableau switching family on $I \times J$".

First, "tableau switching family" should be "a tableau switching family" here.

On a less trivial note, it took me a while to figure out why this extension exists and is unique. A posteriori, the difficulties were illusory: Since there are many orders in which we can compute the values of this extension, my intuition suggested that there can also be many different results and perhaps even some "holonomy" that would need to be proven trivial. In truth, the proof is easy and requires no such work:

- First one proves the **existence** of the tableau switching family $\left(\lambda^{[i,j]}\right)_{k\leq i\leq l;\ m\leq j\leq n}$ with given values on a given lattice path from $[k, m]$ to $[l, n]$. This can be done by recursively computing the entries $\lambda^{[i,j]}$ of the family as follows: Start with the given entries on the lattice path (which are already known). Then, compute all the entries $\lambda^{[i,j]}$ above the lattice path (in the order in which they become computable: until all these entries have been computed, there is always at least one uncomputed $\lambda^{[i,j]}$ whose neighbors $\lambda^{[i+1,j]}$, $\lambda^{[i,j+1]}$, $\lambda^{[i+1,j+1]}$ are all computed already, and so we pick such an entry and compute it). Finally, compute all the entries $\lambda^{[i,j]}$ below the lattice path (in an analogous way). Thus, at least one tableau switching family $\left(\lambda^{[i,j]}\right)_{k\leq i\leq l;\ m\leq j\leq n}$ exists that has the given values on our given lattice path.

- To prove its **uniqueness**, we must show that if $\left(\lambda^{[i,j]}\right)_{k\leq i\leq l;\ m\leq j\leq n}$ and $\left(\mu^{[i,j]}\right)_{k\leq i\leq l;\ m\leq j\leq n}$ are two tableau switching families that agree on a given lattice path from $[k, m]$ to $[l, n]$, then they agree everywhere. This, again, is proved recursively, just as in the proof of existence: We are given that $\lambda^{[i,j]} = \mu^{[i,j]}$ for all (i, j) on the lattice path. Next, we recursively prove that $\lambda^{[i,j]} = \mu^{[i,j]}$ for all points (i, j) above the lattice path (again recursively, e.g., by strong induction on $-i - j$, so that the induction hypothesis already shows that $\lambda^{[i+1,j]} = \mu^{[i+1,j]}$ and $\lambda^{[i,j+1]} = \mu^{[i,j+1]}$ and $\lambda^{[i+1,j+1]} = \mu^{[i+1,j+1]}$, and thus the uniqueness of the tableau-switching rule allows us to conclude that $\lambda^{[i,j]} = \mu^{[i,j]}$). Finally, we prove that $\lambda^{[i,j]} = \mu^{[i,j]}$ for all points (i, j) below the lattice path (similarly). Thus, $\lambda^{[i,j]} = \mu^{[i,j]}$ is proved for all (i, j); this completes the proof of uniqueness.

Page 11, §2.2: There is a typo in the paragraph succeeding Theorem 2.2.2: "exactly that of in inward *jeu de taquin* slide" should be "exactly that of an inward *jeu de taquin* slide".

Page 11, §2.2: I think it would add value to the exposition in the paper if the notion of a jeu de taquin slide would be defined (particularly the "inward slide into" and "outward slide into" terminology). While it is possible to define jeu de taquin slides through tableau switching, I don't think you make it very clear how to do that, and you do use "sliding" metaphors further in the text. You kind-of define jeu de taquin slides in your `pictures.pdf` paper, but for some reason you call them "glissements" there...

Page 12, §2.3: In "indicate the first second and third", add a comma after "first".

Page 12, §2.3: In the Young tableau one line above Proposition 2.3.2, why are the entries $1, 2, 3$ called rather than $0, 1, 2$?

Page 13, §2.3: "If t lies a diagonal" should be "If t lies on a diagonal".

Page 13, §2.3: One line above Proposition 2.3.3, "consequence proposition 2.2.5" should be "consequence of proposition 2.2.5".

Page 13, proof of Theorem 2.4.1: "between those of p and q" should be "between those of p and r".

Page 14, Proposition 2.4.2: This is worded quite ambiguously: Does "any" means "some" or "all"? Is the tableau T supposed to be skew or straight? standard or just semistandard?

In truth, several of these options work and are being used later! Thus, I suggest replacing Proposition 2.4.2 by the following:

> **2.4.2. Proposition.** Let S_1 and S_2 be two skew semistandard tableaux of the same shape λ/μ. Then, the following three statements are equivalent:
>
> **(1)** The tableaux S_1 and S_2 are dual equivalent.
>
> **(2)** There is a standard tableau T of shape μ such that if we set $X(T, S_i) = (S'_i, T'_i)$ for $i = 1, 2$, then $T'_1 = T'_2$.
>
> **(3)** For any skew semistandard tableau T of any shape μ/ν, if we set $X(T, S_i) = (S'_i, T'_i)$ for $i = 1, 2$, then $T'_1 = T'_2$.

The first three sentences of the proof given for Proposition 2.4.2 are proving the implication **(2)** $\Longrightarrow$ **(1)** (and the notion of "reverse dual equivalence" is used for statement **(2)** there). The last sentence of the proof is proving the implication **(1)** $\Longrightarrow$ **(3)**. The implication **(1)** $\Longrightarrow$ **(3)** (and the notion of "reverse dual equivalence" is used for statement **(3)** there). Combining these implications with the obvious implication **(3)** $\Longrightarrow$ **(2)**, we conclude that all three statements are equivalent, and so the updated Proposition 2.4.2 follows.

In the next paragraph, the implication **(2)** $\Longrightarrow$ **(1)** is used (since any application of successive inward slides into S_1 and S_2 until obtaining a straight-shaped Young tableau amounts to computing $X(T, S_i)$ for a standard tableau T of shape μ).

Page 17, Proposition 3.1.1: Remove one of the words "to" from "associating to w to the sequence".

Page 17: When you say "connected component", it might be useful to point out that these words mean a connected component of the **undirected** graph obtained by forgetting the directions of the edges in the coplactic graph.

Page 21, proof of Corollary 3.3.3: There is a "the the" typo here.

Page 28, §4.1: Typo: "estabishes".
