Determinants, Paths, and Plane Partitions
Ira M. Gessel, X. V. Viennot
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version of 28 August 2000
Errata by Darij Grinberg

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The following are my comments on specific places in the preprint "**Determinants, Paths, and Plane Partitions**" by Ira M. Gessel and X. V. Viennot (in its version of 28 August 2000). I have read only parts of the preprint.

- **page 2, proof of Theorem 1:** Replace the big " $\bigcup_{\pi \in S_k}$ " sign (in the second displayed equation of this proof) by a " $\bigsqcup_{\pi \in S_k}$ " sign (which stands for an external disjoint union). In fact, the sets $\mathsf{P}(\mathbf{u}, \pi(\mathbf{v})) - \mathsf{N}(\mathbf{u}, \pi(\mathbf{v}))$ can have nonempty intersection for different permutations $\pi \in S_k$ when some of the v_i's are equal. Thus we must take a disjoint union in order to ensure that each k-path in it "knows" which π it comes from.

- **page 2, proof of Theorem 1:** Before "Then properties (i), (ii), and (iii) are easily verified", I would add the following sentence: "We then define $\mathbf{A}^*$ (that is, the image of $\mathbf{A}$ under our bijection) as the k-path $\left(A_1^*, A_2^*, \ldots, A_k^*\right) \in \mathsf{P}(\mathbf{u}, \sigma(\mathbf{v}))$, where $\sigma = \pi \circ (i, j)$." (This should clarify which permutation $\mathbf{A}^*$ corresponds to when some of the v_i are equal.)

- **page 2:** You write: "Let us say that a pair $(\mathbf{u}, \mathbf{v})$ of k-vertices is *nonpermutable* if $N(\mathbf{u}, \pi(\mathbf{v}))$ is empty" etc.. Here, "N" should be "N".

- **page 3:** On the first line of this page, replace "for $i > 1$" by "for $i \in \{2, 3, \ldots, \ell(\lambda)\}$" (since sufficiently large i would otherwise have to satisfy $0 = 0 + 1$). Also, this isn't how I would define a skew-hook. Your definition forces the skew hook to start in row 1, which is unlike the standard definition that is used in the Murnaghan-Nakayama rule.

- **page 3:** On the first line of this page, replace "*skew hook*" by "*skew-hook*" (since you later use the hyphenated version).

- **page 3:** In "The plane partition (p_{ij}) is *row-strict* if (3.2) is replaced by $p_{ij} > p_{i,j+1}$ and column-strictness is defined similarly", replace "(3.2)" by "(3.1)".

- **page 3:** "by reversing all inequalities" $\to$ "by reversing the inequalities (3.1) and (3.2)" (not the inequalities $\mu_i < j \leq \lambda_i$).

- **page 3:** I am not sure what "with each row shifted one place to the right in relation to the previous row" means.
- **page 4:** In "and k-paths with initial", replace "k-paths" by "disjoint k-paths".
- **page 4:** In "Theorem 1 allows us then to count these tableaux", replace "Theorem 1" by "Corollary 2".
- **page 4:** Add a period after "in all positions on a diagonal".
- **page 4:** "Then by Theorem 1" $\to$ "Then by Corollary 2".
- **page 4:** I think a whitespace is missing in "$\left|P\left(u_i, v_j\right)\right|_1^k$,where".
- **page 5, Theorem 3:** Replace "weights of $\mathbf{f}\left(t\right)$" by "weights of $\mathbf{f}\left(T\right)$".
- **page 5, Corollary 4:** Replace "satisfy b_{i+1}" by "satisfy $b_{i+1} \geq b_i$".
- **page 6, Corollary 5:** I think the equality sign in "$= p_{ij}$" should be removed.
- **page 8, third paragraph:** "height m and width m" $\to$ "height m and width n".
- **page 10, §6:** "Now let h_n be the coefficient" $\to$ "Now let h_n^* be the coefficient".
- **page 11, (7.2):** Remove the period at the end of (7.2).
- **page 11:** You write: "We prove the generalization of Corollary". What corollary?
- **page 11, proof of Theorem 11:** Replace "(but $i \geq 1$)" by "(but $i \geq 1$ and $\mu_i < j$)".
- **page 13, proof of Theorem 11:** You write: "We leave it to the reader to verify that this involution cancels the unwanted terms in $s^R_{\lambda/\mu}$".

  Frankly, some detail would be good at this point, since you have never explained how exactly your arrays correspond to terms in $s^R_{\lambda/\mu}$ to begin with (and the connection to lattice paths is not clear anymore, since X is an arbitrary semitransitive relation). Here are some words which I think would make the argument clearer:

  Let k be the length of λ. For any $\pi \in S_k$, let a *π-array* be an array $(a_{i,j})$ indexed by pairs (i, j) of integers satisfying $\mu_i < j \leq \lambda_{\pi(i)} - \pi(i) + i$, and satisfying $a_{i,j} \ R \ a_{i,j+1}$. However, if some $i \leq k$ satisfies $\mu_i > \lambda_{\pi(i)} - \pi(i) + i$, then we say that there exist no π-arrays. The *weight* of a π-array means the product of x_a for a ranging over all entries of this array.

We have

$$s^R_{\lambda/\mu} = \sum_{\pi \in S_k} (-1)^\pi \cdot (\text{the sum of the weights of all } \pi\text{-arrays})$$

(this follows from the definition of $s^R_{\lambda/\mu}$ by writing the determinant as a sum over permutations). The involution ε cancels unwanted terms in this formula (i.e., those which do not correspond to π being the identity permutation and the π-array being an R-tableau) because it maps any π-array to a $\pi \circ (i, i+1)$-array, where i is the row of the earliest failure.

- **page 13:** You write: "Theorem 12 could easily be generalized to include part restrictions on the rows". I suspect you mean Theorem 11, not Theorem 12, here.
- **page 17, proof of Lemma 18:** In the first displayed equation of this proof, replace "$(a_i)_j$" by "$(\alpha_i)_j$".
- **page 18, proof of Lemma 19:** You write "and the result follows from 18". Probably you mean "and the result follows from Lemma 18.".