The Bhargava greedoid as a Gaussian elimination greedoid

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Abstract. Inspired by Manjul Bhargava's theory of generalized factorials, Fedor Petrov and the author have defined the *Bhargava greedoid* – a greedoid (a matroid-like set system on a finite set) assigned to any "ultra triple" (a somewhat extended variant of a finite ultrametric space). Here we show that the Bhargava greedoid of a finite ultra triple is always a *Gaussian elimination greedoid* over any sufficiently large (e.g., infinite) field; this is a greedoid analogue of a representable matroid. We find necessary and sufficient conditions on the size of the field to ensure this.

Contents

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The notion of a *greedoid* was coined in 1981 by Korte and Lovász, and has since seen significant developments ([KoLoSc91], [BjoZie92]). It is a type of set system (i.e., a set of subsets of a given ground set) that is required to satisfy some axioms weaker than the matroid axioms – so that, in particular, the independent sets of a matroid form a greedoid.

In [GriPet19], Grinberg and Petrov have constructed a greedoid stemming from Bhargava's theory of generalized factorials [Bharga97, §2], albeit in a setting significantly more general than Bhargava's. Roughly speaking, the sets that belong to this greedoid are subsets of maximum perimeter (among all subsets of their size) of a finite ultrametric space (which, in Bhargava's work, was a Dedekind ring with a metric coming from a valuation).

More precisely, the setup is more general than that of an ultrametric space: We consider a finite set E, a *distance function* d that assigns a "distance" $d(e,f)$ to any pair (e,f) of distinct elements of E, and a *weight function* w that assigns a "weight" $w(e)$ to each $e \in E$. The distances and weights are required to belong to a totally ordered abelian group $\mathbb{V}$ (for example, $\mathbb{R}$). The distances are required to satisfy the symmetry axiom $d(e,f) = d(f,e)$ and the "ultrametric triangle inequality" $d(a,b) \leqslant \max\{d(a,c), d(b,c)\}$. In this setting, any subset S of E has a well-defined *perimeter*, obtained by summing the weights and the pairwise distances of all its elements. The subsets S of E that have maximum perimeter (among all $|S|$-element subsets of E) then form a greedoid, which has been called the *Bhargava greedoid* of (E,w,d) in [GriPet19]. This greedoid is furthermore a strong greedoid [GriPet19, Theorem 6.1], which implies in particular that for any given $k \leqslant |E|$, the k-element subsets of E that have maximum perimeter are the bases of a matroid.

In the present paper, we prove that the Bhargava greedoid of (E,w,d) is a *Gaussian elimination greedoid* over any sufficiently large (e.g., infinite) field. Roughly speaking, a Gaussian elimination greedoid is a greedoid analogue of a representable matroid[1]. We quantify the "sufficiently large" by providing a sufficient condition for the size of the field. When all weights $w(e)$ are equal, we show that this condition is also necessary.

We note that the Bhargava greedoid can be seen to arise from an optimization problem in phylogenetics: Given a finite set E of organisms and an integer $k \in \mathbb{N}$, we want to choose a k-element subset of E that maximizes some kind of biodiversity. Depending on the definition of biodiversity used, the properties of the maximizing subsets can differ. It appears natural to define biodiversity in terms of distances on the evolutionary tree (which is a finite ultrametric space), and such a definition has been considered by Moulton, Semple and Steel in [MoSeSt06], leading to the result that the maximum-biodiversity sets form a strong greedoid. The Bhargava greedoid is an analogue of their greedoid using a slightly different definition of biodiversity[2]. The present paper potentially breaks this analogy by showing that the Bhargava greedoid is a Gaussian elimination greedoid, whereas this is unknown for the greedoid of Moulton, Semple and Steel. Whether the latter is a Gaussian elimination greedoid as well remains to be understood[3], as does the question of interpolating between the two notions of biodiversity.

This paper is self-contained (up to some elementary linear algebra), and in particular can be read independently of [GriPet19].

The 12-page extended abstract [GriPet20] summarizes the highlights of both [GriPet19] and this paper; it is thus a convenient starting point for a reader interested in the subject.

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[1]In particular, this entails that all the matroids mentioned in the preceding paragraph are representable.

[2]To be specific: We view the organisms as the leaves of an evolutionary tree $\mathcal{T}$ that obeys a molecular clock assumption (i.e., all its leaves have the same distance from the root). Then, the set E of these organisms is equipped with a distance function (measuring distances along the edges of the tree), which satisfies the "ultrametric triangle inequality". We define the weight function w by setting $w(e) = 0$ for all $e \in E$. Now, the *phylogenetic diversity* of a subset $S \subseteq E$ is defined to be the sum of the edge lengths of the minimal subtree of $\mathcal{T}$ that connects all leaves in S. This phylogenetic diversity is the measure of biodiversity used in [MoSeSt06]. Meanwhile, our notion of perimeter can also be seen as a measure of biodiversity – perhaps even a better one for sustainability questions, as it rewards subsets that are roughly balanced across different clades. To give a trivial example, a zoo optimized for phylogenetic diversity might have dozens of mammals and only one bird, while this would unlikely be considered optimal in terms of perimeter.

The molecular clock assumption can actually be dropped, at the expense of changing the weight function to account for different distances from the root.

[3]This question might have algorithmic significance. At least for polymatroids, representability can make the difference between a problem being NP-hard and in P, as shown by Lovász in [Lovasz80] for polymatroid matching.

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1. Gaussian elimination greedoids

1.1. The definition

Convention 1.1. Here and in the following, $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$.

Convention 1.2. If E is any set, then 2^E will denote the powerset of E (that is, the set of all subsets of E).

Convention 1.3. Let $\mathbb{K}$ be any field, and let $n \in \mathbb{N}$. Then, $\mathbb{K}^n$ shall denote the $\mathbb{K}$-vector space of all column vectors of size n over $\mathbb{K}$.

We recall the definition of a Gaussian elimination greedoid:

Definition 1.4. Let E be a finite set.

Let $m \in \mathbb{N}$ be such that $m \geqslant |E|$. Let $\mathbb{K}$ be a field. For each $k \in \{0, 1, \ldots, m\}$, let $\pi_k : \mathbb{K}^m \to \mathbb{K}^k$ be the projection map that removes all but the first k coordinates of a column vector. (That is, $\pi_k \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$ for each $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \in \mathbb{K}^m$.)

For each $e \in E$, let $v_e \in \mathbb{K}^m$ be a column vector. The family $(v_e)_{e \in E}$ will be called a *vector family* over $\mathbb{K}$.

Let $\mathcal{G}$ be the subset

$$\left\{ F \subseteq E \mid \text{the family } \left(\pi_{|F|} (v_e) \right)_{e \in F} \in \left(\mathbb{K}^{|F|} \right)^F \text{ is linearly independent} \right\}$$

of 2^E. Then, $\mathcal{G}$ is called the *Gaussian elimination greedoid* of the vector family $(v_e)_{e \in E}$. It is furthermore called a *Gaussian elimination greedoid on ground set E*.

Example 1.5. Let $\mathbb{K} = \mathbb{Q}$ and $E = \{1,2,3,4,5\}$ and $m = 6$. Let $v_1, v_2, v_3, v_4, v_5 \in \mathbb{K}^6$ be the columns of the 6×5-matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 1 \end{pmatrix}.$$

Then, the Gaussian elimination greedoid of the vector family $(v_e)_{e \in E} = (v_1, v_2, v_3, v_4, v_5)$ is the set

$$\{\varnothing, \{2\}, \{3\}, \{5\}, \{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,5\}, \\ \{1,2,3\}, \{1,2,5\}, \{1,2,3,5\}\}.$$

For example, the 3-element set $\{1,2,5\}$ belongs to this greedoid because the family $(\pi_3(v_e))_{e \in \{1,2,5\}} \in (\mathbb{K}^3)^{\{1,2,5\}}$ is linearly independent (indeed, this family consists of the vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$).

Our definition of a Gaussian elimination greedoid follows [Knapp18, §1.3], except that we are using vector families instead of matrices (but this is equivalent, since any matrix can be identified with the vector family consisting of its columns) and we are talking about linear independence rather than non-singularity of matrices (but this is again equivalent, since a square matrix is non-singular if and only if its columns are linearly independent). The same definition is given in [KoLoSc91, §IV.2.3].

1.2. Context

In the rest of Section 1, we shall briefly connect Definition 1.4 with known concepts in the theory of greedoids. This is not necessary for the rest of our work, so the impatient reader can well skip to Section 2.

As the name suggests, Gaussian elimination greedoids are instances of greedoids – a class of set systems (i.e., sets of sets) characterized by some simple axioms. We refer to Definition 12.1 below for the definition of a greedoid, and to [KoLoSc91] for the properties of such. A subclass of greedoids that has particular interest to us are the *strong greedoids*; see, e.g., Section 12 below or [GriPet19, §6.1] or [BrySha99, §2] for their definition.[4] The following theorem is implicit in [KoLoSc91, §IX.4][5]:

[4]They also appear in [KoLoSc91, §IX.4] under the name of "Gauss greedoids", but they are defined differently. (The equivalence between the two definitions is proved in [BrySha99, §2].)

[5]A partial proof of Theorem 1.6 also appears in [Knapp18, §1.3]. (Namely, two paragraphs above

Theorem 1.6. The Gaussian elimination greedoid $\mathcal{G}$ in Definition 1.4 is a strong greedoid.

See Section 12 below for a proof of this theorem.

Matroids are a class of set systems more famous than greedoids; see [Oxley11] for their definition. We will not concern ourselves with matroids much in this note, but let us remark one connection to Gaussian elimination greedoids:[6]

Proposition 1.7. Let $\mathcal{G}$ be a Gaussian elimination greedoid on a ground set E. Let $k \in \mathbb{N}$. Let $\mathcal{G}_k$ be the set of all k-element sets in $\mathcal{G}$. Then, $\mathcal{G}_k$ is either empty or is the collection of bases of a representable matroid on the ground set E.

See Section 13 below for a proof of this proposition.

Proposition 1.7 justifies thinking of Gaussian elimination greedoids as a greedoid analogue of representable matroids.

2. $\mathbb{V}$-ultra triples

Definition 2.1. Let E be a set. Then, $E \underline{\times} E$ shall denote the subset $\{(e, f) \in E \times E \mid e \neq f\}$ of $E \times E$.

Convention 2.2. Fix a totally ordered abelian group $(\mathbb{V}, +, 0, \leqslant)$ (with ground set $\mathbb{V}$, group operation $+$, zero 0 and smaller-or-equal relation $\leqslant$). The total order on $\mathbb{V}$ is required to be translation-invariant (i.e., if $a, b, c \in \mathbb{V}$ satisfy $a \leqslant b$, then $a + c \leqslant b + c$).

We shall refer to the ordered abelian group $(\mathbb{V}, +, 0, \leqslant)$ simply as $\mathbb{V}$. We will use the standard additive notations for the abelian group $\mathbb{V}$; in particular, we will use the $\sum$ sign for finite sums inside the group $\mathbb{V}$. We will furthermore use the standard order-theoretical notations for the totally ordered set $\mathbb{V}$; in particular, we will use the symbol $\geqslant$ for the reverse relation of $\leqslant$ (that is, $a \geqslant b$ means $b \leqslant a$), and we will use the symbols $<$ and $>$ for the strict versions of the relations $\leqslant$ and $\geqslant$. We will denote the largest element of a nonempty subset S of $\mathbb{V}$ (with respect to the relation $\leqslant$) by $\max S$. Likewise, $\min S$ will stand for the smallest element of S.

We will keep this group $\mathbb{V}$ fixed throughout this paper.

For almost all examples we are aware of, it suffices to set $\mathbb{V}$ to be the abelian group $\mathbb{R}$, or even the smaller abelian group $\mathbb{Z}$. Nevertheless, we shall work in full generality, as it serves to separate objects that would otherwise easily be confused.

[Knapp18, Example 1.3.15], it is shown that $\mathcal{G}$ is a greedoid.)

[6] See [Oxley11, §1.1] for the definition of a representable matroid.

Definition 2.3. A $\mathbb{V}$-*ultra triple* shall mean a triple (E, w, d) consisting of:

- a set E, called the *ground set* of this $\mathbb{V}$-ultra triple;
- a map $w : E \to \mathbb{V}$, called the *weight function* of this $\mathbb{V}$-ultra triple;
- a map $d : E \underline{\times} E \to \mathbb{V}$, called the *distance function* of this $\mathbb{V}$-ultra triple, and required to satisfy the following axioms:
  - **Symmetry:** We have $d(a, b) = d(b, a)$ for any two distinct elements a and b of E.
  - **Ultrametric triangle inequality:** We have $d(a, b) \leqslant \max\{d(a, c), d(b, c)\}$ for any three distinct elements a, b and c of E.

If (E, w, d) is a $\mathbb{V}$-ultra triple and $e \in E$, then the value $w(e) \in \mathbb{V}$ is called the *weight* of e.

If (E, w, d) is a $\mathbb{V}$-ultra triple and e and f are two distinct elements of E, then the value $d(e, f) \in \mathbb{V}$ is called the *distance* between e and f.

Example 2.4. For this example, let $\mathbb{V} = \mathbb{Z}$, and let E be a subset of $\mathbb{Z}$. Let m be any integer. Define a map $w : E \to \mathbb{V}$ arbitrarily. Define a map $d : E \underline{\times} E \to \mathbb{V}$ by

$$d(a, b) = \begin{cases} 1, & \text{if } a \not\equiv b \bmod m; \\ 0, & \text{if } a \equiv b \bmod m \end{cases} \qquad \text{for all } (a, b) \in E \underline{\times} E.$$

It is easy to see that (E, w, d) is a $\mathbb{V}$-ultra triple.

Example 2.5. For this example, let $\mathbb{V} = \mathbb{Z}$ again, and let E be a subset of $\mathbb{Z}$. Fix a prime number p. For each nonzero integer k, let $v_p(k)$ denote the largest $i \in \mathbb{N}$ such that $p^i \mid k$. (For instance, $v_3(45) = 2$.)

Define a map $w : E \to \mathbb{V}$ arbitrarily. Define a map $d : E \underline{\times} E \to \mathbb{V}$ by

$$d(a, b) = -v_p(a - b) \qquad \text{for all } (a, b) \in E \underline{\times} E.$$

It is easy to see that (E, w, d) is a $\mathbb{V}$-ultra triple.

More generally, we can replace $\mathbb{Z}$ by any integral domain, and v_p by any valuation on this integral domain, and obtain a $\mathbb{V}$-ultra triple, where $\mathbb{V}$ is the target of our valuation.

The notion of a $\mathbb{V}$-ultra triple generalizes the notion of an ultra triple as defined in [GriPet19]. More precisely, if $\mathbb{V}$ is the additive group $(\mathbb{R}, +, 0, \leqslant)$ (with the usual addition and the usual total order on $\mathbb{R}$), then a $\mathbb{V}$-ultra triple is the same as what is called an "ultra triple" in [GriPet19]. Several properties and examples of ultra triples can be found in [GriPet19].

It is straightforward to adapt all the definitions and results stated in [GriPet19] for ultra triples to the more general setting of $\mathbb{V}$-ultra triples[7]. Let us specifically extend two definitions from [GriPet19] to $\mathbb{V}$-ultra triples: the definition of a perimeter ([GriPet19, §3.1]) and the definition of the Bhargava greedoid ([GriPet19, §6.2]):

Definition 2.6. Let (E, w, d) be a $\mathbb{V}$-ultra triple. Let A be a finite subset of E. Then, the *perimeter* of A (with respect to (E, w, d)) is defined to be

$$\sum_{a \in A} w(a) + \sum_{\substack{\{a,b\} \subseteq A; \\ a \neq b}} d(a,b) \in \mathbb{V}.$$

(Here, the second sum ranges over all **unordered** pairs $\{a, b\}$ of distinct elements of A.)

The perimeter of A is denoted by $\operatorname{PER}(A)$.

For example, if $A = \{p, q, r\}$ is a 3-element set, then

$$\operatorname{PER}(A) = w(p) + w(q) + w(r) + d(p,q) + d(p,r) + d(q,r).$$

Definition 2.7. Let S be any set, and let $k \in \mathbb{N}$. A *k-subset* of S means a k-element subset of S (that is, a subset of S having size k).

Definition 2.8. Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. The *Bhargava greedoid* of (E, w, d) is defined to be the subset

$$\begin{aligned} &\{A \subseteq E \mid A \text{ has maximum perimeter among all } |A| \text{-subsets of } E\} \\ &= \{A \subseteq E \mid \operatorname{PER}(A) \geqslant \operatorname{PER}(B) \text{ for all } B \subseteq E \text{ satisfying } |B| = |A|\} \end{aligned}$$

of 2^E.

Some examples of Bhargava greedoids can be found in [GriPet19, §6.2]. Here are two more:

[7] There is one stupid exception: The definition of R in [GriPet19, Remark 8.13] requires $\mathbb{V} \neq 0$. But [GriPet19, Remark 8.13] is just a tangent without concrete use.

Example 2.9. For this example, let $\mathbb{V} = \mathbb{Z}$ and $E = \{0,1,2,3,4\}$. Define a map $w : E \to \mathbb{V}$ by setting $w(e) = \max\{e,1\}$ for each $e \in E$. (Thus, $w(0) = 1$ and $w(e) = e$ for all $e > 0$.) Define a map $d : E \underline{\times} E \to \mathbb{V}$ by setting

$$d(e,f) = \min\{3, \max\{4-e, 4-f\}\} \qquad \text{for all } (e,f) \in E \underline{\times} E.$$

Here is a table of values of d:

d	0	1	2	3	4
0		3	3	3	3
1	3		3	3	3
2	3	3		2	2
3	3	3	2		1
4	3	3	2	1	

.

It is easy to see that (E, w, d) is a $\mathbb{V}$-ultra triple. Let $\mathcal{F}$ be its Bhargava greedoid. Thus, $\mathcal{F}$ consists of the subsets A of E that have maximum perimeter among all $|A|$-subsets of E. What are these subsets?

- Clearly, $\varnothing$ is the only $|\varnothing|$-subset of E, and thus has maximum perimeter among all $|\varnothing|$-subsets of E. Hence, $\varnothing \in \mathcal{F}$.

- The perimeter of a 1-subset $\{e\}$ of E is just the weight $w(e)$. Thus, the 1-subsets of E having maximum perimeter among all 1-subsets of E are precisely the subsets $\{e\}$ where $e \in E$ has maximum weight. In our example, there is only one $e \in E$ having maximum weight, namely 4. Thus, the only 1-subset of E having maximum perimeter among all 1-subsets of E is $\{4\}$. In other words, the only 1-element set in $\mathcal{F}$ is $\{4\}$.

- What about 2-element sets in $\mathcal{F}$? The perimeter $\operatorname{PER}\{e,f\}$ of a 2-subset $\{e,f\}$ of E is $w(e) + w(f) + d(e,f)$. Thus,

  $$\operatorname{PER}\{0,4\} = w(0) + w(4) + d(0,4) = 1 + 4 + 3 = 8$$

  and similarly $\operatorname{PER}\{1,4\} = 8$ and $\operatorname{PER}\{2,4\} = 8$ and $\operatorname{PER}\{3,4\} = 8$ and $\operatorname{PER}\{0,3\} = 7$ and $\operatorname{PER}\{1,3\} = 7$ and $\operatorname{PER}\{2,3\} = 7$ and $\operatorname{PER}\{0,2\} = 6$ and $\operatorname{PER}\{1,2\} = 6$ and $\operatorname{PER}\{0,1\} = 5$. Thus, the 2-subsets of E having maximum perimeter among all 2-subsets of E are $\{0,4\}$ and $\{1,4\}$ and $\{2,4\}$ and $\{3,4\}$. So these four sets are the 2-element sets in $\mathcal{F}$.

- Similarly, the 3-element sets in $\mathcal{F}$ are $\{0,1,4\}$, $\{0,3,4\}$, $\{1,3,4\}$, $\{0,2,4\}$ and $\{1,2,4\}$. They have perimeter 15, while all other 3-subsets of E have perimeter 14 or 13.

- Similarly, the 4-element sets in $\mathcal{F}$ are $\{0,1,2,4\}$ and $\{0,1,3,4\}$.

- Clearly, E is the only $|E|$-subset of E, and thus has maximum perimeter among all $|E|$-subsets of E. Hence, $E \in \mathcal{F}$.

Thus, the Bhargava greedoid of (E, w, d) is

$$\mathcal{F} = \{\varnothing, \{4\}, \{0,4\}, \{1,4\}, \{2,4\}, \{3,4\}, \\ \{0,1,4\}, \{0,3,4\}, \{1,3,4\}, \{0,2,4\}, \{1,2,4\}, \{0,1,2,4\}, \{0,1,3,4\}, E\}.$$

Example 2.10. For this example, let $\mathbb{V} = \mathbb{Z}$ and $E = \{1,2,3\}$. Define a map $w : E \to \mathbb{V}$ by setting $w(e) = e$ for each $e \in E$. Define a map $d : E \underline{\times} E \to \mathbb{V}$ by setting $d(e,f) = 1$ for each $(e,f) \in E \underline{\times} E$. It is easy to see that (E, w, d) is a $\mathbb{V}$-ultra triple. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). What is $\mathcal{F}$?

The same kind of reasoning as in Example 2.9 (but simpler due to the fact that all values of d are the same) shows that

$$\mathcal{F} = \{\varnothing, \{3\}, \{2,3\}, \{1,2,3\}\}.$$

One thing we observed in both of these examples is the following simple fact:

Remark 2.11. Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Then, $E \in \mathcal{F}$.

Proof of Remark 2.11. The set E obviously has maximum perimeter among all $|E|$-subsets of E (since E is the only $|E|$-subset of E).

But $\mathcal{F}$ is the Bhargava greedoid of (E, w, d). In other words,

$$\mathcal{F} = \{A \subseteq E \mid A \text{ has maximum perimeter among all } |A| \text{-subsets of } E\}$$

(by Definition 2.8). Hence, $E \in \mathcal{F}$ (since E is a subset of E that has maximum perimeter among all $|E|$-subsets of E). This proves Remark 2.11. □

3. The main theorem

In [GriPet19, Theorem 6.1], it was proved that the Bhargava greedoid of an ultra triple with finite ground set is a strong greedoid[8]. More generally, this holds for any $\mathbb{V}$-ultra triple with finite ground set (and the same argument can be used to prove this). However, we shall prove a stronger statement:

Theorem 3.1. Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Let $\mathbb{K}$ be a field of size $|\mathbb{K}| \geqslant |E|$. Then, $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$.

We will spend the next few sections working towards a proof of this theorem. First, however, let us extend it somewhat by strengthening the $|\mathbb{K}| \geqslant |E|$ bound.

[8] See [GriPet19, §6.1] for the definition of strong greedoids.

4. Cliques and stronger bounds

For the rest of Section 4, we fix a $\mathbb{V}$-ultra triple (E, w, d).

Let us define a certain kind of subsets of E, which we call *cliques*.

Definition 4.1. Let $\alpha \in \mathbb{V}$. An *α-clique* of (E, w, d) will mean a subset F of E such that any two distinct elements $a, b \in F$ satisfy $d(a, b) = \alpha$.

Definition 4.2. A *clique* of (E, w, d) will mean a subset of E that is an α-clique for some $\alpha \in \mathbb{V}$.

Thus, any 1-element subset of E is a clique (and an α-clique for every $\alpha \in \mathbb{V}$). The same holds for the empty subset. Any 2-element subset $\{a, b\}$ of E is a clique and, in fact, a $d(a, b)$-clique.

Note that the notion of a clique (and of an α-clique) depends only on E and d, not on w.

Example 4.3. For this example, let m, $\mathbb{V}$, E, w and d be as in Example 2.4. Then:

- **(a)** The 0-cliques of E are the subsets of E whose elements are all mutually congruent modulo m.
- **(b)** The 1-cliques of E are the subsets of E that have no two distinct elements congruent to each other modulo m. Thus, any 1-clique has size $\leqslant m$ if m is positive.
- **(c)** If $\alpha \in \mathbb{V}$ is distinct from 0 and 1, then the α-cliques of E are the subsets of E having size $\leqslant 1$.

Using the notion of cliques, we can assign a number $\operatorname{mcs}(E, w, d)$ to our $\mathbb{V}$-ultra triple (E, w, d):

Definition 4.4. Let $\operatorname{mcs}(E, w, d)$ denote the maximum size of a clique of (E, w, d). (This is well-defined whenever E is finite, and sometimes even otherwise.)

Clearly, $\operatorname{mcs}(E, w, d) \leqslant |E|$, since any clique of (E, w, d) is a subset of E.

Example 4.5. Let $\mathbb{V}$, E, w and d be as in Example 2.9. Then, $\{0, 1, 2\}$ is a 3-clique of (E, w, d) and has size 3; no larger cliques exist in (E, w, d). Thus, $\operatorname{mcs}(E, w, d) = 3$.

Example 4.6. For this example, let m, $\mathbb{V}$, E, w and d be as in Example 2.4. Then:

- **(a)** If $m = 2$ and $E = \{1, 2, 3, 4, 5, 6\}$, then $\operatorname{mcs}(E, w, d) = 3$, due to the 0-clique $\{1, 3, 5\}$ having maximum size among all cliques.
- **(b)** If $m = 3$ and $E = \{1, 2, 3, 4, 5, 6\}$, then $\operatorname{mcs}(E, w, d) = 3$, due to the 1-clique $\{1, 2, 3\}$ having maximum size among all cliques.

We can now state a stronger version of Theorem 3.1:

Theorem 4.7. Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Let $\mathbb{K}$ be a field of size $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$. Then, $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$.

Theorem 4.7 is stronger than Theorem 3.1 because $|E| \geqslant \operatorname{mcs}(E, w, d)$.

We shall prove Theorem 4.7 in Section 10.

5. The converse direction

Before that, let us explore the question whether the bound $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$ can be improved. In an important particular case – namely, when the map w is constant[9] –, it cannot, as the following theorem shows:

Theorem 5.1. Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. Assume that the map w is constant. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Let $\mathbb{K}$ be a field such that $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$. Then, $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$.

We shall prove Theorem 5.1 in Section 11.

When the map w in a $\mathbb{V}$-ultra triple (E, w, d) is constant, Theorems 4.7 and 5.1 combined yield an exact characterization of those fields $\mathbb{K}$ for which the Bhargava greedoid of (E, w, d) can be represented as the Gaussian elimination greedoid of a vector family over $\mathbb{K}$: Namely, those fields are precisely the fields $\mathbb{K}$ of size $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$. When w is not constant, Theorem 4.7 gives a sufficient condition; we don't know a necessary condition. Here are two examples:

Example 5.2. Let $\mathbb{V}$, E, w, d and $\mathcal{F}$ be as in Example 2.9. Then, $\operatorname{mcs}(E, w, d) = 3$ (as we saw in Example 4.5). Hence, Theorem 4.7 shows that $\mathcal{F}$ can be represented as the Gaussian elimination greedoid of a vector family over any field $\mathbb{K}$ of size $|\mathbb{K}| \geqslant 3$. This bound on $|\mathbb{K}|$ is optimal, since the Bhargava greedoid $\mathcal{F}$ is not the Gaussian elimination greedoid of any vector family over the 2-element field $\mathbb{F}_2$. (But this does not follow from Theorem 5.1, because w is not constant.)

Example 5.3. Let $\mathbb{V}$, E, w, d and $\mathcal{F}$ be as in Example 2.10. Then, $\operatorname{mcs}(E, w, d) = 3$, since E itself is a clique. Hence, Theorem 4.7 shows that $\mathcal{F}$ can be represented as the Gaussian elimination greedoid of a vector family over any field $\mathbb{K}$ of size $|\mathbb{K}| \geqslant 3$. However, this bound on $|\mathbb{K}|$ is not optimal. Indeed, the Bhargava

[9] A map $f : X \to Y$ between two sets X and Y is said to be *constant* if all values of f are equal (i.e., if every $x_1, x_2 \in X$ satisfy $f(x_1) = f(x_2)$). In particular, if $|X| \leqslant 1$, then $f : X \to Y$ is automatically constant.

greedoid $\mathcal{F}$ is the Gaussian elimination greedoid of the vector family $(v_e)_{e\in E} = (v_1, v_2, v_3)$ over the field $\mathbb{F}_2$, where $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Question 5.4. Let (E, w, d) be a $\mathbb{V}$-ultra triple such that E is finite. How to characterize the fields $\mathbb{K}$ for which the Bhargava greedoid of (E, w, d) is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$? Is there a constant $c(E, w, d)$ such that these fields are precisely the fields of size $\geqslant c(E, w, d)$?

Remark 5.5. Let E, w, d and $\mathcal{F}$ be as in Theorem 3.1. Let $\mathbb{K}$ be any field. For each $k \in \mathbb{N}$, let $\mathcal{F}_k$ be the set of all k-element sets in $\mathcal{F}$.

If $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$, then each $\mathcal{F}_k$ with $k \in \{0, 1, \ldots, |E|\}$ is the collection of bases of a representable matroid on the ground set E. (Indeed, this follows from Proposition 1.7, since $\mathcal{F}_k$ is nonempty.) But the converse is not true: It can happen that each $\mathcal{F}_k$ with $k \in \{0, 1, \ldots, |E|\}$ is the collection of bases of a representable matroid on the ground set E, yet $\mathcal{F}$ is not the Gaussian elimination greedoid of a vector family over $\mathbb{K}$. For example, this happens if $E = \{1, 2, 3\}$ and both maps w and d are constant (so that $\mathcal{F} = 2^E$), and $\mathbb{K} = \mathbb{F}_2$.

6. Valadic $\mathbb{V}$-ultra triples

As a first step towards the proof of Theorem 4.7, we will next introduce a special kind of $\mathbb{V}$-ultra triples which, in a way, are similar to Bhargava's for integers (see [GriPet19, Example 2.5 and §9]). We will call them *valadic*[10], and we will see (in Theorem 6.9) that they satisfy Theorem 3.1. Afterwards (in Theorem 9.2), we will prove that any $\mathbb{V}$-ultra triple with finite ground set is isomorphic (in an appropriate sense) to a valadic one over a sufficiently large field. Combining these two facts, we will then readily obtain Theorem 4.7.

Recall that $\mathbb{V}$ is a totally ordered abelian group (see Convention 2.2 for details). Let us introduce some further notations that will be used throughout Section 6.

[10]The name is a homage to the notion of a valuation ring, which is latent in the argument that follows (although never used explicitly). Indeed, if we define the notion of a valuation ring as in [Eisenb95, Exercise 11.1], then the $\mathbb{K}$-algebra $\mathbb{L}_+$ constructed below is an instance of a valuation ring (with $\mathbb{L}$ being its fraction field, and $\operatorname{ord} : \mathbb{L} \setminus \{0\} \to \mathbb{V}$ being its valuation), and many of its properties that will be used below are instances of general properties of valuation rings. If we extended our argument to the more general setting of valuation rings, we would also recover Bhargava's original ultra triples based on integer divisibility (see [GriPet19, Example 2.5 and §9]). However, we have no need for this generality (as we only need the construction as a stepping stone towards our proof of Theorem 4.7), and prefer to remain elementary and self-contained.

Definition 6.1. We fix a field $\mathbb{K}$. Let $\mathbb{K}[\mathbb{V}]$ denote the group algebra of the group $\mathbb{V}$ over $\mathbb{K}$. This is a free $\mathbb{K}$-module with basis $(t_\alpha)_{\alpha\in\mathbb{V}}$; it becomes a $\mathbb{K}$-algebra with unity t_0 and with multiplication determined by

$$t_\alpha t_\beta = t_{\alpha+\beta} \qquad \text{for all } \alpha, \beta \in \mathbb{V}.$$

This group algebra $\mathbb{K}[\mathbb{V}]$ is commutative, since the group $\mathbb{V}$ is abelian.

Let $\mathbb{V}_{\geqslant 0}$ be the set of all $\alpha \in \mathbb{V}$ satisfying $\alpha \geqslant 0$; this is a submonoid of the group $\mathbb{V}$. Let $\mathbb{K}[\mathbb{V}_{\geqslant 0}]$ be the monoid algebra of this monoid $\mathbb{V}_{\geqslant 0}$ over $\mathbb{K}$. This is a $\mathbb{K}$-algebra defined in the same way as $\mathbb{K}[\mathbb{V}]$, but using $\mathbb{V}_{\geqslant 0}$ instead of $\mathbb{V}$. It is clear that $\mathbb{K}[\mathbb{V}_{\geqslant 0}]$ is the $\mathbb{K}$-subalgebra of $\mathbb{K}[\mathbb{V}]$ spanned by the basis elements t_α with $\alpha \in \mathbb{V}_{\geqslant 0}$.

Example 6.2. If $\mathbb{V} = \mathbb{Z}$ (with the usual addition and total order), then $\mathbb{V}_{\geqslant 0} = \mathbb{N}$. In this case, the group algebra $\mathbb{K}[\mathbb{V}]$ is the Laurent polynomial ring $\mathbb{K}[X, X^{-1}]$ in a single indeterminate X over $\mathbb{K}$ (indeed, t_1 plays the role of X, and more generally, each t_α plays the role of X^α), and its subalgebra $\mathbb{K}[\mathbb{V}_{\geqslant 0}]$ is the polynomial ring $\mathbb{K}[X]$.

This example should be regarded as a guide; even in the general case (where $\mathbb{V}$ does not have to be $\mathbb{Z}$), the reader cannot go wrong thinking of $\mathbb{K}[\mathbb{V}]$ as a generalized Laurent polynomial ring and of $\mathbb{K}[\mathbb{V}_{\geqslant 0}]$ as a generalized polynomial ring (in a single indeterminate) and of t_α as a generalized monomial X^α. This analogy shall clarify much of what follows.

Definition 6.3.

(a) Let $\mathbb{L}$ be the commutative $\mathbb{K}$-algebra $\mathbb{K}[\mathbb{V}]$, and let $\mathbb{L}_+$ be its $\mathbb{K}$-subalgebra $\mathbb{K}[\mathbb{V}_{\geqslant 0}]$. Thus, the $\mathbb{K}$-module $\mathbb{L}$ has basis $(t_\alpha)_{\alpha\in\mathbb{V}}$, while its $\mathbb{K}$-submodule $\mathbb{L}_+$ has basis $(t_\alpha)_{\alpha\in\mathbb{V}_{\geqslant 0}}$.

(b) If $a \in \mathbb{L}$ and $\beta \in \mathbb{V}$, then $[t_\beta]\, a$ shall denote the coefficient of t_β in a (when a is expanded in the basis $(t_\alpha)_{\alpha\in\mathbb{V}}$ of $\mathbb{L}$). This is an element of $\mathbb{K}$. For example, $[t_3](t_2 - t_3 + 5t_6) = -1$ (if $\mathbb{V} = \mathbb{Z}$).

(c) If $a \in \mathbb{L}$ is nonzero, then the *order* of a is defined to be the smallest $\beta \in \mathbb{V}$ such that $[t_\beta]\, a \neq 0$. This order is an element of $\mathbb{V}$, and is denoted by $\operatorname{ord} a$. For example, $\operatorname{ord}(t_2 - t_3 + 5t_6) = 2$ (if $\mathbb{V} = \mathbb{Z}$). Note that $\operatorname{ord}(t_\alpha) = \alpha$ for each $\alpha \in \mathbb{V}$.

The notations we just defined generalize standard features of Laurent polynomials: If $\mathbb{V} = \mathbb{Z}$ as in Example 6.2, then the coefficient $[t_\beta]\, a$ of an element $a \in \mathbb{L} = \mathbb{K}[X, X^{-1}]$ is simply the coefficient of X^β in the Laurent polynomial a, and the order $\operatorname{ord} a$ of a nonzero Laurent polynomial $a \in \mathbb{L}$ is the order of a in the usual sense (i.e., the smallest exponent of a monomial appearing in a). If we substitute X^{-1} for X (thus replacing each monomial X^{β} by $X^{-\beta}$), then the order of a Laurent polynomial becomes its degree (with a negative sign). In light of this, the following properties of orders should not be surprising:

Lemma 6.4.

(a) A nonzero element $a \in \mathbb{L}$ belongs to $\mathbb{L}_+$ if and only if its order $\operatorname{ord} a$ is nonnegative (i.e., we have $\operatorname{ord} a \geqslant 0$).

(b) We have $\operatorname{ord}(-a) = \operatorname{ord} a$ for any nonzero $a \in \mathbb{L}$.

(c) Let a and b be two nonzero elements of $\mathbb{L}$. Then, ab is nonzero and satisfies $\operatorname{ord}(ab) = \operatorname{ord} a + \operatorname{ord} b$.

(d) Let a and b be two nonzero elements of $\mathbb{L}$ such that $a + b$ is nonzero. Then, $\operatorname{ord}(a+b) \geqslant \min\{\operatorname{ord} a, \operatorname{ord} b\}$.

See Section 14 for the (straightforward) proof of this lemma. (This proof is entirely analogous to the proof of the corresponding properties of usual polynomials.)

Corollary 6.5. The ring $\mathbb{L}$ is an integral domain.

Proof of Corollary 6.5. This follows from Lemma 6.4 **(c)**. □

Applying Lemma 6.4 **(c)** many times, we also obtain the following:

Corollary 6.6. The map $\operatorname{ord} : \mathbb{L} \setminus \{0\} \to \mathbb{V}$ transforms (finite) products into sums. In more detail: If $(a_i)_{i \in I}$ is any finite family of nonzero elements of $\mathbb{L}$, then the product $\prod\limits_{i \in I} a_i$ is nonzero and satisfies

$$\operatorname{ord}\left(\prod_{i \in I} a_i\right) = \sum_{i \in I} \operatorname{ord}(a_i).$$

Proof. Induction on $|I|$. The induction step uses Lemma 6.4 **(c)**; the straightforward details are left to the reader. □

We can now assign a $\mathbb{V}$-ultra triple to each subset of $\mathbb{L}$:

Definition 6.7. Let E be a subset of $\mathbb{L}$. Define a distance function $d : E \underline{\times} E \to \mathbb{V}$ by setting

$$d(a,b) = -\operatorname{ord}(a-b) \qquad \text{for all } (a,b) \in E \underline{\times} E.$$

(Recall that $E \underline{\times} E$ means the set $\{(a,b) \in E \times E \mid a \neq b\}$.)

Then, (E, w, d) is a $\mathbb{V}$-ultra triple whenever $w : E \to \mathbb{V}$ is a function (by Lemma 6.8 below). Such a $\mathbb{V}$-ultra triple (E, w, d) will be called *valadic*.

Lemma 6.8. In Definition 6.7, the triple (E, w, d) is indeed a $\mathbb{V}$-ultra triple.

Lemma 6.8 follows easily from Lemma 6.4. (See Section 14 for the details of the proof.)

Now, we claim that the Bhargava greedoid of a valadic $\mathbb{V}$-ultra triple (E, w, d) with finite E is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$:

Theorem 6.9. Let E be a finite subset of $\mathbb{L}$. Define d as in Definition 6.7. Let $w : E \to \mathbb{V}$ be a function. Then, the Bhargava greedoid of the $\mathbb{V}$-ultra triple (E, w, d) is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$.

In order to prove this theorem, we will need a determinantal identity:

Lemma 6.10. Let R be a commutative ring. Consider the polynomial ring $R[X]$. Let $m \in \mathbb{N}$. Let $f_1, f_2, \ldots, f_m$ be m polynomials in $R[X]$. Assume that f_j is a monic polynomial of degree $j-1$ for each $j \in \{1, 2, \ldots, m\}$. Let $u_1, u_2, \ldots, u_m$ be m elements of R. Then,

$$\det\left(\left(f_j(u_i)\right)_{1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant m}\right) = \prod_{\substack{(i,j) \in \{1,2,\ldots,m\}^2; \\ i > j}} \left(u_i - u_j\right).$$

Here, we are using the notation $\left(b_{i,j}\right)_{1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q}$ for the $p \times q$-matrix whose (i, j)-th entry is $b_{i,j}$ for all $i \in \{1, 2, \ldots, p\}$ and all $j \in \{1, 2, \ldots, q\}$.

Lemma 6.10 is a classical generalization of the famous Vandermonde determinant. In this form, it is a particular case of [Grinbe11, Theorem 2] (applied to $P_j = f_j$ and $a_i = u_i$), because the coefficient of X^{j-1} in a monic polynomial of degree $j-1$ is 1. It also appears in [FadSom72, Exercise 267] (where it is stated for the transpose of the matrix we are considering here), in [Muir60, Chapter XI, Exercise 2 in Set XVIII] (where it, too, is stated for the transpose of the matrix), in [Kratte99, Proposition 1], and in [Grinbe22, Exercise 6.62].

We need two more simple lemmas for our proof of Theorem 6.9:

Lemma 6.11. The map

$$\pi : \mathbb{L}_+ \to \mathbb{K},$$
$$x \mapsto [t_0]\, x$$

is a $\mathbb{K}$-algebra homomorphism.

Lemma 6.12. Consider the map $\pi : \mathbb{L}_+ \to \mathbb{K}$ from Lemma 6.11. Let $a \in \mathbb{L}_+$ be nonzero. Then, $\pi(a) \neq 0$ holds if and only if $\operatorname{ord} a = 0$.

See Section 14 for the (easy) proofs of these two lemmas.

Proof of Theorem 6.9. Let $m = |E|$. Consider the $\mathbb{V}$-ultra triple (E, w, d); all perimeters discussed in this proof are defined with respect to this $\mathbb{V}$-ultra triple.

We construct a list $(c_1, c_2, \ldots, c_m)$ of elements of E by the following recursive procedure:

- For each $i \in \{1, 2, \ldots, m\}$, we choose c_i (assuming that all the preceding entries $c_1, c_2, \ldots, c_{i-1}$ of our list are already constructed) to be an element of $E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$ that maximizes the perimeter $\operatorname{PER}\{c_1, c_2, \ldots, c_i\}$.

This procedure can indeed be carried out, since at each step we can find an element $c_i \in E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$ that maximizes the perimeter $\operatorname{PER}\{c_1, c_2, \ldots, c_i\}$. [11] Clearly, this procedure constructs an m-tuple $(c_1, c_2, \ldots, c_m)$ of elements of E. The m entries $c_1, c_2, \ldots, c_m$ of this m-tuple are distinct[12], and thus are m distinct elements of E; but E has only m elements altogether (since $m = |E|$). Hence, the m entries $c_1, c_2, \ldots, c_m$ must cover the whole set E. In other words, $E = \{c_1, c_2, \ldots, c_m\}$.

Furthermore, for each $i \in \{1, 2, \ldots, m\}$ and each $x \in E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$, we have

$$\operatorname{PER}\{c_1, c_2, \ldots, c_i\} \geqslant \operatorname{PER}\{c_1, c_2, \ldots, c_{i-1}, x\} \tag{1}$$

(due to how c_i is chosen). Thus, in the parlance of [GriPet19, §3.2], the m-tuple $(c_1, c_2, \ldots, c_m)$ is a greedy m-permutation of E.

For each $j \in \{1, 2, \ldots, m\}$, define a $\rho_j \in \mathbb{V}$ by

$$\rho_j = w(c_j) + \sum_{i=1}^{j-1} d(c_i, c_j). \tag{2}$$

(This is precisely what is called $\nu_j^\circ(C)$ in [GriPet19], where $C = E$.)

Consider the polynomial ring $\mathbb{L}[X]$. For each $j \in \{1, 2, \ldots, m\}$, define a polynomial $f_j \in \mathbb{L}[X]$ by

$$f_j = (X - c_1)(X - c_2) \cdots (X - c_{j-1}) = \prod_{i=1}^{j-1} (X - c_i).$$

This is a monic polynomial of degree $j - 1$.

Next we claim the following:

Claim 1: Let $e \in E$ and $j \in \{1, 2, \ldots, m\}$. Then, $t_{\rho_j - w(e)} f_j(e) \in \mathbb{L}_+$.

[11] Indeed, the set $E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$ is nonempty (since $|\{c_1, c_2, \ldots, c_{i-1}\}| \leqslant i - 1 < i \leqslant m = |E|$ and thus $\{c_1, c_2, \ldots, c_{i-1}\} \not\supseteq E$) and finite (since E is finite), and thus at least one of its elements will maximize the perimeter in question.

[12] since each c_i is chosen to be an element of $E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$, and thus is distinct from all the preceding entries $c_1, c_2, \ldots, c_{i-1}$

[*Proof of Claim 1:* We have $f_j = \prod_{i=1}^{j-1} (X - c_i)$ and thus $f_j(e) = \prod_{i=1}^{j-1} (e - c_i)$. Hence, if $e \in \{c_1, c_2, \ldots, c_{j-1}\}$, then $f_j(e) = 0$ and thus our claim $t_{\rho_j - w(e)} f_j(e) \in \mathbb{L}_+$ is obvious. Thus, we WLOG assume that $e \notin \{c_1, c_2, \ldots, c_{j-1}\}$. Thus, each $i \in \{1, 2, \ldots, j-1\}$ satisfies $e \neq c_i$ and thus $e - c_i \neq 0$. Hence, $\prod_{i=1}^{j-1} (e - c_i)$ is a product of nonzero elements of $\mathbb{L}$, and thus is itself nonzero (since Corollary 6.5 says that $\mathbb{L}$ is an integral domain). In other words, $f_j(e)$ is nonzero (since $f_j(e) = \prod_{i=1}^{j-1} (e - c_i)$). Hence, $t_{\rho_j - w(e)} f_j(e)$ is nonzero as well (since $t_{\rho_j - w(e)}$ is nonzero, and since $\mathbb{L}$ is an integral domain).

Moreover, from $f_j(e) = \prod_{i=1}^{j-1} (e - c_i)$, we obtain

$$\operatorname{ord}\left(f_j(e)\right) = \operatorname{ord}\left(\prod_{i=1}^{j-1} (e - c_i)\right) = \sum_{i=1}^{j-1} \operatorname{ord}(e - c_i) \tag{3}$$

(by Corollary 6.6).

From $e \in E$ and $e \notin \{c_1, c_2, \ldots, c_{j-1}\}$, we obtain $e \in E \setminus \{c_1, c_2, \ldots, c_{j-1}\}$. Hence, (1) (applied to $i = j$ and $x = e$) yields

$$\operatorname{PER}\{c_1, c_2, \ldots, c_j\} \geqslant \operatorname{PER}\{c_1, c_2, \ldots, c_{j-1}, e\}. \tag{4}$$

But $c_1, c_2, \ldots, c_j$ are distinct[13]. Hence, the definition of the perimeter yields

$$\begin{aligned}
\operatorname{PER}\{c_1, c_2, \ldots, c_j\} &= \underbrace{\sum_{i=1}^{j} w(c_i)}_{= \sum_{i=1}^{j-1} w(c_i) + w(c_j)} + \underbrace{\sum_{1 \leqslant i < p \leqslant j} d(c_i, c_p)}_{= \sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p) + \sum_{i=1}^{j-1} d(c_i, c_j)} \\
&= \sum_{i=1}^{j-1} w(c_i) + w(c_j) + \sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p) + \sum_{i=1}^{j-1} d(c_i, c_j) \\
&= \underbrace{w(c_j) + \sum_{i=1}^{j-1} d(c_i, c_j)}_{\substack{= \rho_j \\ \text{(by (2))}}} + \sum_{i=1}^{j-1} w(c_i) + \sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p) \\
&= \rho_j + \sum_{i=1}^{j-1} w(c_i) + \sum_{1 \leqslant i < p \leqslant j-1} d(c_i, c_p)
\end{aligned}$$

[13]This is because $c_1, c_2, \ldots, c_m$ are distinct.

and

$$\operatorname{PER}\left\{c_1, c_2, \ldots, c_{j-1}, e\right\} = \sum_{i=1}^{j-1} w\left(c_i\right) + w\left(e\right) + \sum_{1 \leqslant i < p \leqslant j-1} d\left(c_i, c_p\right) + \sum_{i=1}^{j-1} d\left(c_i, e\right)$$

(since $c_1, c_2, \ldots, c_{j-1}, e$ are distinct[14]). Hence, (4) rewrites as

$$\begin{aligned} &\rho_j + \sum_{i=1}^{j-1} w\left(c_i\right) + \sum_{1 \leqslant i < p \leqslant j-1} d\left(c_i, c_p\right) \\ &\geqslant \sum_{i=1}^{j-1} w\left(c_i\right) + w\left(e\right) + \sum_{1 \leqslant i < p \leqslant j-1} d\left(c_i, c_p\right) + \sum_{i=1}^{j-1} d\left(c_i, e\right). \end{aligned}$$

After cancelling equal terms, this inequality transforms into

$$\rho_j \geqslant w\left(e\right) + \sum_{i=1}^{j-1} d\left(c_i, e\right).$$

In view of

$$\sum_{i=1}^{j-1} \underbrace{d\left(c_i, e\right)}_{\substack{=d(e,c_i) \\ \text{(by the “Symmetry”} \\ \text{axiom in the definition} \\ \text{of a } \mathbb{V}\text{-ultra triple)}}} = \sum_{i=1}^{j-1} \underbrace{d\left(e, c_i\right)}_{\substack{=-\operatorname{ord}(e-c_i) \\ \text{(by the definition of } d)}} = -\underbrace{\sum_{i=1}^{j-1} \operatorname{ord}\left(e - c_i\right)}_{\substack{=\operatorname{ord}\left(f_j(e)\right) \\ \text{(by (3))}}} = -\operatorname{ord}\left(f_j\left(e\right)\right),$$

this rewrites as

$$\rho_j \geqslant w\left(e\right) - \operatorname{ord}\left(f_j\left(e\right)\right).$$

In other words, $\operatorname{ord}\left(f_j\left(e\right)\right) \geqslant w\left(e\right) - \rho_j$. Now, Lemma 6.4 **(c)** (applied to $a = t_{\rho_j - w(e)}$ and $b = f_j\left(e\right)$) yields

$$\begin{aligned} \operatorname{ord}\left(t_{\rho_j - w(e)} f_j\left(e\right)\right) &= \underbrace{\operatorname{ord}\left(t_{\rho_j - w(e)}\right)}_{=\rho_j - w(e)} + \underbrace{\operatorname{ord}\left(f_j\left(e\right)\right)}_{\geqslant w(e) - \rho_j} \\ &\geqslant \rho_j - w\left(e\right) + w\left(e\right) - \rho_j = 0. \end{aligned}$$

Hence, Lemma 6.4 **(a)** (applied to $a = t_{\rho_j - w(e)} f_j\left(e\right)$) shows that $t_{\rho_j - w(e)} f_j\left(e\right)$ belongs to $\mathbb{L}_+$. Thus, $t_{\rho_j - w(e)} f_j\left(e\right) \in \mathbb{L}_+$. This proves Claim 1.]

For each $e \in E$ and $j \in \{1, 2, \ldots, m\}$, we define an $a\left(e, j\right) \in \mathbb{L}_+$ by

$$a\left(e, j\right) = t_{\rho_j - w(e)} f_j\left(e\right). \tag{5}$$

(This is well-defined, due to Claim 1.)

We now claim the following:

[14]This is because $c_1, c_2, \ldots, c_m$ are distinct and $e \notin \left\{c_1, c_2, \ldots, c_{j-1}\right\}$.

Claim 2: Let $k \in \mathbb{N}$. Let $u_1, u_2, \ldots, u_k$ be any k distinct elements of E. Let $U = \{u_1, u_2, \ldots, u_k\}$. Then,

$$\det\left(\left(a\left(u_i, j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right) \text{ is a nonzero element of } \mathbb{L}_+ \tag{6}$$

and

$$\operatorname{ord}\left(\det\left(\left(a\left(u_i, j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right) = \sum_{j=1}^{k} \rho_j - \operatorname{PER}(U). \tag{7}$$

[*Proof of Claim 2:* The set E has at least k many elements (since $u_1, u_2, \ldots, u_k$ are k distinct elements of E). In other words, $|E| \geqslant k$. Hence, $k \leqslant |E| = m$. Therefore, $\{1, 2, \ldots, k\} \subseteq \{1, 2, \ldots, m\}$. In other words, for each $j \in \{1, 2, \ldots, k\}$, we have $j \in \{1, 2, \ldots, m\}$.

Hence, $a(u_i, j) \in \mathbb{L}_+$ for any $i, j \in \{1, 2, \ldots, k\}$ (since we defined $a(e, j)$ to satisfy $a(e, j) \in \mathbb{L}_+$ for any $e \in E$ and $j \in \{1, 2, \ldots, m\}$). In other words, all entries of the matrix $(a(u_i, j))_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}$ belong to $\mathbb{L}_+$. Hence, its determinant $\det\left((a(u_i, j))_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)$ belongs to $\mathbb{L}_+$ as well (since $\mathbb{L}_+$ is a ring).

Lemma 6.10 (applied to $\mathbb{L}$ and k instead of R and m) yields

$$\det\left(\left(f_j\left(u_i\right)\right)_{1\leqslant i\leqslant k,\ 1\leqslant j\leqslant k}\right) = \prod_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}} \left(u_i - u_j\right). \tag{8}$$

It is known that the determinant of a matrix equals the determinant of its transpose. Thus,

$$\det\left(\left(f_j\left(u_i\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right) = \det\left(\left(f_j\left(u_i\right)\right)_{1\leqslant i\leqslant k,\ 1\leqslant j\leqslant k}\right)$$

(since the matrix $\left(f_j(u_i)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}$ is the transpose of the matrix $\left(f_j(u_i)\right)_{1\leqslant i\leqslant k,\ 1\leqslant j\leqslant k}$).

But when we scale a column of a matrix by a scalar λ, then its determinant also gets multiplied by λ. Hence,

$$\begin{aligned}
\det\left(\left(t_{-w(u_i)} f_j\left(u_i\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right) &= \left(\prod_{i=1}^{k} t_{-w(u_i)}\right) \cdot \underbrace{\det\left(\left(f_j\left(u_i\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)}_{\substack{=\det\left(\left(f_j(u_i)\right)_{1\leqslant i\leqslant k,\ 1\leqslant j\leqslant k}\right) \\ = \prod_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}} \left(u_i - u_j\right) \\ \text{(by (8))}}} \\
&= \left(\prod_{i=1}^{k} t_{-w(u_i)}\right) \cdot \prod_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}} \left(u_i - u_j\right).
\end{aligned}$$

Furthermore, when we scale a row of a matrix by a scalar λ, then its determinant also gets multiplied by λ. Hence,

$$\begin{aligned}
&\det\left(\left(t_{\rho_j} t_{-w(u_i)} f_j(u_i)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right) \\
&= \left(\prod_{j=1}^{k} t_{\rho_j}\right) \cdot \underbrace{\det\left(\left(t_{-w(u_i)} f_j(u_i)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)}_{=\left(\prod_{i=1}^{k} t_{-w(u_i)}\right)\cdot \prod_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}} (u_i-u_j)} \\
&= \left(\prod_{j=1}^{k} t_{\rho_j}\right)\left(\prod_{i=1}^{k} t_{-w(u_i)}\right) \cdot \prod_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}} (u_i-u_j).
\end{aligned}$$

However, for every $i \in \{1,2,\ldots,k\}$ and $j \in \{1,2,\ldots,k\}$, we have

$$\begin{aligned}
a(u_i, j) &= \underbrace{t_{\rho_j - w(u_i)}}_{=t_{\rho_j} t_{-w(u_i)}} f_j(u_i) \qquad \text{(by the definition of } a(u_i,j)\text{)} \\
&= t_{\rho_j} t_{-w(u_i)} f_j(u_i).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\det\left(\left(\underbrace{a(u_i,j)}_{=t_{\rho_j} t_{-w(u_i)} f_j(u_i)}\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right) \\
&= \det\left(\left(t_{\rho_j} t_{-w(u_i)} f_j(u_i)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right) \\
&= \left(\prod_{j=1}^{k} t_{\rho_j}\right)\left(\prod_{i=1}^{k} t_{-w(u_i)}\right) \cdot \prod_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}} (u_i-u_j). \qquad (9)
\end{aligned}$$

The right hand side of this equality is a product of nonzero elements of $\mathbb{L}$ (since $u_1, u_2, \ldots, u_k$ are distinct), and thus is nonzero (by Corollary 6.5). Hence, the left hand side is nonzero. In other words, $\det\left(\left(a(u_i,j)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)$ is nonzero. This proves (6) (since we already know that $\det\left(\left(a(u_i,j)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)$ belongs to $\mathbb{L}_+$).

Moreover, (9) yields

$$\begin{aligned}
&\operatorname{ord}\left(\det\left(\left(a\left(u_i,j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right)\\
&=\operatorname{ord}\left(\left(\prod_{j=1}^{k} t_{\rho_j}\right)\left(\prod_{i=1}^{k} t_{-w(u_i)}\right)\cdot\prod_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}}\left(u_i-u_j\right)\right)\\
&=\sum_{j=1}^{k}\underbrace{\operatorname{ord}\left(t_{\rho_j}\right)}_{=\rho_j}+\sum_{i=1}^{k}\underbrace{\operatorname{ord}\left(t_{-w(u_i)}\right)}_{=-w(u_i)}+\sum_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}}\operatorname{ord}\left(u_i-u_j\right)\\
&\qquad\text{(by Lemma 6.4 \textbf{(c)} and Corollary 6.6)}\\
&=\sum_{j=1}^{k}\rho_j-\sum_{i=1}^{k}w\left(u_i\right)+\sum_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}}\operatorname{ord}\left(u_i-u_j\right)\\
&=\sum_{j=1}^{k}\rho_j-\left(\sum_{i=1}^{k}w\left(u_i\right)-\sum_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}}\operatorname{ord}\left(u_i-u_j\right)\right).
\end{aligned}\tag{10}$$

But recall that $U=\{u_1,u_2,\ldots,u_k\}$ with $u_1,u_2,\ldots,u_k$ distinct. The definition of perimeter thus yields

$$\begin{aligned}
\operatorname{PER}(U)&=\sum_{i=1}^{k}w\left(u_i\right)+\sum_{1\leqslant i<j\leqslant k}d\left(u_i,u_j\right)\\
&=\sum_{i=1}^{k}w\left(u_i\right)+\underbrace{\sum_{1\leqslant j<i\leqslant k}}_{=\sum_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}}}\underbrace{d\left(u_j,u_i\right)}_{\substack{=d\left(u_i,u_j\right)\\ \text{(by the ``Symmetry''}\\ \text{axiom in the definition}\\ \text{of a }\mathbb{V}\text{-ultra triple)}}}\\
&\qquad\left(\begin{array}{c}\text{here, we have renamed}\\ \text{the index }(i,j)\text{ as }(j,i)\text{ in the second sum}\end{array}\right)\\
&=\sum_{i=1}^{k}w\left(u_i\right)+\sum_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}}\underbrace{d\left(u_i,u_j\right)}_{\substack{=-\operatorname{ord}\left(u_i-u_j\right)\\ \text{(by the definition of }d)}}\\
&=\sum_{i=1}^{k}w\left(u_i\right)-\sum_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}}\operatorname{ord}\left(u_i-u_j\right).
\end{aligned}\tag{11}$$

Hence, (10) becomes

$$\begin{aligned}
&\operatorname{ord}\left(\det\left(\left(a\left(u_i,j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right)\\
&=\sum_{j=1}^{k}\rho_j-\underbrace{\left(\sum_{i=1}^{k}w\left(u_i\right)-\sum_{\substack{(i,j)\in\{1,2,\ldots,k\}^2;\\ i>j}}\operatorname{ord}\left(u_i-u_j\right)\right)}_{\substack{=\operatorname{PER}(U)\\ \text{(by (11))}}}\\
&=\sum_{j=1}^{k}\rho_j-\operatorname{PER}\left(U\right).
\end{aligned}$$

Hence, (7) is proved. This proves Claim 2.]

As a consequence of Claim 2, we obtain the following:

> *Claim 3:* Let $k\in\{0,1,\ldots,m\}$. Then, $\sum\limits_{j=1}^{k}\rho_j$ is the maximum perimeter of a k-subset of E.

[*Proof of Claim 3:* The elements $c_1,c_2,\ldots,c_m$ are distinct; thus, the elements $c_1,c_2,\ldots,c_k$ are distinct. Hence, $\{c_1,c_2,\ldots,c_k\}$ is a k-subset of E.

Adding up the equalities (2) for all $j\in\{1,2,\ldots,k\}$, we obtain

$$\begin{aligned}
\sum_{j=1}^{k}\rho_j&=\sum_{j=1}^{k}\left(w\left(c_j\right)+\sum_{i=1}^{j-1}d\left(c_i,c_j\right)\right)\\
&=\sum_{j=1}^{k}w\left(c_j\right)+\sum_{1\leqslant i<j\leqslant k}d\left(c_i,c_j\right)=\operatorname{PER}\left\{c_1,c_2,\ldots,c_k\right\}
\end{aligned}$$

(since $c_1,c_2,\ldots,c_k$ are distinct). Since $\{c_1,c_2,\ldots,c_k\}$ is a k-subset of E, we thus conclude that $\sum\limits_{j=1}^{k}\rho_j$ is the perimeter of some k-subset of E. Thus, in order to prove Claim 3, we need only to show that $\sum\limits_{j=1}^{k}\rho_j\geqslant\operatorname{PER}(U)$ for every k-subset U of E.

So let U be a k-subset of E. We must prove $\sum\limits_{j=1}^{k}\rho_j\geqslant\operatorname{PER}(U)$.

Write the k-subset U in the form $U=\{u_1,u_2,\ldots,u_k\}$ for k distinct elements $u_1,u_2,\ldots,u_k$ of E. Claim 2 thus yields that

$$\det\left(\left(a\left(u_i,j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\text{ is a nonzero element of }\mathbb{L}_+$$

and

$$\operatorname{ord}\left(\det\left(\left(a\left(u_i,j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right)=\sum_{j=1}^{k}\rho_j-\operatorname{PER}\left(U\right). \tag{12}$$

Thus, in particular, $\det\left(\left(a\left(u_i,j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)$ belongs to $\mathbb{L}_+$. Hence,

$$\operatorname{ord}\left(\det\left(\left(a\left(u_i,j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right)\geqslant 0$$

(by Lemma 6.4 **(a)**, applied to $\det\left(\left(a\left(u_i,j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)$ instead of a). In view of (12), this rewrites as $\sum_{j=1}^{k}\rho_j-\operatorname{PER}\left(U\right)\geqslant 0$. In other words, $\sum_{j=1}^{k}\rho_j\geqslant\operatorname{PER}\left(U\right)$. This completes the proof of Claim 3.]

Now, recall the $\mathbb{K}$-algebra homomorphism $\pi:\mathbb{L}_+\to\mathbb{K}$ from Lemma 6.11. For each $e\in E$, define a column vector $v_e\in\mathbb{K}^m$ by

$$v_e=\begin{pmatrix}\pi\left(a\left(e,1\right)\right)\\ \pi\left(a\left(e,2\right)\right)\\ \vdots\\ \pi\left(a\left(e,m\right)\right)\end{pmatrix}=\left(\pi\left(a\left(e,j\right)\right)\right)_{1\leqslant j\leqslant m}.$$

We thus have a vector family $\left(v_e\right)_{e\in E}$ over $\mathbb{K}$. Let $\mathcal{G}$ be the Gaussian elimination greedoid of this family. Let $\mathcal{F}$ be the Bhargava greedoid of $\left(E,w,d\right)$. Our goal is to prove that $\mathcal{F}=\mathcal{G}$ (since this will yield Theorem 6.9).

In order to do so, it suffices to show that if U is any subset of E, then we have the logical equivalence

$$\left(U\in\mathcal{F}\right)\iff\left(U\in\mathcal{G}\right). \tag{13}$$

So let us do this. Let U be a subset of E. We must prove the equivalence (13).

Write the subset U in the form $U=\left\{u_1,u_2,\ldots,u_k\right\}$ with $u_1,u_2,\ldots,u_k$ distinct. Thus, $\left|U\right|=k$. In other words, U is a k-subset of E. Claim 2 thus yields that

$$\det\left(\left(a\left(u_i,j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\text{ is a nonzero element of }\mathbb{L}_+ \tag{14}$$

and

$$\operatorname{ord}\left(\det\left(\left(a\left(u_i,j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right)=\sum_{j=1}^{k}\rho_j-\operatorname{PER}\left(U\right). \tag{15}$$

Also, recall that π is a $\mathbb{K}$-algebra homomorphism (by Lemma 6.11), thus a ring homomorphism. Hence,

$$\det\left(\left(\pi\left(a\left(u_i,j\right)\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)=\pi\left(\det\left(\left(a\left(u_i,j\right)\right)_{1\leqslant j\leqslant k,\ 1\leqslant i\leqslant k}\right)\right) \tag{16}$$

(since ring homomorphisms respect determinants).

Also, U is a subset of E; hence, $|U| \leqslant |E| = m$. Thus, $k = |U| \leqslant m$. Hence, $k \in \{0, 1, \ldots, m\}$. Therefore, Claim 3 shows that $\sum_{j=1}^{k} \rho_j$ is the maximum perimeter of a k-subset of E. In other words, $\sum_{j=1}^{k} \rho_j$ is the maximum perimeter of a $|U|$-subset of E (since $|U| = k$).

The definition of the Gaussian elimination greedoid $\mathcal{G}$ shows that we have the following equivalence:[15]

$$\begin{aligned}
&(U \in \mathcal{G}) \\
\Longleftrightarrow\ & \left(\text{the family } \left(\pi_{|U|}(v_e)\right)_{e \in U} \in \left(\mathbb{K}^{|U|}\right)^U \text{ is linearly independent}\right) \\
\Longleftrightarrow\ & \left(\text{the family } \left(\pi_k(v_e)\right)_{e \in U} \in \left(\mathbb{K}^k\right)^U \text{ is linearly independent}\right) \\
& \quad (\text{since } |U| = k) \\
\Longleftrightarrow\ & \left(\text{the vectors } \pi_k(v_{u_1}), \pi_k(v_{u_2}), \ldots, \pi_k(v_{u_k}) \text{ are linearly independent}\right) \\
& \quad (\text{since } U = \{u_1, u_2, \ldots, u_k\} \text{ with } u_1, u_2, \ldots, u_k \text{ distinct}) \\
\Longleftrightarrow\ & \left(\text{the columns of the matrix } \left(\pi(a(u_i, j))\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k} \text{ are linearly independent}\right) \\
& \quad \left(\begin{array}{c}\text{since the vectors } \pi_k(v_{u_1}), \pi_k(v_{u_2}), \ldots, \pi_k(v_{u_k}) \\ \text{are the columns of the matrix } \left(\pi(a(u_i, j))\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\end{array}\right) \\
\Longleftrightarrow\ & \left(\text{the matrix } \left(\pi(a(u_i, j))\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k} \text{ is invertible}\right) \\
& \quad \left(\begin{array}{c}\text{since a square matrix over the field } \mathbb{K} \text{ is invertible} \\ \text{if and only if its columns are linearly independent}\end{array}\right) \\
\Longleftrightarrow\ & \left(\det\left(\left(\pi(a(u_i, j))\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right) \neq 0 \text{ in } \mathbb{K}\right) \\
\Longleftrightarrow\ & \left(\pi\left(\det\left(\left(a(u_i, j)\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right)\right) \neq 0 \text{ in } \mathbb{K}\right) \qquad (\text{by (16)}) \\
\Longleftrightarrow\ & \left(\operatorname{ord}\left(\det\left(\left(a(u_i, j)\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right)\right) = 0\right) \\
& \quad \left(\text{by Lemma 6.12, applied to } \det\left(\left(a(u_i, j)\right)_{1 \leqslant j \leqslant k,\ 1 \leqslant i \leqslant k}\right) \text{ instead of } a\right) \\
\Longleftrightarrow\ & \left(\sum_{j=1}^{k} \rho_j - \operatorname{PER}(U) = 0\right) \qquad (\text{by (15)}) \\
\Longleftrightarrow\ & \left(\operatorname{PER}(U) = \sum_{j=1}^{k} \rho_j\right) \\
\Longleftrightarrow\ & \left(\operatorname{PER}(U) \text{ is the maximum perimeter of a } |U|\text{-subset of } E\right) \\
& \quad \left(\text{since } \sum_{j=1}^{k} \rho_j \text{ is the maximum perimeter of a } |U|\text{-subset of } E\right) \\
\Longleftrightarrow\ & \left(U \text{ has maximum perimeter among all } |U|\text{-subsets of } E\right) \\
\Longleftrightarrow\ & (U \in \mathcal{F})
\end{aligned}$$

(by the definition of the Bhargava greedoid $\mathcal{F}$). Thus, the equivalence (13) is proven. This concludes the proof of Theorem 6.9. □

[15]The words "linearly independent" should always be understood to mean "$\mathbb{K}$-linearly independent" here.

7. Isomorphism

Next, we introduce the notion of a set system. This elementary notion will play a purely technical role in what follows.

Definition 7.1. Let E be a set.

(a) We let 2^E denote the powerset of E.

(b) A *set system* on ground set E shall mean a subset of 2^E.

Thus:

- The Gaussian elimination greedoid of a vector family $(v_e)_{e \in E}$ (over any field $\mathbb{K}$) is a set system on ground set E.
- The Bhargava greedoid of any $\mathbb{V}$-ultra triple (E, w, d) is a set system on ground set E.

(More generally, any greedoid is a set system, but we shall not need this.)
We shall use the following two simple concepts of isomorphism:

Definition 7.2. Let (E, w, d) and (F, v, c) be two $\mathbb{V}$-ultra triples.

(a) A bijective map $f : E \to F$ is said to be an *isomorphism of* $\mathbb{V}$*-ultra triples from* (E, w, d) *to* (F, v, c) if it satisfies $v \circ f = w$ and

$$c(f(a), f(b)) = d(a, b) \qquad \text{for all } (a, b) \in E \underline{\times} E.$$

(b) The $\mathbb{V}$-ultra triples (E, w, d) and (F, v, c) are said to be *isomorphic* if there exists an isomorphism $f : E \to F$ of $\mathbb{V}$-ultra triples from (E, w, d) to (F, v, c). (Note that being isomorphic is clearly a symmetric relation, because if $f : E \to F$ is an isomorphism of $\mathbb{V}$-ultra triples from (E, w, d) to (F, v, c), then $f^{-1} : F \to E$ is an isomorphism of $\mathbb{V}$-ultra triples from (F, v, c) to (E, w, d).)

Definition 7.3. Let $\mathcal{E}$ and $\mathcal{F}$ be two set systems on ground sets E and F, respectively.

(a) A bijective map $f : E \to F$ is said to be an *isomorphism of set systems from* $\mathcal{E}$ *to* $\mathcal{F}$ if the bijection $2^f : 2^E \to 2^F$ induced by it (i.e., the bijection that sends each $S \in 2^E$ to $f(S) \in 2^F$) satisfies $2^f(\mathcal{E}) = \mathcal{F}$.

(b) The set systems $\mathcal{E}$ and $\mathcal{F}$ are said to be *isomorphic* if there exists an isomorphism $f : E \to F$ of set systems from $\mathcal{E}$ to $\mathcal{F}$. (Note that being isomorphic is clearly a symmetric relation, because if $f : E \to F$ is an isomorphism of set systems from $\mathcal{E}$ to $\mathcal{F}$, then $f^{-1} : F \to E$ is an isomorphism of set systems from $\mathcal{F}$ to $\mathcal{E}$.)

The intuitive meaning of both of these two definitions is simple: Two $\mathbb{V}$-ultra triples are isomorphic if and only if one can be obtained from the other by relabeling the elements of the ground set. The same holds for two set systems.

The following two propositions are obvious:

Proposition 7.4. Let (E, w, d) and (F, v, c) be two isomorphic $\mathbb{V}$-ultra triples such that E and F are finite. Then, the Bhargava greedoids of (E, w, d) and (F, v, c) are isomorphic as set systems.

Proposition 7.5. Let $\mathbb{K}$ be a field. Let $\mathcal{E}$ and $\mathcal{F}$ be two isomorphic set systems. If $\mathcal{E}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$, then so is $\mathcal{F}$.

8. Decomposing a $\mathbb{V}$-ultra triple

For the rest of Section 8, we fix a $\mathbb{V}$-ultra triple (E, w, d).

We are going to study the structure of this $\mathbb{V}$-ultra triple. We recall the notions of α-cliques and cliques (defined in Definition 4.1 and Definition 4.2, respectively). We shall next define another kind of subsets of E: the *open balls*.

Definition 8.1. Let $\alpha \in \mathbb{V}$ and $e \in E$. The *open ball* $B^\circ_\alpha(e)$ is defined to be the subset

$$\{f \in E \mid f = e \text{ or else } d(f, e) < \alpha\}$$

of E.

Clearly, for each $\alpha \in \mathbb{V}$ and each $e \in E$, we have $e \in B^\circ_\alpha(e)$, so that the open ball $B^\circ_\alpha(e)$ contains at least the element e.

Proposition 8.2. Let $\alpha \in \mathbb{V}$ and $e, f \in E$ be such that $e \neq f$ and $d(e, f) < \alpha$. Then, $B^\circ_\alpha(e) = B^\circ_\alpha(f)$.

Proof. We have $e \neq f$, so that $f \neq e$. Hence, the "symmetry" axiom in Definition 2.3 yields that $d(f, e) = d(e, f) < \alpha$.

From $e \neq f$ and $d(e, f) < \alpha$, we obtain $e \in B^\circ_\alpha(f)$ (by the definition of $B^\circ_\alpha(f)$). Also, $f \in B^\circ_\alpha(f)$ (by the definition of $B^\circ_\alpha(f)$).

Let $x \in B^\circ_\alpha(e)$. We shall show that $x \in B^\circ_\alpha(f)$.

If $x = e$, then this follows immediately from $e \in B^\circ_\alpha(f)$. Hence, for the rest of the proof of $x \in B^\circ_\alpha(f)$, we WLOG assume that $x \neq e$. Thus, from $x \in B^\circ_\alpha(e)$, we obtain $d(x, e) < \alpha$ (by the definition of $B^\circ_\alpha(e)$). If $x = f$, then $x \in B^\circ_\alpha(f)$

follows immediately from the fact that $f \in B_\alpha^\circ(f)$. Thus, for the rest of the proof of $x \in B_\alpha^\circ(f)$, we WLOG assume that $x \neq f$. Now, the points e, f, x are distinct (since $x \neq e$, $x \neq f$ and $e \neq f$). Hence, the ultrametric triangle inequality yields $d(x,f) \leqslant \max\{d(x,e), d(f,e)\} < \alpha$ (since $d(x,e) < \alpha$ and $d(f,e) < \alpha$). Thus, $x \in B_\alpha^\circ(f)$ (by the definition of $B_\alpha^\circ(f)$).

Now, forget that we fixed x. We thus have shown that $x \in B_\alpha^\circ(f)$ for each $x \in B_\alpha^\circ(e)$. In other words, $B_\alpha^\circ(e) \subseteq B_\alpha^\circ(f)$.

But our situation is symmetric in e and f (since $f \neq e$ and $d(f,e) < \alpha$). Hence, the same argument that let us prove $B_\alpha^\circ(e) \subseteq B_\alpha^\circ(f)$ can be applied with the roles of e and f interchanged; thus we obtain $B_\alpha^\circ(f) \subseteq B_\alpha^\circ(e)$. Combining this with $B_\alpha^\circ(e) \subseteq B_\alpha^\circ(f)$, we obtain $B_\alpha^\circ(e) = B_\alpha^\circ(f)$. This proves Proposition 8.2. □

Corollary 8.3. Let $\alpha \in \mathbb{V}$ and $e \in E$. Let $f \in B_\alpha^\circ(e)$. Then, $B_\alpha^\circ(e) = B_\alpha^\circ(f)$.

Proof. If $e = f$, then this is obvious. Hence, WLOG assume that $e \neq f$. Thus, $f \neq e$. Hence, $d(f,e) < \alpha$ (since $f \in B_\alpha^\circ(e)$). Thus, the "symmetry" axiom in Definition 2.3 yields that $d(e,f) = d(f,e) < \alpha$. Hence, Proposition 8.2 yields $B_\alpha^\circ(e) = B_\alpha^\circ(f)$. Qed. □

Proposition 8.4. Let $\alpha \in \mathbb{V}$ and $e, f \in E$ be such that $e \neq f$ and $d(e,f) \geqslant \alpha$. Then:

- **(a)** If $a \in B_\alpha^\circ(e)$ and $b \in B_\alpha^\circ(f)$, then $a \neq b$ and $d(a,b) \geqslant \alpha$.
- **(b)** The open balls $B_\alpha^\circ(e)$ and $B_\alpha^\circ(f)$ are disjoint.

Proof. **(a)** Let $a \in B_\alpha^\circ(e)$ and $b \in B_\alpha^\circ(f)$. We must prove that $a \neq b$ and $d(a,b) \geqslant \alpha$.

Assume the contrary. Thus, either $a = b$ or else $d(a,b) < \alpha$. Hence, $a \in B_\alpha^\circ(b)$ (by the definition of $B_\alpha^\circ(b)$). Thus, $B_\alpha^\circ(b) = B_\alpha^\circ(a)$ (by Corollary 8.3, applied to b and a instead of e and f). Also, from $a \in B_\alpha^\circ(e)$, we obtain $B_\alpha^\circ(e) = B_\alpha^\circ(a)$ (by Corollary 8.3, applied to a instead of f). Furthermore, from $b \in B_\alpha^\circ(f)$, we obtain $B_\alpha^\circ(f) = B_\alpha^\circ(b)$ (by Corollary 8.3, applied to f and b instead of e and f). Finally, the definition of $B_\alpha^\circ(e)$ yields $e \in B_\alpha^\circ(e) = B_\alpha^\circ(a) = B_\alpha^\circ(b) = B_\alpha^\circ(f)$. Since $e \neq f$, this entails $d(e,f) < \alpha$ (by the definition of $B_\alpha^\circ(f)$). This contradicts $d(e,f) \geqslant \alpha$. This contradiction shows that our assumption was false. Hence, Proposition 8.4 **(a)** follows.

(b) This is simply the "$a \neq b$" part of Proposition 8.4 **(a)**. □

The next proposition is trivial:

Proposition 8.5. Let F be a subset of E. Let (F, w', d') be the $\mathbb{V}$-ultra triple $\left(F, w\mid_F, d\mid_{F\times F}\right)$. Then:

- **(a)** If a subset of F is a clique of the $\mathbb{V}$-ultra triple (E, w, d), then this subset is also a clique of the $\mathbb{V}$-ultra triple (F, w', d').

(b) Any clique of the $\mathbb{V}$-ultra triple (F, w', d') is a clique of the $\mathbb{V}$-ultra triple (E, w, d).

Proof. **(a)** Let C be a subset of F that is a clique of the $\mathbb{V}$-ultra triple (E, w, d). We must show that C is also a clique of the $\mathbb{V}$-ultra triple (F, w', d').

The definition of (F, w', d') shows that $d' = d \mid_{F \underline{\times} F}$. Thus, any two distinct elements $a, b \in F$ satisfy

$$d'(a,b) = d(a,b). \tag{17}$$

We know that C is a clique of (E, w, d). In other words, C is an α-clique of (E, w, d) for some $\alpha \in \mathbb{V}$. Consider this α. Thus, C is an α-clique of (E, w, d). In other words, any two distinct elements $a, b \in C$ satisfy

$$d(a,b) = \alpha \tag{18}$$

(by the definition of an "α-clique"). Hence, for any two distinct elements $a, b \in C$, we have

$$\begin{aligned} d'(a,b) &= d(a,b) \qquad (\text{by (17), since } a, b \in C \subseteq F) \\ &= \alpha \qquad (\text{by (18)}). \end{aligned}$$

In other words, any two distinct elements $a, b \in C$ satisfy $d'(a,b) = \alpha$. Since C is a subset of F, we thus conclude that C is an α-clique of the $\mathbb{V}$-ultra triple (F, w', d') (by the definition of an "α-clique"). Hence, C is a clique of the $\mathbb{V}$-ultra triple (F, w', d'). This proves Proposition 8.5 **(a)**.

(b) This is proved essentially by reading the above proof of Proposition 8.5 **(a)** backwards (as we now have to derive $d(a,b) = \alpha$ from $d'(a,b) = \alpha$). □

The next proposition shows how a finite $\mathbb{V}$-ultra triple (of size > 1) can be decomposed into several smaller $\mathbb{V}$-ultra triples; this will later be used for recursive reasoning:[16]

Proposition 8.6. Assume that E is finite and $|E| > 1$. Let $\alpha = \max(d(E \underline{\times} E))$.

Pick any maximum-size α-clique, and write it in the form $\{e_1, e_2, \ldots, e_m\}$ for some distinct elements $e_1, e_2, \ldots, e_m$ of E.

For each $i \in \{1, 2, \ldots, m\}$, let E_i be the open ball $B^{\circ}_{\alpha}(e_i)$, and let (E_i, w_i, d_i) be the $\mathbb{V}$-ultra triple $\left(E_i, w \mid_{E_i}, d \mid_{E_i \underline{\times} E_i}\right)$.

Then:

(a) We have $m > 1$.

(b) The sets $E_1, E_2, \ldots, E_m$ form a set partition of E. (This means that these sets $E_1, E_2, \ldots, E_m$ are disjoint and nonempty and their union is E.)

[16] See Definition 4.4 for the meaning of $\operatorname{mcs}(E, w, d)$.

(c) We have $|E_i| < |E|$ for each $i \in \{1, 2, \ldots, m\}$.

(d) If $i \in \{1, 2, \ldots, m\}$, and if $a, b \in E_i$ are distinct, then $d(a, b) < \alpha$.

(e) If i and j are two distinct elements of $\{1, 2, \ldots, m\}$, and if $a \in E_i$ and $b \in E_j$, then $a \neq b$ and $d(a, b) = \alpha$.

(f) Let $n_i = \operatorname{mcs}(E_i, w_i, d_i)$ for each $i \in \{1, 2, \ldots, m\}$. Then,

$$\operatorname{mcs}(E, w, d) = \max\{m, n_1, n_2, \ldots, n_m\}.$$

Proof. Let us first check that α is well-defined. Indeed, the set $E \underline{\times} E$ is nonempty (since $|E| > 1$) and finite (since E is finite). Hence, the set $d(E \underline{\times} E)$ is nonempty and finite as well. Thus, its largest element $\max(d(E \underline{\times} E))$ is well-defined. In other words, α is well-defined.

We have

$$d(a, b) \leqslant \alpha \qquad \text{for each } (a, b) \in E \underline{\times} E \tag{19}$$

(since $\alpha = \max(d(E \underline{\times} E))$).

The set $\{e_1, e_2, \ldots, e_m\}$ is a maximum-size α-clique (by its definition), but its size is m (since $e_1, e_2, \ldots, e_m$ are distinct). Hence, the maximum size of an α-clique is m. Thus, every α-clique has size $\leqslant m$.

(a) From $\alpha = \max(d(E \underline{\times} E)) \in d(E \underline{\times} E)$, we conclude that there exist two distinct elements u and v of E satisfying $d(u, v) = \alpha$. Consider these u and v. Then, $\{u, v\}$ is an α-clique. This α-clique must have size $\leqslant m$ (since every α-clique has size $\leqslant m$). Hence, $|\{u, v\}| \leqslant m$. Thus, $m \geqslant |\{u, v\}| = 2$ (since u and v are distinct), so that $m \geqslant 2 > 1$. This proves Proposition 8.6 **(a)**.

(b) If i and j are two distinct elements of $\{1, 2, \ldots, m\}$, then $e_i \neq e_j$ (since $e_1, e_2, \ldots, e_m$ are distinct) and thus $d(e_i, e_j) = \alpha$ (since $\{e_1, e_2, \ldots, e_m\}$ is an α-clique), and therefore the open balls $B^\circ_\alpha(e_i)$ and $B^\circ_\alpha(e_j)$ are disjoint (by Proposition 8.4 **(b)**, applied to $e = e_i$ and $f = e_j$). In other words, if i and j are two distinct elements of $\{1, 2, \ldots, m\}$, then the open balls E_i and E_j are disjoint (since $E_i = B^\circ_\alpha(e_i)$ and $E_j = B^\circ_\alpha(e_j)$).

Hence, the open balls $E_1, E_2, \ldots, E_m$ are disjoint. Furthermore, these balls are nonempty (since each open ball $E_i = B^\circ_\alpha(e_i)$ contains at least the element e_i).

Furthermore, the union of these balls $E_1, E_2, \ldots, E_m$ is the whole set E.

[*Proof:* Assume the contrary. Thus, the union of the balls $E_1, E_2, \ldots, E_m$ must be a **proper** subset of E (since it is clearly a subset of E). In other words, there exists some $a \in E$ that belongs to none of these balls $E_1, E_2, \ldots, E_m$. Consider this a. Then, for each $i \in \{1, 2, \ldots, m\}$, we have $a \notin E_i = B^\circ_\alpha(e_i)$. By the definition of $B^\circ_\alpha(e_i)$, this entails that $a \neq e_i$ and $d(a, e_i) \geqslant \alpha$, hence $d(a, e_i) = \alpha$ (since (19) yields $d(a, e_i) \leqslant \alpha$). Thus, we have shown that $d(a, e_i) = \alpha$ for all $i \in \{1, 2, \ldots, m\}$. Therefore, $\{a, e_1, e_2, \ldots, e_m\}$ is an α-clique. This α-clique has size $m + 1$ (since $e_1, e_2, \ldots, e_m$ are distinct, and since $a \neq e_i$ for each $i \in \{1,2,\ldots,m\}$). But this contradicts the fact that every α-clique has size $\leqslant m$. This contradiction shows that our assumption was wrong. Hence, the union of the balls $E_1, E_2, \ldots, E_m$ is the whole set E.]

We have now proved that the balls $E_1, E_2, \ldots, E_m$ are disjoint and nonempty and their union is the whole set E. In other words, they form a set partition of E. This proves Proposition 8.6 **(b)**.

(c) Proposition 8.6 **(b)** shows that the m sets $E_1, E_2, \ldots, E_m$ form a set partition of E. Thus, these m sets are m disjoint nonempty subsets of E; hence, each of them is a **proper** subset of E (since $m > 1$). In other words, E_i is a proper subset of E for each $i \in \{1,2,\ldots,m\}$. Hence, $|E_i| < |E|$ for each $i \in \{1,2,\ldots,m\}$. This proves Proposition 8.6 **(c)**.

(d) Let $i \in \{1,2,\ldots,m\}$. Let $a, b \in E_i$ be distinct. We must prove that $d(a,b) < \alpha$.

We have $b \in E_i = B^\circ_\alpha(e_i)$ (by the definition of E_i) and thus $B^\circ_\alpha(e_i) = B^\circ_\alpha(b)$ (by Corollary 8.3, applied to e_i and b instead of e and f). But $a \in E_i = B^\circ_\alpha(e_i) = B^\circ_\alpha(b)$. In other words, $a = b$ or else $d(a,b) < \alpha$ (by the definition of $B^\circ_\alpha(b)$). Hence, $d(a,b) < \alpha$ (since $a \neq b$). This proves Proposition 8.6 **(d)**.

(e) Let i and j be two distinct elements of $\{1,2,\ldots,m\}$. Let $a \in E_i$ and $b \in E_j$. We must prove that $a \neq b$ and $d(a,b) = \alpha$.

We have $a \in E_i = B^\circ_\alpha(e_i)$ (by the definition of E_i) and similarly $b \in B^\circ_\alpha(e_j)$. Furthermore, $e_i \neq e_j$ (since $i \neq j$ and since $e_1, e_2, \ldots, e_m$ are distinct) and thus $d(e_i, e_j) = \alpha$ (since $\{e_1, e_2, \ldots, e_m\}$ is an α-clique). Hence, Proposition 8.4 **(a)** (applied to e_i and e_j instead of e and f) yields $a \neq b$ and $d(a,b) \geqslant \alpha$. Combining $d(a,b) \geqslant \alpha$ with (19), we obtain $d(a,b) = \alpha$. This proves Proposition 8.6 **(e)**.

(f) Let us first notice that the map d_i (for each $i \in \{1,2,\ldots,m\}$) is defined to be a restriction of the map d. Thus, for any $i \in \{1,2,\ldots,m\}$, we have

$$d_i(e,f) = d(e,f) \qquad \text{for any two distinct } e, f \in E_i. \tag{20}$$

The $\mathbb{V}$-ultra triple (E, w, d) has a clique of size m (namely, $\{e_1, e_2, \ldots, e_m\}$), and a clique of size n_i for each $i \in \{1,2,\ldots,m\}$ (indeed, $n_i = \operatorname{mcs}(E_i, w_i, d_i)$ shows that the $\mathbb{V}$-ultra triple (E_i, w_i, d_i) has such a clique; but this clique must of course be a clique of (E, w, d) as well[17]). Thus, the $\mathbb{V}$-ultra triple (E, w, d) has a clique of size $\max\{m, n_1, n_2, \ldots, n_m\}$ (since $\max\{m, n_1, n_2, \ldots, n_m\}$ is one of the numbers $m, n_1, n_2, \ldots, n_m$). Thus,

$$\operatorname{mcs}(E, w, d) \geqslant \max\{m, n_1, n_2, \ldots, n_m\}.$$

It remains to prove the reverse inequality – i.e., to prove that $\operatorname{mcs}(E, w, d) \leqslant \max\{m, n_1, n_2, \ldots, n_m\}$.

Assume the contrary. Thus, $\operatorname{mcs}(E, w, d) > \max\{m, n_1, n_2, \ldots, n_m\}$.

[17] since Proposition 8.5 **(b)** (applied to $(F, w', d') = (E_i, w_i, d_i)$) shows that any clique of the $\mathbb{V}$-ultra triple (E_i, w_i, d_i) is a clique of the $\mathbb{V}$-ultra triple (E, w, d)

The $\mathbb{V}$-ultra triple (E, w, d) has a clique C of size $\operatorname{mcs}(E, w, d)$ (by the definition of $\operatorname{mcs}(E, w, d)$). Consider this C. Thus, $|C| = \operatorname{mcs}(E, w, d) > \max\{m, n_1, n_2, \ldots, n_m\} \geqslant 0$. Hence, the set C is nonempty. In other words, there exists some $a \in C$. Consider this a.

But recall that $E_1, E_2, \ldots, E_m$ form a set partition of E. Thus, $E_1 \cup E_2 \cup \cdots \cup E_m = E$. Now, $a \in C \subseteq E = E_1 \cup E_2 \cup \cdots \cup E_m$. In other words, $a \in E_i$ for some $i \in \{1, 2, \ldots, m\}$. Consider this i.

We are in one of the following two cases:

Case 1: We have $C \subseteq E_i$.

Case 2: We have $C \not\subseteq E_i$.

Let us first consider Case 1. In this case, we have $C \subseteq E_i$. In other words, C is a subset of E_i.

Recall that C is a clique of (E, w, d). Since C is a subset of E_i, we can thus conclude that C is a clique of the $\mathbb{V}$-ultra triple (E_i, w_i, d_i) (since Proposition 8.5 **(a)** (applied to $(F, w', d') = (E_i, w_i, d_i)$) shows that if a subset of E_i is a clique of (E, w, d), then this subset is also a clique of the $\mathbb{V}$-ultra triple (E_i, w_i, d_i)). But the maximum size of such a clique is $\operatorname{mcs}(E_i, w_i, d_i)$ (by the definition of $\operatorname{mcs}(E_i, w_i, d_i)$). Hence, $|C| \leqslant \operatorname{mcs}(E_i, w_i, d_i) = n_i$. This contradicts $|C| > \max\{m, n_1, n_2, \ldots, n_m\} \geqslant n_i$. Thus, we have obtained a contradiction in Case 1.

Let us next consider Case 2. In this case, we have $C \not\subseteq E_i$. In other words, there exists some $b \in C$ such that $b \notin E_i$. Consider this b. From $b \in C \subseteq E = E_1 \cup E_2 \cup \cdots \cup E_m$, we conclude that $b \in E_j$ for some $j \in \{1, 2, \ldots, m\}$. Consider this j. If we had $i = j$, then we would have $b \in E_j = E_i$ (since $j = i$), which would contradict $b \notin E_i$. Hence, we cannot have $i = j$. Hence, $i \neq j$. Thus, Proposition 8.6 **(e)** yields that $a \neq b$ and $d(a, b) = \alpha$. Now, $a \neq b$ shows that a and b are two distinct elements of C. But C is a clique, and thus is a γ-clique for some $\gamma \in \mathbb{V}$. Consider this γ. Since C is a γ-clique, we have $d(a, b) = \gamma$ (since a and b are two distinct elements of C). Hence, $\gamma = d(a, b) = \alpha$. Thus, C is an α-clique (since C is a γ-clique). Thus, $|C| \leqslant m$ (since every α-clique has size $\leqslant m$). This contradicts $|C| > \max\{m, n_1, n_2, \ldots, n_m\} \geqslant m$. Thus, we have obtained a contradiction in Case 2.

We have thus found a contradiction in each of the two Cases 1 and 2. Thus, we always have a contradiction. This completes the proof of Proposition 8.6 **(f)**. □

Remark 8.7. Here are some additional observations on Proposition 8.6, which we will not need (and thus will not prove):

- **(a)** The set partition $\{E_1, E_2, \ldots, E_m\}$ constructed in Proposition 8.6 **(b)** does not depend on the choice of $e_1, e_2, \ldots, e_m$. Indeed, $E_1, E_2, \ldots, E_m$ are precisely the maximal (with respect to inclusion) subsets F of E satisfying $(d(a, b) < \alpha$ for any distinct $a, b \in F)$.
- **(b)** Applying Proposition 8.6 iteratively, we can see that a $\mathbb{V}$-ultra triple (E, w, d) with finite E has a recursive structure governed by a tree. This idea is not new; see [Lemin03] and [PetDov14, §2–§3] for related results.

9. Valadic representation of $\mathbb{V}$-ultra triples

For the whole Section 9, we fix a field $\mathbb{K}$, and we let $\mathbb{V}_{\geqslant 0}$, $\mathbb{L}$, $\mathbb{L}_+$ and t_α be as in Section 6. We also recall Definition 6.7.

Definition 9.1. Let $\gamma \in \mathbb{V}$ and $u \in \mathbb{L}$. We say that a valadic $\mathbb{V}$-ultra triple (E, w, d) is (γ, u)*-positioned* if

$$E \subseteq u + t_{-\gamma}\mathbb{L}_+.$$

In other words, a valadic $\mathbb{V}$-ultra triple (E, w, d) is (γ, u)-positioned if and only if each element of E has the form $u + \sum\limits_{\substack{\beta \in \mathbb{V}; \\ \beta \geqslant -\gamma}} p_\beta t_\beta$ for some $p_\beta \in \mathbb{K}$.

Theorem 9.2. Let (E, w, d) be a $\mathbb{V}$-ultra triple such that the set E is finite. Let $\gamma \in \mathbb{V}$ be such that

$$d(a, b) \leqslant \gamma \qquad \text{for each } (a, b) \in E \underline{\times} E. \tag{21}$$

Let $u \in \mathbb{L}$.

Assume that $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$. Then, there exists a (γ, u)-positioned valadic $\mathbb{V}$-ultra triple isomorphic to (E, w, d).

Proof of Theorem 9.2. We proceed by strong induction on $|E|$.

If $|E| = 1$, then this is clear (just take the obvious valadic $\mathbb{V}$-ultra triple on the set $\{u\} \subseteq \mathbb{L}$, which is clearly (γ, u)-positioned). The case $|E| = 0$ is even more obvious. Thus, WLOG assume that $|E| > 1$. Thus, the set $E \underline{\times} E$ is nonempty. Hence, $d(E \underline{\times} E)$ is a nonempty finite subset of $\mathbb{V}$, and therefore has a largest element. In other words, $\max(d(E \underline{\times} E))$ is well-defined.

Let $\alpha = \max(d(E \underline{\times} E))$. Thus, $\alpha \leqslant \gamma$ (by (21)).

Pick any maximum-size α-clique, and write it in the form $\{e_1, e_2, \ldots, e_m\}$ for some distinct elements $e_1, e_2, \ldots, e_m$ of E. For each $i \in \{1, 2, \ldots, m\}$, let E_i be the open ball $B^\circ_\alpha(e_i)$, and let (E_i, w_i, d_i) be the $\mathbb{V}$-ultra triple $\left(E_i, w\mid_{E_i}, d\mid_{E_i \underline{\times} E_i}\right)$.

Then, Proposition 8.6 **(a)** shows that $m > 1$. Moreover, Proposition 8.6 **(b)** shows that the m sets $E_1, E_2, \ldots, E_m$ form a set partition of E. Hence, $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_m$ (an internal disjoint union). Each set E_i is the open ball $B^\circ_\alpha(e_i)$ and thus contains e_i.

Let $n_i = \operatorname{mcs}(E_i, w_i, d_i)$ for each $i \in \{1, 2, \ldots, m\}$. Then,

$$|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d) = \max\{m, n_1, n_2, \ldots, n_m\} \qquad (\text{by Proposition 8.6 } \textbf{(f)}).$$

For any subset Z of $\mathbb{L}$, we define a distance function $\overline{d}_Z : Z \underline{\times} Z \to \mathbb{V}$ by setting

$$\overline{d}_Z(a, b) = -\operatorname{ord}(a - b) \qquad \text{for all } (a, b) \in Z \underline{\times} Z. \tag{22}$$

Note that the distance function of any valadic $\mathbb{V}$-ultra triple is precisely $\overline{d}_Z$, where Z is the ground set of this $\mathbb{V}$-ultra triple.

We have $|\mathbb{K}| \geqslant \max\{m, n_1, n_2, \ldots, n_m\} \geqslant m$. Hence, there exist m distinct elements $\lambda_1, \lambda_2, \ldots, \lambda_m$ of $\mathbb{K}$. Fix m such elements. Define m elements $u_1, u_2, \ldots, u_m$ of $\mathbb{L}$ by

$$u_i = u + \lambda_i t_{-\alpha} \qquad \text{for each } i \in \{1, 2, \ldots, m\}. \tag{23}$$

Let $\mathbb{L}_{++}$ denote the $\mathbb{K}$-submodule of $\mathbb{L}_+$ generated by t_δ for all positive $\delta \in \mathbb{V}$. (Of course, $\delta \in \mathbb{V}$ is said to be positive if and only if $\delta > 0$.) It is easy to see that $\mathbb{L}_{++}$ is an ideal of $\mathbb{L}_+$. (Actually, $\mathbb{L}_{++} = \operatorname{Ker} \pi$, where π is as defined in Lemma 6.11.) Hence, $\mathbb{L}_{++}\mathbb{L}_+ \subseteq \mathbb{L}_{++}$.

Let β be the second-largest element of $d(E \underline{\times} E)$. (If this second-largest element does not exist, then we leave β undefined and should interpret $t_{-\beta}$ to mean 0 from now on.)

The definition of β yields $\beta < \alpha$ (since $\alpha = \max(d(E \underline{\times} E))$). Hence, $-\beta > -\alpha$, so that

$$t_{-\beta} \in t_{-\alpha}\mathbb{L}_{++}. \tag{24}$$

(This holds even when β is undefined, because $t_{-\beta} = 0$ in this case.)

Let $i \in \{1, 2, \ldots, m\}$. We shall work under the assumption that β is well-defined; we will later explain how to proceed without it.

Let $(a, b) \in E_i \underline{\times} E_i$. Then, a and b are two distinct elements of E_i. Hence, Proposition 8.6 **(d)** yields

$$d(a, b) < \alpha = \max(d(E \underline{\times} E)) = (\text{the largest element of } d(E \underline{\times} E)).$$

But any element of $d(E \underline{\times} E)$ that is smaller than the largest element of $d(E \underline{\times} E)$ must be at most as large as the second-largest element of $d(E \underline{\times} E)$ (since there exist no elements of $d(E \underline{\times} E)$ between the largest and the second-largest elements). Hence, from $d(a, b) \in d(E \underline{\times} E)$ and $d(a, b) < \max(d(E \underline{\times} E))$, we obtain

$$d(a, b) \leqslant (\text{the second-largest element of } d(E \underline{\times} E)) = \beta$$

(since the second-largest element of $d(E \underline{\times} E)$ is β). Since $d_i = d\mid_{E_i \underline{\times} E_i}$, we now have $d_i(a, b) = d(a, b) \leqslant \beta$.

Forget that we fixed (a, b). We thus have proved that

$$d_i(a, b) \leqslant \beta \qquad \text{for each } (a, b) \in E_i \underline{\times} E_i.$$

Also, $|E_i| < |E|$ (by Proposition 8.6 **(c)**) and

$$|\mathbb{K}| \geqslant \max\{m, n_1, n_2, \ldots, n_m\} \geqslant n_i = \operatorname{mcs}(E_i, w_i, d_i).$$

Hence, the induction hypothesis shows that we can apply Theorem 9.2 to β, u_i and (E_i, w_i, d_i) instead of γ, u and (E, w, d). We thus conclude that there exists a (β, u_i)-positioned valadic $\mathbb{V}$-ultra triple $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ isomorphic to (E_i, w_i, d_i). Consider this $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$. The $\mathbb{V}$-ultra triple (E_i, w_i, d_i) is isomorphic to $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ (since

$\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ is isomorphic to (E_i, w_i, d_i), but being isomorphic is a symmetric relation). In other words, there exists an isomorphism $f_i : E_i \to \overline{E}_i$ of $\mathbb{V}$-ultra triples from (E_i, w_i, d_i) to $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$. Consider this f_i. Since the $\mathbb{V}$-ultra triple $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ is (β, u_i)-positioned, we have

$$\overline{E}_i \subseteq u_i + \underbrace{t_{-\beta}}_{\substack{\in t_{-\alpha}\mathbb{L}_{++} \\ \text{(by (24))}}} \mathbb{L}_+ \subseteq u_i + t_{-\alpha} \underbrace{\mathbb{L}_{++}\mathbb{L}_+}_{\subseteq \mathbb{L}_{++}} \subseteq u_i + t_{-\alpha}\mathbb{L}_{++}.$$

Hence, we have found a $\mathbb{V}$-ultra triple (E_i, w_i, d_i) and a valadic $\mathbb{V}$-ultra triple $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ satisfying

$$\overline{E}_i \subseteq u_i + t_{-\alpha}\mathbb{L}_{++},$$

and an isomorphism $f_i : E_i \to \overline{E}_i$ of $\mathbb{V}$-ultra triples from (E_i, w_i, d_i) to $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$. We have done so assuming that β is well-defined; but this is even easier when β is undefined[18].

Forget that we fixed i. Thus, for each $i \in \{1, 2, \ldots, m\}$, we have constructed a $\mathbb{V}$-ultra triple (E_i, w_i, d_i) and a valadic $\mathbb{V}$-ultra triple $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ satisfying

$$\overline{E}_i \subseteq u_i + t_{-\alpha}\mathbb{L}_{++}$$

and an isomorphism $f_i : E_i \to \overline{E}_i$ of $\mathbb{V}$-ultra triples from (E_i, w_i, d_i) to $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$.

For later use, let us observe the following:

> *Claim 1:* Let i and j be two distinct elements of $\{1, 2, \ldots, m\}$. Let $a \in \overline{E}_i$ and $b \in \overline{E}_j$. Then, $a - b \neq 0$ and $\operatorname{ord}(a - b) = -\alpha$.

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[18] *Proof.* Assume that β is undefined. In other words, the set $d(E \underline{\times} E)$ has no second-largest element. Hence, all elements of $d(E \underline{\times} E)$ are equal (since $d(E \underline{\times} E)$ is a nonempty finite set). Thus, all elements of $d(E \underline{\times} E)$ are α (since $\alpha = \max(d(E \underline{\times} E))$). In other words, $d(a,b) = \alpha$ for each $(a,b) \in E \underline{\times} E$. In other words, $d(a,b) = \alpha$ for any two distinct elements a and b of E. Thus, in particular, $d(a,b) = \alpha$ for any two distinct elements a and b of E_i. But Proposition 8.6 **(d)** shows that $d(a,b) < \alpha$ for any two distinct elements a and b of E_i. The previous two sentences would contradict each other if the set E_i had two distinct elements. Thus, the set E_i has no two distinct elements; in other words, $|E_i| \leqslant 1$. Since e_i is an element of E_i (because $E_i = B^{\circ}_{\alpha}(e_i)$), we thus have $E_i = \{e_i\}$. Now, set $\overline{E}_i = \{u_i\}$; let $\overline{w}_i : \overline{E}_i \to \mathbb{V}$ be the map that sends u_i to $w_i(e_i)$; let $\overline{d}_i : \overline{E}_i \underline{\times} \overline{E}_i \to \mathbb{V}$ be the distance function $\overline{d}_{\{u_i\}}$; and let $f_i : E_i \to \overline{E}_i$ be the map that sends e_i to u_i. Then, it is easy to see that $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ is a valadic $\mathbb{V}$-ultra triple satisfying $\overline{E}_i \subseteq u_i + t_{-\alpha}\mathbb{L}_{++}$ (since $\overline{E}_i = \{u_i\} = u_i + \underbrace{0}_{\subseteq t_{-\alpha}\mathbb{L}_{++}} \subseteq u_i + t_{-\alpha}\mathbb{L}_{++}$), and that the map $f_i : E_i \to \overline{E}_i$ is an isomorphism of $\mathbb{V}$-ultra triples from (E_i, w_i, d_i) to $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ (since $E_i = \{e_i\}$ and $\overline{E}_i = \{u_i\}$, so that the maps d_i and $\overline{d}_i$ both have no values, whereas the map $\overline{w}_i$ is defined in such a way that $\overline{w}_i(u_i) = w_i(e_i)$ and thus $\overline{w}_i \circ f_i = w_i$). Thus, everything we constructed above still exists when β is undefined.

[*Proof of Claim 1:* From $i \neq j$, we conclude that λ_i and λ_j are two distinct elements of $\mathbb{K}$ (since $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct elements of $\mathbb{K}$). Hence, $\lambda_i - \lambda_j$ is a nonzero element of $\mathbb{K}$.

We have $a \in \overline{E}_i \subseteq u_i + t_{-\alpha}\mathbb{L}_{++}$, so that $a - u_i \in t_{-\alpha}\mathbb{L}_{++}$. Similarly, $b - u_j \in t_{-\alpha}\mathbb{L}_{++}$. Also, (23) yields $u_i = u + \lambda_i t_{-\alpha}$. Likewise, $u_j = u + \lambda_j t_{-\alpha}$. Subtracting the latter two equalities from one another, we obtain

$$u_i - u_j = (u + \lambda_i t_{-\alpha}) - (u + \lambda_j t_{-\alpha}) = (\lambda_i - \lambda_j)\, t_{-\alpha}.$$

Now,

$$\begin{aligned} a - b &= \underbrace{(u_i - u_j)}_{=(\lambda_i - \lambda_j)t_{-\alpha}} + \underbrace{(a - u_i)}_{\in t_{-\alpha}\mathbb{L}_{++}} - \underbrace{(b - u_j)}_{\in t_{-\alpha}\mathbb{L}_{++}} \\ &\in (\lambda_i - \lambda_j)\, t_{-\alpha} + \underbrace{t_{-\alpha}\mathbb{L}_{++} - t_{-\alpha}\mathbb{L}_{++}}_{\subseteq t_{-\alpha}\mathbb{L}_{++}} \subseteq (\lambda_i - \lambda_j)\, t_{-\alpha} + t_{-\alpha}\mathbb{L}_{++}. \end{aligned}$$

In other words, $a - b$ can be written in the form

$$a - b = (\lambda_i - \lambda_j)\, t_{-\alpha} + \left(\text{a } \mathbb{K}\text{-linear combination of } t_\eta \text{ with } \eta > -\alpha\right)$$

(since the elements of $t_{-\alpha}\mathbb{L}_{++}$ are precisely the $\mathbb{K}$-linear combinations of t_η with $\eta > -\alpha$).

From this, we obtain two things: First, we obtain that $[t_{-\alpha}](a - b) = \lambda_i - \lambda_j$ (since $\lambda_i - \lambda_j \in \mathbb{K}$), hence $[t_{-\alpha}](a - b) = \lambda_i - \lambda_j \neq 0$ (since $\lambda_i - \lambda_j$ is nonzero), and thus $a - b \neq 0$. Furthermore, we obtain that $\operatorname{ord}(a - b) = -\alpha$ (again since $\lambda_i - \lambda_j \neq 0$). Thus, Claim 1 is proved.]

Let $\overline{E}$ denote the subset

$$\overline{E}_1 \cup \overline{E}_2 \cup \cdots \cup \overline{E}_m$$

of $\mathbb{L}$. Note that the sets $\overline{E}_1, \overline{E}_2, \ldots, \overline{E}_m$ are disjoint[19]. Hence, $\overline{E} = \overline{E}_1 \sqcup \overline{E}_2 \sqcup \cdots \sqcup \overline{E}_m$ (an internal disjoint union). Moreover, each $i \in \{1, 2, \ldots, m\}$ satisfies

$$\begin{aligned} \overline{E}_i &\subseteq \underbrace{u_i}_{\substack{=u+\lambda_i t_{-\alpha} \\ \text{(by (23))}}} + t_{-\alpha}\mathbb{L}_{++} = u + \underbrace{\lambda_i t_{-\alpha} + t_{-\alpha}\mathbb{L}_{++}}_{=t_{-\alpha}(\lambda_i + \mathbb{L}_{++})} \\ &= u + \underbrace{t_{-\alpha}}_{\substack{\in t_{-\gamma}\mathbb{L}_+ \\ \text{(since } -\alpha \geqslant -\gamma \\ \text{(because } \alpha \leqslant \gamma))}} \underbrace{(\lambda_i + \mathbb{L}_{++})}_{\substack{\subseteq \mathbb{L}_+ \\ \text{(since } \lambda_i \in \mathbb{K} \subseteq \mathbb{L}_+)}} \subseteq u + t_{-\gamma} \underbrace{\mathbb{L}_+ \mathbb{L}_+}_{\subseteq \mathbb{L}_+} \\ &\subseteq u + t_{-\gamma}\mathbb{L}_+. \end{aligned} \tag{25}$$

[19] *Proof.* Assume the contrary. Thus, there exist some distinct $i, j \in \{1, 2, \ldots, m\}$ such that $\overline{E}_i \cap \overline{E}_j \neq \varnothing$. Consider these i and j. There exists some $a \in \overline{E}_i \cap \overline{E}_j$ (since $\overline{E}_i \cap \overline{E}_j \neq \varnothing$). Consider this a. We have $a \in \overline{E}_i \cap \overline{E}_j \subseteq \overline{E}_i$ and similarly $a \in \overline{E}_j$. Hence, Claim 1 (applied to $b = a$) yields $a - a \neq 0$ and $\operatorname{ord}(a - a) = -\alpha$. But $a - a \neq 0$ clearly contradicts $a - a = 0$. This contradiction proves that our assumption was false, qed.

Now,

$$\overline{E} = \overline{E}_1 \cup \overline{E}_2 \cup \cdots \cup \overline{E}_m = \bigcup_{i=1}^{m} \underbrace{\overline{E}_i}_{\substack{\subseteq u + t_{-\gamma}\mathbb{L}_+ \\ \text{(by (25))}}} \subseteq \bigcup_{i=1}^{m} \left(u + t_{-\gamma}\mathbb{L}_+\right) \subseteq u + t_{-\gamma}\mathbb{L}_+.$$

Now, recall that $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_m$ and $\overline{E} = \overline{E}_1 \sqcup \overline{E}_2 \sqcup \cdots \sqcup \overline{E}_m$. Thus, we can glue the bijections $f_i : E_i \to \overline{E}_i$ together to a single bijection

$$f : E \to \overline{E},$$
$$a \mapsto f_i(a), \qquad \text{where } i \in \{1, 2, \ldots, m\} \text{ is such that } a \in E_i.$$

Consider this f. Define a weight function $\overline{w} : \overline{E} \to \mathbb{V}$ by setting $\overline{w} = w \circ f^{-1}$. Then, $\left(\overline{E}, \overline{w}, \overline{d}_{\overline{E}}\right)$ is a (γ, u)-positioned valadic $\mathbb{V}$-ultra triple. (It is (γ, u)-positioned, since $\overline{E} \subseteq u + t_{-\gamma}\mathbb{L}_+$.) Moreover, it is easy to see that every $i, j \in \{1, 2, \ldots, m\}$ and any two distinct elements $a \in \overline{E}_i$ and $b \in \overline{E}_j$ satisfy

$$\overline{d}_{\overline{E}}(a, b) = \begin{cases} \alpha, & \text{if } i \neq j; \\ \overline{d}_{\overline{E}_i}(a, b), & \text{if } i = j \end{cases} \tag{26}$$

[20]. From this, it is easy to see that

$$\overline{d}_{\overline{E}}(f(a), f(b)) = d(a, b) \qquad \text{for all } (a, b) \in E \underline{\times} E \tag{27}$$

[21]. Moreover, $\overline{w} \circ f = w$ (since $\overline{w} = w \circ f^{-1}$). Hence, the bijection $f : E \to \overline{E}$ is an isomorphism of $\mathbb{V}$-ultra triples from (E, w, d) to $\left(\overline{E}, \overline{w}, \overline{d}_{\overline{E}}\right)$. Hence, there exists a

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[20] *Proof of (26):* Let $i, j \in \{1, 2, \ldots, m\}$, and let $a \in \overline{E}_i$ and $b \in \overline{E}_j$ be two distinct elements. We must prove (26). It clearly suffices to verify the equality

$$\operatorname{ord}(a - b) = \begin{cases} -\alpha, & \text{if } i \neq j; \\ \operatorname{ord}(a - b), & \text{if } i = j \end{cases}$$

(since both $\overline{d}_{\overline{E}}$ and $\overline{d}_{\overline{E}_i}$ are given by the formula (22)). This equality is obviously true in the case when $i = j$; on the other hand, it follows from Claim 1 in the case when $i \neq j$. Hence, (26) is proved.

[21] *Proof of (27):* Let $(a, b) \in E \underline{\times} E$. Thus, a and b are two distinct elements of E. Let $i, j \in \{1, 2, \ldots, m\}$ be such that $a \in E_i$ and $b \in E_j$. (These i and j clearly exist, because a and b both belong to $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_m$.) Then, the definition of f yields $f(a) = f_i(a)$ and $f(b) = f_j(b)$. Hence,

$$\begin{aligned} \overline{d}_{\overline{E}}(f(a), f(b)) &= \overline{d}_{\overline{E}}\left(f_i(a), f_j(b)\right) \\ &= \begin{cases} \alpha, & \text{if } i \neq j; \\ \overline{d}_{\overline{E}_i}\left(f_i(a), f_j(b)\right), & \text{if } i = j \end{cases} \end{aligned} \tag{28}$$

(by (26), applied to $f_i(a)$ and $f_j(b)$ instead of a and b). If $i \neq j$, then this becomes

$$\overline{d}_{\overline{E}}(f(a), f(b)) = \alpha = d(a, b) \qquad \text{(since Proposition 8.6 **(e)** yields } d(a, b) = \alpha\text{)},$$

(γ, u)-positioned valadic $\mathbb{V}$-ultra triple isomorphic to (E, w, d) (namely, $\left(\overline{E}, \overline{w}, \overline{d}_{\overline{E}}\right)$). This proves Theorem 9.2 for our (E, w, d). Thus, the induction step is complete, and Theorem 9.2 is proven. $\square$

10. Proof of the main theorem

We can now prove Theorem 4.7 (from which we will immediately obtain Theorem 3.1).

Proof of Theorem 4.7. Pick some $\gamma \in \mathbb{V}$ such that

$$d(a, b) \leqslant \gamma \qquad \text{for each } (a, b) \in E \underline{\times} E.$$

(Such a γ clearly exists, since the set $E \underline{\times} E$ is finite and the set $\mathbb{V}$ is totally ordered.)

Theorem 9.2 (applied to $u = 0$) thus yields that there exists a $(\gamma, 0)$-positioned valadic $\mathbb{V}$-ultra triple isomorphic to (E, w, d). Consider this valadic $\mathbb{V}$-ultra triple, and denote it by (F, v, c). Let $\mathcal{E}$ denote the Bhargava greedoid of this $\mathbb{V}$-ultra triple (F, v, c). The set F is a subset of $\mathbb{L}$ (since (F, v, c) is a valadic $\mathbb{V}$-ultra triple) and is finite (since (F, v, c) is isomorphic to (E, w, d), whence $|F| = |E| < \infty$). Moreover, the distance function c of the $\mathbb{V}$-ultra triple (F, v, c) is the function d from Definition 6.7 (since (F, v, c) is a valadic $\mathbb{V}$-ultra triple). Hence, Theorem 6.9 (applied to (F, v, c) instead of (E, w, d)) yields that the Bhargava greedoid of (F, v, c) is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$. In other words, $\mathcal{E}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$ (since the Bhargava greedoid of (F, v, c) is $\mathcal{E}$).

But Proposition 7.4 yields that the Bhargava greedoids of (E, w, d) and (F, v, c) are isomorphic. In other words, the set systems $\mathcal{F}$ and $\mathcal{E}$ are isomorphic (since $\mathcal{F}$

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and thus (27) is proven in this case. Hence, for the rest of this proof of (27), we WLOG assume that $i = j$. Hence, $b \in E_j = E_i$ (since $j = i$). Recall that $f_i : E_i \to \overline{E}_i$ is an isomorphism of $\mathbb{V}$-ultra triples from (E_i, w_i, d_i) to $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$. Hence,

$$\overline{d}_i (f_i(a), f_i(b)) = d_i(a, b) = d(a, b) \qquad (\text{by the definition of } d_i).$$

Also, $\overline{d}_i = \overline{d}_{\overline{E}_i}$ (since the $\mathbb{V}$-ultra triple $\left(\overline{E}_i, \overline{w}_i, \overline{d}_i\right)$ is valadic). Now, (28) becomes

$$\begin{aligned} \overline{d}_{\overline{E}} (f(a), f(b)) &= \begin{cases} \alpha, & \text{if } i \neq j; \\ \overline{d}_{\overline{E}_i} \left(f_i(a), f_j(b)\right), & \text{if } i = j \end{cases} \\ &= \overline{d}_{\overline{E}_i} \left(f_i(a), f_j(b)\right) \qquad (\text{since } i = j) \\ &= \underbrace{\overline{d}_{\overline{E}_i}}_{= \overline{d}_i} \left(f_i(a), f_i(b)\right) \qquad (\text{since } j = i) \\ &= \overline{d}_i (f_i(a), f_i(b)) = d(a, b). \end{aligned}$$

Thus, (27) is proven.

and $\mathcal{E}$ are the Bhargava greedoids of (E, w, d) and (F, v, c), respectively). In other words, the set systems $\mathcal{E}$ and $\mathcal{F}$ are isomorphic.

Hence, Proposition 7.5 shows that $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$. This proves Theorem 4.7. □

Proof of Theorem 3.1. We have $|\mathbb{K}| \geqslant |E| \geqslant \operatorname{mcs}(E, w, d)$. Thus, Theorem 4.7 yields that $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$. This proves Theorem 3.1. □

11. Proof of Theorem 5.1

For the rest of Section 11, we fix a $\mathbb{V}$-ultra triple (E, w, d).

Our next goal is to prove Theorem 5.1. We have to build several tools to this purpose.

11.1. Closed balls

We will use a counterpart to the concept of open balls: the notion of *closed balls*. To wit, it is defined as follows:

Definition 11.1. Let $\alpha \in \mathbb{V}$ and $e \in E$. The *closed ball* $B_\alpha(e)$ is defined to be the subset

$$\{f \in E \mid f = e \text{ or else } d(f, e) \leqslant \alpha\}$$

of E.

Clearly, for each $\alpha \in \mathbb{V}$ and each $e \in E$, we have $e \in B_\alpha(e)$, so that the closed ball $B_\alpha(e)$ contains at least the element e.

Most properties of open balls have analogues for closed balls. In particular, here is an analogue of Proposition 8.2:

Proposition 11.2. Let $\alpha \in \mathbb{V}$ and $e, f \in E$ be such that $e \neq f$ and $d(e, f) \leqslant \alpha$. Then, $B_\alpha(e) = B_\alpha(f)$.

Proof. This can be proved by a straightforward modification of the above proof of Proposition 8.2 (namely, all "$<$" signs are replaced by "$\leqslant$" signs). □

Next comes an analogue of Corollary 8.3:

Corollary 11.3. Let $\alpha \in \mathbb{V}$ and $e \in E$. Let $f \in B_\alpha(e)$. Then, $B_\alpha(e) = B_\alpha(f)$.

Proof. This can be proved by a straightforward modification of the above proof of Corollary 8.3 (namely, all "$<$" signs are replaced by "$\leqslant$" signs). □

Knowing these properties, we can easily obtain the following lemma:

Lemma 11.4. Let $\beta \in \mathbb{V}$. Let C be a β-clique.

(a) The closed balls $B_\beta(c)$ for all $c \in C$ are identical.

Now, let $B = B_\beta(c)$ for some $c \in C$. Then:

(b) We have $C \subseteq B$.

(c) For any distinct elements $p, q \in B$, we have $d(p,q) \leqslant \beta$.

(d) For any $n \in E \setminus B$ and any $p, q \in B$, we have $d(n,p) = d(n,q)$.

(Intuitively, it helps to think of a clique C as an ultrametric analogue of a sphere, and of the set B constructed in Lemma 11.4 as being the whole closed ball whose boundary is this sphere. Of course, this must not be taken literally; in particular, every point in this ball serves as the "center" of this ball, so to speak.)

Proof of Lemma 11.4. We know that C is a β-clique. In other words, C is a subset of E such that

$$\text{any two distinct elements } a, b \in C \text{ satisfy } d(a,b) = \beta \tag{29}$$

(by the definition of a "β-clique").

(a) We must prove that $B_\beta(e) = B_\beta(f)$ for any $e, f \in C$.

So let $e, f \in C$. We must prove that $B_\beta(e) = B_\beta(f)$. If $e = f$, then this is obvious. Hence, we WLOG assume that $e \neq f$. Thus, (29) (applied to $a = e$ and $b = f$) yields $d(e,f) = \beta$. Thus, the "symmetry" axiom in Definition 2.3 yields that $d(f,e) = d(e,f) = \beta \leqslant \beta$. Hence, $f = e$ or else $d(f,e) \leqslant \beta$. In other words, $f \in B_\beta(e)$ (by the definition of $B_\beta(e)$). Hence, Corollary 11.3 (applied to $\alpha = \beta$) yields $B_\beta(e) = B_\beta(f)$.

Hence, we have proved $B_\beta(e) = B_\beta(f)$; thus, our proof of Lemma 11.4 **(a)** is complete.

In preparation for the proofs of parts **(b)**, **(c)** and **(d)**, let us observe the following: We have defined B to be $B_\beta(c)$ for some $c \in C$. Consider this c. Thus, $B = B_\beta(c)$. Lemma 11.4 **(a)** says that

$$B_\beta(a) = B_\beta(b) \qquad \text{for each } a, b \in C. \tag{30}$$

Thus, for each $a \in C$, we have

$$\begin{aligned} B_\beta(a) &= B_\beta(c) && \text{(by (30), applied to } b = c) \\ &= B && \left(\text{since } B = B_\beta(c)\right). \end{aligned} \tag{31}$$

(b) Let $a \in C$. Then, $a \in B_\beta(a)$ (by the definition of $B_\beta(a)$, since $a = a$). Hence, $a \in B_\beta(a) = B$ (by (31)).

Forget that we fixed a. We thus have proved that $a \in B$ for each $a \in C$. In other words, $C \subseteq B$. This proves Lemma 11.4 **(b)**.

(c) Let $p, q \in B$ be distinct. We must prove that $d(p,q) \leqslant \beta$.

We have $p \in B = B_\beta(c)$. Hence, Corollary 11.3 (applied to $\alpha = \beta$, $e = c$ and $f = p$) yields $B_\beta(c) = B_\beta(p)$. Now, $q \in B = B_\beta(c) = B_\beta(p)$. In other words, we have $q = p$ or else $d(q,p) \leqslant \beta$ (by the definition of $B_\beta(p)$). Since $q = p$ is impossible (because p and q are distinct), we thus obtain $d(q,p) \leqslant \beta$. Thus, the "symmetry" axiom in Definition 2.3 yields that $d(p,q) = d(q,p) \leqslant \beta$. This proves Lemma 11.4 **(c)**.

(d) Let $n \in E \setminus B$ and $p, q \in B$. We must prove that $d(n,p) = d(n,q)$. If $p = q$, then this is obvious. Thus, we WLOG assume that $p \neq q$. Hence, Lemma 11.4 **(c)** yields $d(p,q) \leqslant \beta$.

We have $n \in E \setminus B$. In other words, $n \in E$ and $n \notin B$.

But we have $q \in B = B_\beta(c)$. Hence, Corollary 11.3 (applied to $\alpha = \beta$, $e = c$ and $f = q$) yields $B_\beta(c) = B_\beta(q)$. Now, $n \notin B = B_\beta(c) = B_\beta(q)$. In other words, we don't have ($n = q$ or else $d(n,q) \leqslant \beta$) (by the definition of $B_\beta(q)$). In other words, we have $n \neq q$ and $d(n,q) > \beta$. Thus, $d(n,q) > \beta \geqslant d(p,q)$ (since $d(p,q) \leqslant \beta$).

We have $p \neq n$ (since $p \in B$ but $n \notin B$) and similarly $q \neq n$. Hence, the elements n, p and q of E are distinct (since $p \neq n$ and $q \neq n$ and $p \neq q$). The ultrametric triangle inequality (applied to n, p and q instead of a, b and c) thus yields

$$d(n,p) \leqslant \max\{d(n,q), d(p,q)\} = d(n,q) \qquad (\text{since } d(n,q) > d(p,q)).$$

The same argument (with the roles of p and q interchanged) yields $d(n,q) \leqslant d(n,p)$. Combining these two inequalities, we obtain $d(n,p) = d(n,q)$. This proves Lemma 11.4 **(d)**. □

11.2. Exchange results for sets intersecting a ball

From now on, for the rest of Section 11, we assume that E is finite.

Corollary 11.5. Let $\beta \in \mathbb{V}$. Let C be a β-clique. Let $B = B_\beta(c)$ for some $c \in C$.

Let N be a subset of $E \setminus B$.

Let P and Q be two subsets of B such that $|P| = |Q|$. Then:

(a) We have $\operatorname{PER}(N \cup Q) - \operatorname{PER}(N \cup P) = \operatorname{PER}(Q) - \operatorname{PER}(P)$.

(b) Assume that the map $w : E \to \mathbb{V}$ is constant. Assume further that Q is a subset of C. Then, $\operatorname{PER}(N \cup Q) \geqslant \operatorname{PER}(N \cup P)$.

Proof of Corollary 11.5. Let $m = |P| = |Q|$. Let $p_1, p_2, \ldots, p_m$ be all the m elements of P (listed without repetition). Let $q_1, q_2, \ldots, q_m$ be all the m elements of Q (listed without repetition).

(a) We have

$$d(n, p_i) = d(n, q_i) \qquad \text{for each } n \in N \text{ and } i \in \{1, 2, \ldots, m\}. \tag{32}$$

[*Proof of (32):* Let $n \in N$ and $i \in \{1, 2, \ldots, m\}$. Then, $n \in N \subseteq E \setminus B$ and $p_i \in P \subseteq B$ and $q_i \in Q \subseteq B$. Thus, Lemma 11.4 **(d)** (applied to $p = p_i$ and $q = q_i$) yields $d(n, p_i) = d(n, q_i)$. This proves (32).]

The set N is disjoint from B (since N is a subset of $E \setminus B$), and thus disjoint from P as well (since $P \subseteq B$). Hence, the definition of perimeter yields

$$\begin{aligned}
\operatorname{PER}(N \cup P) &= \operatorname{PER}(N) + \operatorname{PER}(P) + \sum_{n \in N} \underbrace{\sum_{p \in P} d(n, p)}_{\substack{= \sum_{i=1}^{m} d(n, p_i) \\ \text{(since } p_1, p_2, \ldots, p_m \text{ are all} \\ \text{the } m \text{ elements of } P \\ \text{(listed without repetition))}}} \\
&= \operatorname{PER}(N) + \operatorname{PER}(P) + \sum_{n \in N} \sum_{i=1}^{m} d(n, p_i).
\end{aligned}$$

Likewise,

$$\operatorname{PER}(N \cup Q) = \operatorname{PER}(N) + \operatorname{PER}(Q) + \sum_{n \in N} \sum_{i=1}^{m} d(n, q_i).$$

Subtracting the first of these two equalities from the second, we obtain

$$\begin{aligned}
&\operatorname{PER}(N \cup Q) - \operatorname{PER}(N \cup P) \\
&= \left(\operatorname{PER}(N) + \operatorname{PER}(Q) + \sum_{n \in N} \sum_{i=1}^{m} d(n, q_i) \right) \\
&\qquad - \left(\operatorname{PER}(N) + \operatorname{PER}(P) + \sum_{n \in N} \sum_{i=1}^{m} d(n, p_i) \right) \\
&= \operatorname{PER}(Q) - \operatorname{PER}(P) + \sum_{n \in N} \sum_{i=1}^{m} d(n, q_i) - \sum_{n \in N} \sum_{i=1}^{m} \underbrace{d(n, p_i)}_{\substack{= d(n, q_i) \\ \text{(by (32))}}} \\
&= \operatorname{PER}(Q) - \operatorname{PER}(P) + \sum_{n \in N} \sum_{i=1}^{m} d(n, q_i) - \sum_{n \in N} \sum_{i=1}^{m} d(n, q_i) \\
&= \operatorname{PER}(Q) - \operatorname{PER}(P).
\end{aligned}$$

This proves Corollary 11.5 **(a)**.

(b) We know that the map $w : E \to \mathbb{V}$ is constant. Hence,

$$w(a) = w(b) \qquad \text{for any } a, b \in E. \tag{33}$$

Each $i \in \{1, 2, \ldots, m\}$ satisfies

$$w(p_i) = w(c) \tag{34}$$

(by (33), applied to $a = p_i$ and $b = c$) and

$$w(q_i) = w(c) \tag{35}$$

(by (33), applied to $a = q_i$ and $b = c$).

We know that C is a β-clique. In other words, C is a subset of E such that

$$\text{any two distinct elements } a, b \in C \text{ satisfy } d(a, b) = \beta \tag{36}$$

(by the definition of a "β-clique").

But $p_1, p_2, \ldots, p_m$ are m distinct elements of P (by their definition). Hence, $p_1, p_2, \ldots, p_m$ are m distinct elements of B (since $P \subseteq B$). Thus, if i and j are two distinct elements of $\{1, 2, \ldots, m\}$, then p_i and p_j are two distinct elements of B, and therefore satisfy

$$d\left(p_i, p_j\right) \leqslant \beta \tag{37}$$

(by Lemma 11.4 **(c)**, applied to $p = p_i$ and $q = p_j$).

On the other hand, $q_1, q_2, \ldots, q_m$ are m distinct elements of Q (by their definition). Hence, $q_1, q_2, \ldots, q_m$ are m distinct elements of C (since $Q \subseteq C$). Thus, if i and j are two distinct elements of $\{1, 2, \ldots, m\}$, then q_i and q_j are two distinct elements of C, and therefore satisfy

$$d\left(q_i, q_j\right) = \beta \tag{38}$$

(by (36), applied to $a = q_i$ and $b = q_j$).

Recall that $p_1, p_2, \ldots, p_m$ are all the m elements of P (listed without repetition). Hence, the definition of perimeter yields

$$\begin{aligned}
\operatorname{PER}(P) &= \sum_{i=1}^{m} \underbrace{w(p_i)}_{\substack{=w(c) \\ \text{(by (34))}}} + \sum_{1 \leqslant i < j \leqslant m} \underbrace{d\left(p_i, p_j\right)}_{\substack{\leqslant \beta \\ \text{(by (37))}}} \leqslant \sum_{i=1}^{m} \underbrace{w(c)}_{\substack{=w(q_i) \\ \text{(by (35))}}} + \sum_{1 \leqslant i < j \leqslant m} \underbrace{\beta}_{\substack{=d\left(q_i, q_j\right) \\ \text{(by (38))}}} \\
&= \sum_{i=1}^{m} w(q_i) + \sum_{1 \leqslant i < j \leqslant m} d\left(q_i, q_j\right) = \operatorname{PER}(Q)
\end{aligned}$$

(by the definition of Q, since $q_1, q_2, \ldots, q_m$ are all the m elements of Q (listed without repetition)).

But Corollary 11.5 **(a)** yields

$$\operatorname{PER}(N \cup Q) - \operatorname{PER}(N \cup P) = \operatorname{PER}(Q) - \operatorname{PER}(P) \geqslant 0$$

(since $\operatorname{PER}(P) \leqslant \operatorname{PER}(Q)$). In other words, $\operatorname{PER}(N \cup Q) \geqslant \operatorname{PER}(N \cup P)$. This proves Corollary 11.5 **(b)**. □

Corollary 11.6. Let $\mathcal{F}$ be the Bhargava greedoid of (E, w, d). Assume that the map $w : E \to \mathbb{V}$ is constant.

Let $\beta \in \mathbb{V}$. Let C be a β-clique. Let $B = B_\beta(c)$ for some $c \in C$.

Let N be a subset of $E \setminus B$.

Let P and Q be two subsets of B such that $|P| = |Q|$ and $Q \subseteq C$ and $N \cup P \in \mathcal{F}$. Then, $N \cup Q \in \mathcal{F}$.

Proof of Corollary 11.6. Corollary 11.5 **(b)** yields $\operatorname{PER}(N \cup Q) \geqslant \operatorname{PER}(N \cup P)$.

Also, the set N is disjoint from B (since $N \subseteq E \setminus B$), and thus is disjoint from P as well (since P is a subset of B). Hence, $|N \cup P| = |N| + |P|$. The same argument (applied to Q instead of P) shows that $|N \cup Q| = |N| + |Q|$. Hence, $|N \cup P| = |N| + \underbrace{|P|}_{=|Q|} = |N| + |Q| = |N \cup Q|$.

We know that $\mathcal{F}$ is the Bhargava greedoid of (E, w, d). In other words,

$$\mathcal{F} = \{A \subseteq E \mid A \text{ has maximum perimeter among all } |A| \text{-subsets of } E\} \qquad (39)$$

(by Definition 2.8). Hence, from $N \cup P \in \mathcal{F}$, we conclude that the set $N \cup P$ has maximum perimeter among all $|N \cup P|$-subsets of E. In other words, the set $N \cup P$ has maximum perimeter among all $|N \cup Q|$-subsets of E (since $|N \cup P| = |N \cup Q|$). Since $N \cup Q$ is a further $|N \cup Q|$-subset of E, we thus conclude that $\operatorname{PER}(N \cup P) \geqslant \operatorname{PER}(N \cup Q)$. Combining this with $\operatorname{PER}(N \cup Q) \geqslant \operatorname{PER}(N \cup P)$, we obtain $\operatorname{PER}(N \cup Q) = \operatorname{PER}(N \cup P)$. In other words, the subsets $N \cup Q$ and $N \cup P$ of E have the same perimeter. Therefore, the set $N \cup Q$ has maximum perimeter among all $|N \cup Q|$-subsets of E (because the set $N \cup P$ has maximum perimeter among all $|N \cup Q|$-subsets of E). In view of (39), this entails that $N \cup Q \in \mathcal{F}$. This proves Corollary 11.6. □

11.3. Gaussian elimination greedoids in terms of determinants

Next, we introduce some notations for matrices.

Definition 11.7. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $A = (a_{i,j})_{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m}$ be an $n \times m$-matrix. Let $i_1, i_2, \ldots, i_u$ be some elements of $\{1, 2, \ldots, n\}$; let $j_1, j_2, \ldots, j_v$ be some elements of $\{1, 2, \ldots, m\}$. Then, we define $\operatorname{sub}_{i_1, i_2, \ldots, i_u}^{j_1, j_2, \ldots, j_v} A$ to be the $u \times v$-matrix $\left(a_{i_x, j_y}\right)_{1 \leqslant x \leqslant u,\ 1 \leqslant y \leqslant v}$.

When $i_1 < i_2 < \cdots < i_u$ and $j_1 < j_2 < \cdots < j_v$, the matrix $\operatorname{sub}_{i_1, i_2, \ldots, i_u}^{j_1, j_2, \ldots, j_v} A$ can be obtained from A by crossing out all rows other than the i_1-th, the i_2-th, etc., the i_u-th row and crossing out all columns other than the j_1-th, the j_2-th, etc., the j_v-th column. Thus, in this case, $\operatorname{sub}_{i_1, i_2, \ldots, i_u}^{j_1, j_2, \ldots, j_v} A$ is called a *submatrix* of A.

Example 11.8. If $n = 3$ and $m = 4$ and $A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \end{pmatrix}$, then $\operatorname{sub}_{1,3}^{1,3,4} A = \begin{pmatrix} a & c & d \\ i & k & \ell \end{pmatrix}$ (this is a submatrix of A) and $\operatorname{sub}_{2,3}^{3,2,1} A = \begin{pmatrix} g & f & e \\ k & j & i \end{pmatrix}$ (this is not, in general, a submatrix of A).

We can now describe Gaussian elimination greedoids in terms of determinants:

Lemma 11.9. Let $n \in \mathbb{N}$. Let E be the set $\{1, 2, \ldots, n\}$.

Let $m \in \mathbb{N}$ be such that $m \geqslant |E|$. Let $\mathbb{K}$ be a field. For each $e \in E$, let $v_e \in \mathbb{K}^m$ be a column vector. Let A be the $m \times n$-matrix whose columns (from left to right) are $v_1, v_2, \ldots, v_n$.

Let $\mathcal{G}$ be the Gaussian elimination greedoid of the vector family $(v_e)_{e \in E}$.

Let $p \in \mathbb{N}$. Let $i_1, i_2, \ldots, i_p \in E$ be p distinct numbers. Let $I = \{i_1, i_2, \ldots, i_p\}$. Then,

$$I \in \mathcal{G} \text{ holds if and only if } \det\left(\operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A\right) \neq 0.$$

Proof of Lemma 11.9. We have $I = \{i_1, i_2, \ldots, i_p\}$. Thus, $|I| = p$ (since $i_1, i_2, \ldots, i_p$ are distinct) and $I \subseteq E$ (since $i_1, i_2, \ldots, i_p \in E$).

Define the maps π_k for all $k \in \{0, 1, \ldots, m\}$ as in Definition 1.4. Since $I \subseteq E$, we have $|I| \leqslant |E|$. Hence, $p = |I| \leqslant |E| \leqslant m$ (since $m \geqslant |E|$), so that $p \in \{0, 1, \ldots, m\}$. Thus, the map $\pi_p : \mathbb{K}^m \to \mathbb{K}^p$ is well-defined.

Write the $m \times n$-matrix A in the form $A = (a_{i,j})_{1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n}$. Then, the definition of $\operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A$ yields

$$\operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A = \left(a_{x,i_y}\right)_{1 \leqslant x \leqslant p,\ 1 \leqslant y \leqslant p}.$$

Thus, for each $k \in \{1, 2, \ldots, p\}$, we have

$$\left(\text{the } k\text{-th column of } \operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A\right) = \begin{pmatrix} a_{1,i_k} \\ a_{2,i_k} \\ \vdots \\ a_{p,i_k} \end{pmatrix}. \tag{40}$$

The columns of the matrix A (from left to right) are $v_1, v_2, \ldots, v_n$. In other words, the ℓ-th column of A is v_ℓ for each $\ell \in \{1, 2, \ldots, n\}$. In other words, for each $\ell \in \{1, 2, \ldots, n\}$, we have (the ℓ-th column of A) $= v_\ell$. Thus, for each $\ell \in \{1, 2, \ldots, n\}$, we have

$$v_\ell = (\text{the } \ell\text{-th column of } A) = \begin{pmatrix} a_{1,\ell} \\ a_{2,\ell} \\ \vdots \\ a_{m,\ell} \end{pmatrix}$$

(since $A = \left(a_{i,j}\right)_{1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n}$) and therefore

$$\pi_p\left(v_\ell\right) = \pi_p\left(\begin{pmatrix} a_{1,\ell} \\ a_{2,\ell} \\ \vdots \\ a_{m,\ell} \end{pmatrix}\right) = \begin{pmatrix} a_{1,\ell} \\ a_{2,\ell} \\ \vdots \\ a_{p,\ell} \end{pmatrix} \tag{41}$$

(by the definition of π_p). Hence, for each $k \in \{1, 2, \ldots, p\}$, we have

$$\begin{aligned} \pi_p\left(v_{i_k}\right) &= \begin{pmatrix} a_{1,i_k} \\ a_{2,i_k} \\ \vdots \\ a_{p,i_k} \end{pmatrix} && \left(\text{by (41), applied to } \ell = i_k\right) \\ &= \left(\text{the } k\text{-th column of } \operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A\right) && \left(\text{by (40)}\right). \end{aligned}$$

In other words, the vectors $\pi_p\left(v_{i_1}\right), \pi_p\left(v_{i_2}\right), \ldots, \pi_p\left(v_{i_p}\right)$ are the columns of the matrix $\operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A$.

We know that $\mathcal{G}$ is the Gaussian elimination greedoid of the vector family $\left(v_e\right)_{e \in E}$. Thus, Definition 1.4 shows that

$$\mathcal{G} = \left\{F \subseteq E \mid \text{the family } \left(\pi_{|F|}\left(v_e\right)\right)_{e \in F} \in \left(\mathbb{K}^{|F|}\right)^F \text{ is linearly independent}\right\}.$$

Hence, we have the following chain of logical equivalences:

$$
\begin{aligned}
& (I \in \mathcal{G}) \\
\Longleftrightarrow\ & \left(\text{the family } \left(\pi_{|I|}(v_e)\right)_{e\in I} \in \left(\mathbb{K}^{|I|}\right)^I \text{ is linearly independent}\right) \\
\Longleftrightarrow\ & \left(\text{the family } \left(\pi_p(v_e)\right)_{e\in I} \in \left(\mathbb{K}^p\right)^I \text{ is linearly independent}\right) \\
& \qquad (\text{since } |I| = p) \\
\Longleftrightarrow\ & \left(\text{the vectors } \pi_p(v_e) \text{ for } e \in I \text{ are linearly independent}\right) \\
\Longleftrightarrow\ & \left(\text{the vectors } \pi_p(v_{i_1}), \pi_p(v_{i_2}), \ldots, \pi_p(v_{i_p}) \text{ are linearly independent}\right) \\
& \qquad \left(\begin{array}{c} \text{since the vectors } \pi_p(v_e) \text{ for } e \in I \text{ are precisely} \\ \text{the vectors } \pi_p(v_{i_1}), \pi_p(v_{i_2}), \ldots, \pi_p(v_{i_p}) \end{array}\right) \\
\Longleftrightarrow\ & \left(\text{the columns of the matrix } \operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A \text{ are linearly independent}\right) \\
& \qquad \left(\begin{array}{c} \text{since the vectors } \pi_p(v_{i_1}), \pi_p(v_{i_2}), \ldots, \pi_p(v_{i_p}) \text{ are} \\ \text{the columns of the matrix } \operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A \end{array}\right) \\
\Longleftrightarrow\ & \left(\text{the matrix } \operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A \text{ is invertible}\right) \\
& \qquad \left(\begin{array}{c} \text{since } \operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A \text{ is a square matrix, and thus is invertible} \\ \text{if and only if its columns are linearly independent} \end{array}\right) \\
\Longleftrightarrow\ & \left(\det\left(\operatorname{sub}_{1,2,\ldots,p}^{i_1,i_2,\ldots,i_p} A\right) \neq 0\right).
\end{aligned}
$$

This proves Lemma 11.9. □

We can leverage Lemma 11.9 to obtain a criterion that, roughly speaking, says that if a Gaussian elimination greedoid over a field $\mathbb{K}$ contains a certain "constellation" (in an appropriate sense), then $|\mathbb{K}|$ must be $\geqslant$ to a certain value. Namely:

Lemma 11.10. Let $\mathbb{K}$ be a field. Let E be a finite set. Let $\mathcal{F}$ be the Gaussian elimination greedoid of a vector family $(v_e)_{e\in E}$ over $\mathbb{K}$. Let N and C be two disjoint subsets of E. Assume that the following three conditions hold:

(i) For any $i \in C$, we have $N \cup \{i\} \in \mathcal{F}$.

(ii) For any distinct $i, j \in C$, we have $N \cup \{i, j\} \in \mathcal{F}$.

(iii) For any $p \in N$ and any distinct $i, j \in C$, we have $(N \cup \{i, j\}) \setminus \{p\} \notin \mathcal{F}$.

Then, $|\mathbb{K}| \geqslant |C|$.

Proof of Lemma 11.10. Let $m \in \mathbb{N}$ be such that the vectors v_e belong to $\mathbb{K}^m$. Then, $m \geqslant |E|$ (since otherwise, the Gaussian elimination greedoid of the vector family $(v_e)_{e\in E}$ would not be well-defined).

Let $n = |E|$ and $r = |N|$. Hence, N is an r-element subset of the n-element set E. Thus, the set E consists of the r elements of N and of the $n - r$ remaining elements of E.

Clearly, the claim we are proving will not change if we rename the elements of E. Thus, we can rename the elements of E arbitrarily. In particular, we can rename them in such a way that the r elements of the subset N will become $1, 2, \ldots, r$ whereas all remaining $n - r$ elements of E will become $r + 1, r + 2, \ldots, n$. Thus, we can WLOG assume that the r elements of N are $1, 2, \ldots, r$ and the remaining $n - r$ elements of E are $r + 1, r + 2, \ldots, n$. Assume this. Hence, $N = \{1, 2, \ldots, r\}$ and $E \setminus N = \{r + 1, r + 2, \ldots, n\}$ and therefore $E = \{1, 2, \ldots, n\}$.

Since the subsets N and C of E are disjoint, we have $C \subseteq E \setminus N = \{r + 1, r + 2, \ldots, n\}$.

We must prove that $|\mathbb{K}| \geqslant |C|$. If $|C| \leqslant 1$, then this is obvious (since $|\mathbb{K}| \geqslant 1$). Hence, we WLOG assume that $|C| > 1$ from now on. However, from $C \subseteq E \setminus N$, we obtain

$$|C| \leqslant |E \setminus N| = \underbrace{|E|}_{=n} - \underbrace{|N|}_{=r} \qquad \text{(since } N \subseteq E\text{)}$$
$$= n - r.$$

Thus, $n - r \geqslant |C| > 1$, so that $n - r \geqslant 2$ and thus $n \geqslant r + 2$. Thus, $r + 2 \leqslant n = |E| \leqslant m$ (since $m \geqslant |E|$).

Let A be the $m \times n$-matrix whose columns (from left to right) are $v_1, v_2, \ldots, v_n$. Write this matrix A in the form $A = (a_{i,j})_{1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n}$.

We now make a few observations:

Claim 1: We have $\det\left(\operatorname{sub}^{1,2,\ldots,r,i}_{1,2,\ldots,r+1} A\right) \neq 0$ for each $i \in C$.

[*Proof of Claim 1:* Let $i \in C$. Then, $i \in C \subseteq \{r + 1, r + 2, \ldots, n\}$. Thus, the $r + 1$ numbers $1, 2, \ldots, r, i$ are distinct. Also, $N \cup \{i\} \in \mathcal{F}$ (by condition **(i)** in Lemma 11.10). But $N \cup \{i\} = \{1, 2, \ldots, r, i\}$ (since $N = \{1, 2, \ldots, r\}$). Thus, Lemma 11.9 (applied to $\mathcal{F}$, $N \cup \{i\}$, $r + 1$ and $(1, 2, \ldots, r, i)$ instead of $\mathcal{G}$, I, p and $(i_1, i_2, \ldots, i_p)$) yields that $N \cup \{i\} \in \mathcal{F}$ holds if and only if $\det\left(\operatorname{sub}^{1,2,\ldots,r,i}_{1,2,\ldots,r+1} A\right) \neq 0$. Thus, we have $\det\left(\operatorname{sub}^{1,2,\ldots,r,i}_{1,2,\ldots,r+1} A\right) \neq 0$ (since $N \cup \{i\} \in \mathcal{F}$). This proves Claim 1.]

Now, for each $i \in C$, we define a scalar $r_i \in \mathbb{K}$ by

$$r_i = \frac{a_{r+2,i}}{\det\left(\operatorname{sub}^{1,2,\ldots,r,i}_{1,2,\ldots,r+1} A\right)}. \qquad (42)$$

This is well-defined, since Claim 1 yields $\det\left(\operatorname{sub}^{1,2,\ldots,r,i}_{1,2,\ldots,r+1} A\right) \neq 0$ (and since we have $r + 2 \leqslant m$).

Claim 2: The scalars r_i for all $i \in C$ are distinct.

[*Proof of Claim 2:* We need to prove that $r_i \neq r_j$ for any two distinct $i, j \in C$. So let us fix two distinct $i, j \in C$. We must prove that $r_i \neq r_j$.

We have $i, j \in C \subseteq \{r+1, r+2, \ldots, n\}$. Thus, the $r+2$ numbers $1, 2, \ldots, r, i, j$ are distinct (since i and j are distinct). Also, $N \cup \{i, j\} \in \mathcal{F}$ (by condition **(ii)** in Lemma 11.10). But $N \cup \{i, j\} = \{1, 2, \ldots, r, i, j\}$ (since $N = \{1, 2, \ldots, r\}$). Thus, Lemma 11.9 (applied to $\mathcal{F}$, $N \cup \{i, j\}$, $r+2$ and $(1, 2, \ldots, r, i, j)$ instead of $\mathcal{G}$, I, p and $(i_1, i_2, \ldots, i_p)$) yields that $N \cup \{i, j\} \in \mathcal{F}$ holds if and only if $\det\left(\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A\right) \neq 0$. Thus, we have

$$\det\left(\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A\right) \neq 0 \tag{43}$$

(since $N \cup \{i, j\} \in \mathcal{F}$).

Let us agree to use the following notation: If $p \in \{1, 2, \ldots, r\}$ is arbitrary, then "$1, 2, \ldots, \widehat{p}, \ldots, r, i, j$" will denote the list $1, 2, \ldots, r, i, j$ with the entry p omitted (i.e., the list $1, 2, \ldots, p-1, p+1, p+2, \ldots, r, i, j$).

Now, let us use Laplace expansion to expand the determinant of the $(r+2) \times (r+2)$-matrix $\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A$ along its last row. We thus obtain

$$\begin{aligned}
&\det\left(\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A\right) \\
&= \sum_{p=1}^{r} (-1)^{(r+2)+p} a_{r+2,p} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,\widehat{p},\ldots,r,i,j} A\right) \\
&\qquad + (-1)^{(r+2)+(r+1)} a_{r+2,i} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \\
&\qquad + (-1)^{(r+2)+(r+2)} a_{r+2,j} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right).
\end{aligned} \tag{44}$$

(Indeed, the entries of the last row of $\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A$ are

$$\underbrace{a_{r+2,1}, a_{r+2,2}, \ldots, a_{r+2,r}}_{\text{these are the } a_{r+2,p} \text{ for all } p \in \{1,2,\ldots,r\}}, a_{r+2,i}, a_{r+2,j},$$

and the cofactors corresponding to the first r of these entries are

$$(-1)^{(r+2)+p} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,\widehat{p},\ldots,r,i,j} A\right) \qquad \text{for all } p \in \{1, 2, \ldots, r\},$$

whereas the cofactors corresponding to the last two entries are

$$(-1)^{(r+2)+(r+1)} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \qquad \text{and} \qquad (-1)^{(r+2)+(r+2)} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right).$$

)

Now, let us simplify the entries in the $\sum_{p=1}^{r}$ sum on the right hand side of (44).

Let $p \in \{1, 2, \ldots, r\}$. Then, $p \in \{1, 2, \ldots, r\} = N$. Hence, $(N \cup \{i, j\}) \setminus \{p\} \notin \mathcal{F}$ (by condition **(iii)** in Lemma 11.10). In other words, we **don't** have $(N \cup \{i, j\}) \setminus \{p\} \in \mathcal{F}$. But from $N \cup \{i, j\} = \{1, 2, \ldots, r, i, j\}$, we obtain $(N \cup \{i, j\}) \setminus \{p\} = \{1, 2, \ldots, r, i, j\} \setminus \{p\} = \{1, 2, \ldots, \widehat{p}, \ldots, r, i, j\}$ (since the $r + 2$ numbers $1, 2, \ldots, r, i, j$ are distinct). Of course, the $r + 1$ numbers $1, 2, \ldots, \widehat{p}, \ldots, r, i, j$ are distinct (since the $r + 2$ numbers $1, 2, \ldots, r, i, j$ are distinct). Thus, Lemma 11.9 (applied to $\mathcal{F}$, $(N \cup \{i, j\}) \setminus \{p\}$, $r+1$ and $(1, 2, \ldots, \widehat{p}, \ldots, r, i, j)$ instead of $\mathcal{G}$, I, p and $(i_1, i_2, \ldots, i_p)$) yields that $(N \cup \{i, j\}) \setminus \{p\} \in \mathcal{F}$ holds if and only if $\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,\widehat{p},\ldots,r,i,j} A\right) \neq 0$. Thus, we don't have $\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,\widehat{p},\ldots,r,i,j} A\right) \neq 0$ (since we don't have $(N \cup \{i, j\}) \setminus \{p\} \in \mathcal{F}$). In other words, we have

$$\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,\widehat{p},\ldots,r,i,j} A\right) = 0. \tag{45}$$

Forget that we fixed p. We thus have proved (45) for each $p \in \{1, 2, \ldots, r\}$. Now,

(44) becomes

$$\begin{aligned}
&\det\left(\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A\right) \\
&= \sum_{p=1}^{r} (-1)^{(r+2)+p} a_{r+2,p} \underbrace{\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,\widehat{p},\ldots,r,i,j} A\right)}_{\substack{=0 \\ \text{(by (45))}}} \\
&\qquad + \underbrace{(-1)^{(r+2)+(r+1)}}_{=-1} a_{r+2,i} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \\
&\qquad + \underbrace{(-1)^{(r+2)+(r+2)}}_{=1} a_{r+2,j} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \\
&= \underbrace{\sum_{p=1}^{r} (-1)^{(r+2)+p} a_{r+2,p} 0}_{=0} - a_{r+2,i} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) + a_{r+2,j} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \\
&= - \underbrace{a_{r+2,i}}_{\substack{=\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right)\cdot r_i \\ \text{(by (42))}}} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \\
&\qquad + \underbrace{a_{r+2,j}}_{\substack{=\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right)\cdot r_j \\ \text{(since (42) (applied to } j \text{ instead of } i\text{)} \\ \text{yields } r_j = \frac{a_{r+2,j}}{\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right)}\text{)}}} \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \\
&= -\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \cdot r_i \cdot \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \\
&\qquad + \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \cdot r_j \cdot \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \\
&= \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \cdot \left(r_j - r_i\right).
\end{aligned}$$

Hence,

$$\det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,i} A\right) \det\left(\operatorname{sub}_{1,2,\ldots,r+1}^{1,2,\ldots,r,j} A\right) \cdot \left(r_j - r_i\right) = \det\left(\operatorname{sub}_{1,2,\ldots,r+2}^{1,2,\ldots,r,i,j} A\right) \neq 0$$

(by (43)). Thus, $r_j - r_i \neq 0$, so that $r_i \neq r_j$. This proves Claim 2.]

Claim 2 shows that the scalars r_i for all $i \in C$ are distinct. Thus, we have found $|C|$ distinct elements of $\mathbb{K}$ (namely, these scalars r_i for all $i \in C$). Therefore, $\mathbb{K}$ must have at least $|C|$ elements. In other words, $|\mathbb{K}| \geqslant |C|$. Thus, Lemma 11.10 is proved. □

11.4. Proving the theorem

We shall need one more lemma about Gaussian elimination greedoids:

Lemma 11.11. Let $\mathbb{K}$ be a field. Let $\mathcal{F}$ be the Gaussian elimination greedoid of a vector family over $\mathbb{K}$. If $B \in \mathcal{F}$ satisfies $|B| > 0$, then there exists $b \in B$ such that $B \setminus \{b\} \in \mathcal{F}$.

Proof of Lemma 11.11. We shall use the notion of a "strong greedoid", as defined in Definition 12.1.

Theorem 1.6 shows that every Gaussian elimination greedoid is a strong greedoid. Hence, $\mathcal{F}$ is a strong greedoid. In other words, $\mathcal{F}$ is a set system that satisfies the four axioms **(i)**, **(ii)**, **(iii)** and **(iv)** of Definition 12.1. Thus, in particular, $\mathcal{F}$ satisfies axiom **(ii)**. But this axiom says precisely that if $B \in \mathcal{F}$ satisfies $|B| > 0$, then there exists $b \in B$ such that $B \setminus \{b\} \in \mathcal{F}$. Thus, Lemma 11.11 is proved. □

We can now prove Theorem 5.1:

Proof of Theorem 5.1. The definition of $\operatorname{mcs}(E, w, d)$ shows that $\operatorname{mcs}(E, w, d)$ is the maximum size of a clique of (E, w, d). Thus, there exists a clique C of size $\operatorname{mcs}(E, w, d)$. Consider this C. Thus, $\operatorname{mcs}(E, w, d) = |C|$ (since C has size $\operatorname{mcs}(E, w, d)$).

We must prove that $|\mathbb{K}| \geqslant \operatorname{mcs}(E, w, d)$. In other words, we must prove that $|\mathbb{K}| \geqslant |C|$ (since $\operatorname{mcs}(E, w, d) = |C|$). If $|C| \leqslant 1$, then this is obvious (since $|\mathbb{K}| \geqslant 1$). Thus, we WLOG assume that $|C| > 1$. Hence, $|C| \geqslant 2$.

The set C is a clique. In other words, there exists a $\beta \in \mathbb{V}$ such that C is a β-clique. Consider this β.

Choose some $c \in C$. (We can do this, since $|C| \geqslant 2 > 0$.) Set $B = B_\beta(c)$. Lemma 11.4 **(b)** yields $C \subseteq B$, so that $B \supseteq C$.

A *secant set* shall mean a subset S of E satisfying $S \in \mathcal{F}$ and $|S \cap B| \geqslant 2$.

The set E itself satisfies $E \in \mathcal{F}$ (by Remark 2.11) and $|E \cap B| \geqslant 2$ [22]. In other words, E is a secant set. Hence, there exists at least one secant set. Thus, there exists a secant set of smallest size (since there are only finitely many secant sets). Consider such a secant set, and call it S.

Thus, S is a secant set of smallest size. Hence, S is a secant set; in other words, S is a subset of E satisfying $S \in \mathcal{F}$ and $|S \cap B| \geqslant 2$. Since $S \cap B$ is a subset of S, we have $|S \cap B| \leqslant |S|$, so that $|S| \geqslant |S \cap B| \geqslant 2 > 0$. Thus, Lemma 11.11 (applied to $B = S$) yields that there exists $b \in S$ such that $S \setminus \{b\} \in \mathcal{F}$. Consider this b.

We now shall show several claims:

Claim 1: We have $|(S \cap B) \setminus \{b\}| < 2$.

[*Proof of Claim 1:* The set $S \setminus \{b\}$ has smaller size than S (since $b \in S$), and thus cannot be a secant set (since S is a secant set of smallest size). In other words, $S \setminus \{b\}$ cannot be a subset of E satisfying $S \setminus \{b\} \in \mathcal{F}$ and $|(S \setminus \{b\}) \cap B| \geqslant 2$ (by the definition of "secant set"). Hence, we cannot have $|(S \setminus \{b\}) \cap B| \geqslant 2$ (because $S \setminus \{b\}$ is a subset of E satisfying $S \setminus \{b\} \in \mathcal{F}$). In other words, we have $|(S \setminus \{b\}) \cap B| < 2$.

[22] *Proof.* Since B is a subset of E, we have $E \cap B = B \supseteq C$. Thus, $|E \cap B| \geqslant |C| \geqslant 2$.

But it is known from basic set theory that $(X \cap Y) \setminus Z = (X \setminus Z) \cap Y$ for any three sets X, Y and Z. Applying this to $X = S$, $Y = B$ and $Z = \{b\}$, we obtain $(S \cap B) \setminus \{b\} = (S \setminus \{b\}) \cap B$. Hence, $|(S \cap B) \setminus \{b\}| = |(S \setminus \{b\}) \cap B| < 2$. This proves Claim 1.]

Claim 2: We have $b \in S \cap B$.

[*Proof of Claim 2:* Claim 1 yields $|(S \cap B) \setminus \{b\}| < 2 \leqslant |S \cap B|$ (since $|S \cap B| \geqslant 2$). Hence, $|(S \cap B) \setminus \{b\}| \neq |S \cap B|$, so that $(S \cap B) \setminus \{b\} \neq S \cap B$. Therefore, $b \in S \cap B$. This proves Claim 2.]

Claim 3: We have $|S \cap B| = 2$.

[*Proof of Claim 3:* Claim 1 says that $|(S \cap B) \setminus \{b\}| < 2$. But Claim 2 yields $b \in S \cap B$. Thus, $|(S \cap B) \setminus \{b\}| = |S \cap B| - 1$. Hence, $|S \cap B| = \underbrace{|(S \cap B) \setminus \{b\}|}_{<2} + 1 < 2 + 1 = 3$. In other words, $|S \cap B| \leqslant 2$ (since $|S \cap B|$ is an integer). Combining this with $|S \cap B| \geqslant 2$, we find $|S \cap B| = 2$. This proves Claim 3.]

Claim 3 shows that the set $S \cap B$ has exactly two elements. One of these two elements is b (since Claim 2 says that $b \in S \cap B$); let a be the other element. Thus, $a \neq b$ and $S \cap B = \{a, b\}$. Hence, $\{a, b\} = S \cap B \subseteq B$. Also, $|\{a, b\}| = 2$ (since $a \neq b$).

Let $N = S \setminus B$. Then, it is easy to see that $S = N \cup \{a, b\}$ [23] and $N \cap \{a, b\} = \varnothing$ [24]. From $S = N \cup \{a, b\}$, we obtain

$$\begin{aligned} |S| &= |N \cup \{a, b\}| = |N| + \underbrace{|\{a, b\}|}_{=2} \qquad (\text{since } N \cap \{a, b\} = \varnothing) \\ &= |N| + 2. \end{aligned} \tag{46}$$

Moreover,

$$N = \underbrace{S}_{\subseteq E} \setminus B \subseteq E \setminus \underbrace{B}_{\supseteq C} \subseteq E \setminus C.$$

Hence, the two subsets N and C of E are disjoint.

Now, we have the following:

Claim 4: Let $i \in C$. Then, $N \cap \{i\} = \varnothing$ and $N \cup \{i\} \in \mathcal{F}$.

[23] *Proof.* Any two sets X and Y satisfy $X = (X \setminus Y) \cup (X \cap Y)$. Applying this to $X = S$ and $Y = B$, we obtain $S = \underbrace{(S \setminus B)}_{=N} \cup \underbrace{(S \cap B)}_{=\{a,b\}} = N \cup \{a, b\}$.

[24] *Proof.* We have $\{a, b\} \subseteq B$. Hence, $\underbrace{N}_{=S \setminus B} \cap \underbrace{\{a, b\}}_{\subseteq B} \subseteq (S \setminus B) \cap B = \varnothing$, so that $N \cap \{a, b\} = \varnothing$.

[*Proof of Claim 4:* If we had $i \in N$, then we would have $i \notin B$ (since $i \in N = S \setminus B$), which would contradict $i \in C \subseteq B$. Hence, we cannot have $i \in N$. Thus, $N \cap \{i\} = \varnothing$. It remains to prove that $N \cup \{i\} \in \mathcal{F}$.

If we had $b \in N$, then we would have $b \notin B$ (since $b \in N = S \setminus B$), which would contradict $b \in \{a, b\} \subseteq B$. Hence, we cannot have $b \in N$. Thus, $b \notin N$, so that $N \setminus \{b\} = N$.

We have $S = N \cup \{a, b\}$, thus

$$S \setminus \{b\} = (N \cup \{a,b\}) \setminus \{b\} = \underbrace{(N \setminus \{b\})}_{=N} \cup \underbrace{(\{a,b\} \setminus \{b\})}_{\substack{=\{a\} \\ (\text{since } a \neq b)}} = N \cup \{a\}.$$

Hence, $N \cup \{a\} = S \setminus \{b\} \in \mathcal{F}$. But $\{a\}$ and $\{i\}$ are subsets of B (since $a \in \{a, b\} \subseteq B$ and $i \in C \subseteq B$), and satisfy $|\{a\}| = |\{i\}|$ (since both $|\{a\}|$ and $|\{i\}|$ equal 1) and $\{i\} \subseteq C$ (since $i \in C$) and $N \cup \{a\} \in \mathcal{F}$. Hence, Corollary 11.6 (applied to $P = \{a\}$ and $Q = \{i\}$) yields $N \cup \{i\} \in \mathcal{F}$. This finishes the proof of Claim 4.]

Claim 5: Let $i, j \in C$ be distinct. Then, $N \cap \{i, j\} = \varnothing$ and $N \cup \{i, j\} \in \mathcal{F}$.

[*Proof of Claim 5:* If we had $i \in N$, then we would have $i \notin B$ (since $i \in N = S \setminus B$), which would contradict $i \in C \subseteq B$. Hence, we cannot have $i \in N$. In other words, $i \notin N$. Likewise, $j \notin N$. Combining $i \notin N$ with $j \notin N$, we obtain $N \cap \{i, j\} = \varnothing$. It remains to prove that $N \cup \{i, j\} \in \mathcal{F}$.

Note that $|\{a, b\}| = 2$ and $|\{i, j\}| = 2$ (since i and j are distinct). Hence, $|\{a, b\}| = 2 = |\{i, j\}|$. Also, $i, j \in C$, so that $\{i, j\} \subseteq C \subseteq B$.

We have $S = N \cup \{a, b\}$, thus $N \cup \{a, b\} = S \in \mathcal{F}$. But $\{a, b\}$ and $\{i, j\}$ are subsets of B (since $\{a, b\} \subseteq B$ and $\{i, j\} \subseteq B$), and satisfy $|\{a, b\}| = |\{i, j\}|$ and $\{i, j\} \subseteq C$ and $N \cup \{a, b\} \in \mathcal{F}$. Hence, Corollary 11.6 (applied to $P = \{a, b\}$ and $Q = \{i, j\}$) yields $N \cup \{i, j\} \in \mathcal{F}$. This finishes the proof of Claim 5.]

Claim 6: Let $p \in N$. Let $i, j \in C$ be distinct. Then, $(N \cup \{i, j\}) \setminus \{p\} \notin \mathcal{F}$.

[*Proof of Claim 6:* Assume the contrary. Hence, $(N \cup \{i, j\}) \setminus \{p\} \in \mathcal{F}$. Also, $|((N \cup \{i, j\}) \setminus \{p\}) \cap B| \geqslant 2$ [25]. Therefore, $(N \cup \{i, j\}) \setminus \{p\}$ is a secant set (by the definition of "secant set"). Hence,

$$|(N \cup \{i, j\}) \setminus \{p\}| \geqslant |S| \tag{47}$$

[25]*Proof.* We have $p \in N = S \setminus B$, so that $p \notin B$. Hence, $B \setminus \{p\} = B$. Also, $i, j \in C$, so that $\{i, j\} \subseteq C \subseteq B$.

But it is known from set theory that $(X \setminus Y) \cap Z = X \cap (Z \setminus Y)$ for any three sets X, Y and Z. Applying this to $X = N \cup \{i, j\}$, $Y = \{p\}$ and $Z = B$, we obtain

$$((N \cup \{i,j\}) \setminus \{p\}) \cap B = \underbrace{(N \cup \{i,j\})}_{\supseteq \{i,j\}} \cap \underbrace{(B \setminus \{p\})}_{=B} \supseteq \{i,j\} \cap B = \{i,j\}$$

(since $\{i, j\} \subseteq B$). Hence, $|((N \cup \{i, j\}) \setminus \{p\}) \cap B| \geqslant |\{i, j\}| = 2$ (since i and j are distinct). Qed.

(since S was defined to be a secant set of smallest size).

We have $|\{i,j\}| = 2$ (since i and j are distinct). But Claim 5 yields $N \cap \{i,j\} = \varnothing$, so that $|N \cup \{i,j\}| = |N| + \underbrace{|\{i,j\}|}_{=2} = |N| + 2 = |S|$ (by (46)). But $p \in N \subseteq N \cup \{i,j\}$ and thus $|(N \cup \{i,j\}) \setminus \{p\}| = \underbrace{|N \cup \{i,j\}|}_{=|S|} - 1 = |S| - 1 < |S|$. This contradicts (47). This contradiction shows that our assumption was false. Hence, Claim 6 is proved.]

But recall that $\mathcal{F}$ is the Gaussian elimination greedoid of a vector family over $\mathbb{K}$ (by assumption). Let $(v_e)_{e \in E}$ be this vector family. Recall that N and C are two disjoint subsets of E. Moreover, the following facts hold:

(i) For any $i \in C$, we have $N \cup \{i\} \in \mathcal{F}$ (by Claim 4).

(ii) For any distinct $i,j \in C$, we have $N \cup \{i,j\} \in \mathcal{F}$ (by Claim 5).

(iii) For any $p \in N$ and any distinct $i,j \in C$, we have $(N \cup \{i,j\}) \setminus \{p\} \notin \mathcal{F}$ (by Claim 6).

Hence, Lemma 11.10 shows that $|\mathbb{K}| \geqslant |C|$. This proves Theorem 5.1. □

12. Appendix: Gaussian elimination greedoids are strong

We shall now prove Theorem 1.6. In order to do so, we will first need to recall the definition of a strong greedoid (see, e.g., [GriPet19, §6.1]):

Definition 12.1. Let E be a finite set. A set system $\mathcal{F}$ on ground set E is said to be a *greedoid* if it satisfies the following three axioms:

(i) We have $\varnothing \in \mathcal{F}$.

(ii) If $B \in \mathcal{F}$ satisfies $|B| > 0$, then there exists $b \in B$ such that $B \setminus \{b\} \in \mathcal{F}$.

(iii) If $A, B \in \mathcal{F}$ satisfy $|B| = |A| + 1$, then there exists $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{F}$.

Furthermore, $\mathcal{F}$ is said to be a *strong greedoid* if it satisfies the following axiom (in addition to axioms **(i)**, **(ii)** and **(iii)**):

(iv) If $A, B \in \mathcal{F}$ satisfy $|B| = |A| + 1$, then there exists $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{F}$ and $B \setminus \{b\} \in \mathcal{F}$.

Note that axiom **(iii)** in Definition 12.1 clearly follows from axiom **(iv)**; thus, only axioms **(i)**, **(ii)** and **(iv)** need to be checked in order to convince ourselves that a set system is a strong greedoid.

We shall furthermore use a determinantal identity due to Plücker (one of several facts known as the Plücker identities). To state this identity, we will use the following notations:

Definition 12.2. Let $\mathbb{K}$ be a commutative ring. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $A \in \mathbb{K}^{n\times m}$ be an $n \times m$-matrix. (Here and in the following, $\mathbb{K}^{n\times m}$ denotes the set of all $n \times m$-matrices with entries in $\mathbb{K}$.)

- **(a)** If $v \in \mathbb{K}^{n\times 1}$ is a column vector with n entries, then $(A \mid v)$ will denote the $n \times (m+1)$-matrix whose $m+1$ columns are $A_{\bullet,1}, A_{\bullet,2}, \ldots, A_{\bullet,m}, v$ (from left to right). (Informally speaking, $(A \mid v)$ is the matrix obtained when the column vector v is "attached" to A at the right edge.)
- **(b)** If $j \in \{1,2,\ldots,m\}$, then $A_{\bullet,j}$ shall mean the j-th column of the matrix A. This is a column vector with n entries, i.e., an $n \times 1$-matrix.
- **(c)** If $i \in \{1,2,\ldots,n\}$, then $A_{\sim i,\bullet}$ shall mean the matrix obtained from the matrix A by removing the i-th row. This is an $(n-1) \times m$-matrix.
- **(d)** If $i \in \{1,2,\ldots,n\}$ and $j \in \{1,2,\ldots,m\}$, then $A_{\sim i,\sim j}$ shall mean the matrix obtained from A by removing the i-th row and the j-th column. This is an $(n-1) \times (m-1)$-matrix.

We can now state our Plücker identity:

Proposition 12.3. Let $\mathbb{K}$ be a commutative ring. Let n be a positive integer. Let $X \in \mathbb{K}^{n\times(n-1)}$ and $Y \in \mathbb{K}^{n\times n}$. Let $i \in \{1,2,\ldots,n\}$. Then,

$$\det\left(X_{\sim i,\bullet}\right)\det Y = \sum_{q=1}^{n} (-1)^{n+q} \det\left(X \mid Y_{\bullet,q}\right) \det\left(Y_{\sim i,\sim q}\right).$$

Example 12.4. If we set $n = 3$, $X = \begin{pmatrix} a & a' \\ b & b' \\ c & c' \end{pmatrix}$, $Y = \begin{pmatrix} x & x' & x'' \\ y & y' & y'' \\ z & z' & z'' \end{pmatrix}$ and $i = 2$, then Proposition 12.3 states that

$$\det\begin{pmatrix} a & a' \\ c & c' \end{pmatrix}\det\begin{pmatrix} x & x' & x'' \\ y & y' & y'' \\ z & z' & z'' \end{pmatrix}$$
$$= \det\begin{pmatrix} a & a' & x \\ b & b' & y \\ c & c' & z \end{pmatrix}\det\begin{pmatrix} x' & x'' \\ z' & z'' \end{pmatrix} - \det\begin{pmatrix} a & a' & x' \\ b & b' & y' \\ c & c' & z' \end{pmatrix}\det\begin{pmatrix} x & x'' \\ z & z'' \end{pmatrix}$$
$$+ \det\begin{pmatrix} a & a' & x'' \\ b & b' & y'' \\ c & c' & z'' \end{pmatrix}\det\begin{pmatrix} x & x' \\ z & z' \end{pmatrix}.$$

Proposition 12.3 is precisely [Grinbe22, Proposition 6.137] (with A, C and v renamed as X, Y and i).

Proof of Theorem 1.6. We need to prove that $\mathcal{G}$ is a strong greedoid. In other words, we need to prove that the axioms **(i)**, **(ii)**, **(iii)** and **(iv)** from Definition 12.1 are satisfied for $\mathcal{F} = \mathcal{G}$. Let us do this:

*Proof of the axiom **(i)** for $\mathcal{F} = \mathcal{G}$:* The vector family $\left(\pi_{|\varnothing|}(v_e)\right)_{e\in\varnothing} \in \left(\mathbb{K}^{|\varnothing|}\right)^{\varnothing}$ is empty, and thus is linearly independent. In other words, $\varnothing \in \mathcal{G}$. This proves that the axiom **(i)** from Definition 12.1 is satisfied for $\mathcal{F} = \mathcal{G}$.

*Proof of the axiom **(ii)** for $\mathcal{F} = \mathcal{G}$:* Let $B \in \mathcal{G}$ satisfy $|B| > 0$. We shall show that there exists $b \in B$ such that $B \setminus \{b\} \in \mathcal{G}$.

Let $n = |B|$. Thus, $n = |B| > 0$.

We have $B \in \mathcal{G}$. In other words, the family $\left(\pi_{|B|}(v_e)\right)_{e\in B} \in \left(\mathbb{K}^{|B|}\right)^{B}$ is linearly independent (by the definition of $\mathcal{G}$). In other words, the family $(\pi_n(v_e))_{e\in B} \in (\mathbb{K}^n)^B$ is linearly independent (since $|B| = n$).

Let Y be the $n \times n$-matrix whose columns are the vectors $\pi_n(v_e)$ for all $e \in B$ (in some order, with no repetition). The columns of this $n \times n$-matrix Y are thus linearly independent (since the family $(\pi_n(v_e))_{e\in B} \in \left(\mathbb{K}^{|B|}\right)^{B}$ is linearly independent). Hence, this $n \times n$-matrix Y is invertible. In other words, $\det Y \neq 0$.

Now, let $y_{i,j}$ be the (i,j)-th entry of the matrix Y for each $i,j \in \{1,2,\ldots,n\}$. Then, Laplace expansion along the n-th row yields

$$\det Y = \sum_{q=1}^{n} (-1)^{n+q} y_{n,q} \det\left(Y_{\sim n,\sim q}\right).$$

If every $q \in \{1,2,\ldots,n\}$ satisfied $\det\left(Y_{\sim n,\sim q}\right) = 0$, then this would become

$$\det Y = \sum_{q=1}^{n} (-1)^{n+q} y_{n,q} \underbrace{\det\left(Y_{\sim n,\sim q}\right)}_{=0} = 0,$$

which would contradict $\det Y \neq 0$. Hence, not every $q \in \{1, 2, \ldots, n\}$ satisfies $\det \left(Y_{\sim n, \sim q}\right) = 0$. In other words, there exists at least one $q \in \{1, 2, \ldots, n\}$ such that $\det \left(Y_{\sim n, \sim q}\right) \neq 0$. Consider this q. The q-th column of Y is the vector $\pi_n \left(v_f\right)$ for some $f \in B$ (by the definition of Y); consider this f. Note that $|B \setminus \{f\}| = |B| - 1 = n - 1$ (since $|B| = n$).

The $(n-1) \times (n-1)$-matrix $Y_{\sim n, \sim q}$ is invertible (since $\det \left(Y_{\sim n, \sim q}\right) \neq 0$); thus, its $n - 1$ columns are linearly independent. But these $n - 1$ columns are precisely the $n - 1$ vectors $\pi_{n-1} (v_e)$ for all $e \in B \setminus \{f\}$ (by the definition of Y and the choice of f). Hence, the $n - 1$ vectors $\pi_{n-1} (v_e)$ for all $e \in B \setminus \{f\}$ are linearly independent. In other words, the family $(\pi_{n-1} (v_e))_{e \in B \setminus \{f\}}$ is linearly independent. In other words, the family $\left(\pi_{|B \setminus \{f\}|} (v_e)\right)_{e \in B \setminus \{f\}}$ is linearly independent (since $|B \setminus \{f\}| = n - 1$). In other words, $B \setminus \{f\} \in \mathcal{G}$ (by the definition of $\mathcal{G}$). Hence, there exists $b \in B$ such that $B \setminus \{b\} \in \mathcal{G}$ (namely, $b = f$). This shows that the axiom **(ii)** from Definition 12.1 is satisfied for $\mathcal{F} = \mathcal{G}$.

*Proof of the axiom **(iv)** for $\mathcal{F} = \mathcal{G}$:* Let $A, B \in \mathcal{G}$ satisfy $|B| = |A| + 1$. We shall show that there exists $b \in B \setminus A$ such that $A \cup \{b\} \in \mathcal{G}$ and $B \setminus \{b\} \in \mathcal{G}$.

Let $n = |B|$. Thus, $n = |B| = |A| + 1 \geqslant 1 > 0$. Also, from $n = |A| + 1$, we obtain $|A| = n - 1$.

We have $B \in \mathcal{G}$. In other words, the family $\left(\pi_{|B|} (v_e)\right)_{e \in B} \in \left(\mathbb{K}^{|B|}\right)^B$ is linearly independent (by the definition of $\mathcal{G}$). In other words, the family $(\pi_n (v_e))_{e \in B} \in (\mathbb{K}^n)^B$ is linearly independent (since $|B| = n$). Similarly, the family $(\pi_{n-1} (v_e))_{e \in A} \in \left(\mathbb{K}^{n-1}\right)^A$ is linearly independent.

Let Y be the $n \times n$-matrix whose columns are the vectors $\pi_n (v_e)$ for all $e \in B$ (in some order, with no repetition). The columns of this $n \times n$-matrix Y are thus linearly independent (since the family $(\pi_n (v_e))_{e \in B} \in \left(\mathbb{K}^{|B|}\right)^B$ is linearly independent). Hence, this $n \times n$-matrix Y is invertible. In other words, $\det Y \neq 0$.

Let X be the $n \times (n-1)$-matrix whose columns are the vectors $\pi_n (v_e)$ for all $e \in A$ (in some order, with no repetition). Then, the columns of the $(n-1) \times (n-1)$-matrix $X_{\sim n, \bullet}$ are the vectors $\pi_{n-1} (v_e)$ for all $e \in A$; thus, they are linearly independent (since the family $(\pi_{n-1} (v_e))_{e \in A} \in \left(\mathbb{K}^{n-1}\right)^A$ is linearly independent). Hence, this $(n-1) \times (n-1)$-matrix $X_{\sim n, \bullet}$ is invertible. In other words, $\det (X_{\sim n, \bullet}) \neq 0$.

But $\mathbb{K}$ is a field and thus an integral domain. Hence, from $\det (X_{\sim n, \bullet}) \neq 0$ and $\det Y \neq 0$, we obtain

$$\det (X_{\sim n, \bullet}) \det Y \neq 0. \tag{48}$$

If every $q \in \{1, 2, \ldots, n\}$ satisfied $\det \left(X \mid Y_{\bullet, q}\right) \det \left(Y_{\sim n, \sim q}\right) = 0$, then we would have

$$\det\left(X_{\sim n,\bullet}\right)\det Y=\sum_{q=1}^{n}\left(-1\right)^{n+q}\underbrace{\det\left(X\mid Y_{\bullet,q}\right)\det\left(Y_{\sim n,\sim q}\right)}_{\substack{=0\\ \text{(by Proposition 12.3, applied to } i=n\text{)}}}=0,$$

which would contradict (48). Hence, not every $q\in\{1,2,\ldots,n\}$ satisfies $\det\left(X\mid Y_{\bullet,q}\right)\det\left(Y_{\sim n,\sim q}\right)=0$. In other words, some $q\in\{1,2,\ldots,n\}$ satisfies $\det\left(X\mid Y_{\bullet,q}\right)\det\left(Y_{\sim n,\sim q}\right)\neq 0$. Consider this q. The q-th column of Y is the vector $\pi_n\left(v_f\right)$ for some $f\in B$ (by the definition of Y); consider this f. Thus, $Y_{\bullet,q}=\pi_n\left(v_f\right)$.

Note that $|B\setminus\{f\}|=|B|-1=n-1$ (since $|B|=n$).

From $\det\left(X\mid Y_{\bullet,q}\right)\det\left(Y_{\sim n,\sim q}\right)\neq 0$, we obtain $\det\left(X\mid Y_{\bullet,q}\right)\neq 0$ and $\det\left(Y_{\sim n,\sim q}\right)\neq 0$.

The $(n-1)\times(n-1)$-matrix $Y_{\sim n,\sim q}$ is invertible (since $\det\left(Y_{\sim n,\sim q}\right)\neq 0$); thus, its $n-1$ columns are linearly independent. But these $n-1$ columns are precisely the $n-1$ vectors $\pi_{n-1}\left(v_e\right)$ for all $e\in B\setminus\{f\}$ (by the definition of Y and the choice of f). Hence, the $n-1$ vectors $\pi_{n-1}\left(v_e\right)$ for all $e\in B\setminus\{f\}$ are linearly independent. In other words, the family $\left(\pi_{n-1}\left(v_e\right)\right)_{e\in B\setminus\{f\}}$ is linearly independent. In other words, the family $\left(\pi_{|B\setminus\{f\}|}\left(v_e\right)\right)_{e\in B\setminus\{f\}}$ is linearly independent (since $|B\setminus\{f\}|=n-1$). In other words, $B\setminus\{f\}\in\mathcal{G}$ (by the definition of $\mathcal{G}$).

The $n\times n$-matrix $\left(X\mid Y_{\bullet,q}\right)$ is invertible (since $\det\left(X\mid Y_{\bullet,q}\right)\neq 0$); thus, its n columns are linearly independent. But these n columns are precisely the $n-1$ vectors $\pi_n\left(v_e\right)$ for all $e\in A$ as well as the extra vector $Y_{\bullet,q}=\pi_n\left(v_f\right)$. Thus, if we had $f\in A$, then these n columns would contain two equal vectors (namely, the extra vector $Y_{\bullet,q}=\pi_n\left(v_f\right)$ would be equal to one of the vectors $\pi_n\left(v_e\right)$ with $e\in A$), and thus would be linearly dependent. But this would contradict the fact that these n columns are linearly independent. Hence, we cannot have $f\in A$. Thus, $f\notin A$. Combining $f\in B$ with $f\notin A$, we obtain $f\in B\setminus A$. From $f\notin A$, we also obtain $|A\cup\{f\}|=|A|+1=n$ (since $|A|=n-1$).

Recall that the n columns of the $n\times n$-matrix $\left(X\mid Y_{\bullet,q}\right)$ are the $n-1$ vectors $\pi_n\left(v_e\right)$ for all $e\in A$ as well as the extra vector $Y_{\bullet,q}=\pi_n\left(v_f\right)$. In other words, they are the n vectors $\pi_n\left(v_e\right)$ for all $e\in A\cup\{f\}$ (since $f\notin A$). Hence, the n vectors $\pi_n\left(v_e\right)$ for all $e\in A\cup\{f\}$ are linearly independent (since the n columns of the $n\times n$-matrix $\left(X\mid Y_{\bullet,q}\right)$ are linearly independent). In other words, the family $\left(\pi_n\left(e\right)\right)_{e\in A\cup\{f\}}$ is linearly independent. In other words, the family $\left(\pi_{|A\cup\{f\}|}\left(e\right)\right)_{e\in A\cup\{f\}}$ is linearly independent (since $n=|A\cup\{f\}|$). In other words, $A\cup\{f\}\in\mathcal{G}$ (by the definition of $\mathcal{G}$).

We now know that $f\in B\setminus A$ and $A\cup\{f\}\in\mathcal{G}$ and $B\setminus\{f\}\in\mathcal{G}$. Hence, there exists $b\in B\setminus A$ such that $A\cup\{b\}\in\mathcal{G}$ and $B\setminus\{b\}\in\mathcal{G}$ (namely, $b=f$). This shows that the axiom **(iv)** from Definition 12.1 is satisfied for $\mathcal{F}=\mathcal{G}$.

*Proof of the axiom **(iii)** for $\mathcal{F} = \mathcal{G}$:* Clearly, axiom **(iii)** from Definition 12.1 follows from axiom **(iv)**. Thus, axiom **(iii)** holds for $\mathcal{F} = \mathcal{G}$ (since we already know that axiom **(iv)** holds for $\mathcal{F} = \mathcal{G}$).

We have now showed that the system $\mathcal{G}$ satisfies the four axioms **(i)**, **(ii)**, **(iii)** and **(iv)** from Definition 12.1. Thus, $\mathcal{G}$ is a strong greedoid. This proves Theorem 1.6. □

13. Appendix: Proof of Proposition 1.7

We shall now prove Proposition 1.7. This proof is mostly bookkeeping.

Proof of Proposition 1.7 (sketched). If $\mathcal{G}_k$ is empty, then the claim is obvious. Thus, we WLOG assume that $\mathcal{G}_k$ is nonempty.

We assumed that $\mathcal{G}$ is a Gaussian elimination greedoid. In other words, there exist a field $\mathbb{K}$ and a vector family $(v_e)_{e\in E}$ over $\mathbb{K}$ such that $\mathcal{G}$ is the Gaussian elimination greedoid of this vector family $(v_e)_{e\in E}$. Consider these $\mathbb{K}$ and $(v_e)_{e\in E}$. Let X be the $k \times |E|$-matrix whose columns are the vectors $\pi_k(v_e) \in \mathbb{K}^k$ for all $e \in E$ (in some order). Then, there is a bijection $\phi : \{1, 2, \ldots, |E|\} \to E$ such that the columns of the matrix X are $\pi_k\left(v_{\phi(1)}\right), \pi_k\left(v_{\phi(2)}\right), \ldots, \pi_k\left(v_{\phi(|E|)}\right)$ (in this order). Consider this ϕ. Hence,

$$(f\text{-th column of } X) = \pi_k\left(v_{\phi(f)}\right) \tag{49}$$

for each $f \in \{1, 2, \ldots, |E|\}$. Let $\mathcal{V}$ be the vector matroid[26] of the matrix X. This is a matroid on the set $\{1, 2, \ldots, |E|\}$; its rank is at most k (since X has k rows and therefore rank $\leqslant k$).

The bijection ϕ can be used to transport the matroid $\mathcal{V}$ onto the set E. More precisely, we can define a matroid $\mathcal{V}'$ on the set E by

$$\mathcal{V}' = \{\phi(I) \mid I \in \mathcal{V}\}$$

(where we regard a matroid as a collection of independent sets). The independent sets of this matroid $\mathcal{V}'$ are the subsets I of E for which $\phi^{-1}(I)$ is an independent set of $\mathcal{V}$. Moreover, the map ϕ is an isomorphism from the matroid $\mathcal{V}$ to the matroid $\mathcal{V}'$. Thus, the matroid $\mathcal{V}'$ is isomorphic to the vector matroid $\mathcal{V}$, and hence is representable. Our goal is now to show that $\mathcal{G}_k$ is the collection of bases of $\mathcal{V}$.

Let F be a k-element subset of E. Then, we have the following chain of logical

[26] See [Oxley11, §1.1] for the definition of a vector matroid.

equivalences:

$$\begin{aligned}
& (F \in \mathcal{G}_k) \\
\Longleftrightarrow\ & (F \in \mathcal{G}) \qquad \text{(since } \mathcal{G}_k \text{ is the set of all } k\text{-element sets in } \mathcal{G}) \\
\Longleftrightarrow\ & \left(\text{the family } \left(\pi_{|F|}(v_e)\right)_{e\in F} \in \left(\mathbb{K}^{|F|}\right)^F \text{ is linearly independent}\right) \\
& \left(\begin{array}{c}\text{by the definition of a Gaussian elimination greedoid,} \\ \text{since } \mathcal{G} \text{ is the Gaussian elimination greedoid of } (v_e)_{e\in E}\end{array}\right) \\
\Longleftrightarrow\ & \left(\text{the family } (\pi_k(v_e))_{e\in F} \in \left(\mathbb{K}^k\right)^F \text{ is linearly independent}\right) \\
& (\text{since } |F| = k \text{ (because } F \text{ is a } k\text{-element set))} \\
\Longleftrightarrow\ & \left(\text{the family } \left(\pi_k\left(v_{\phi(f)}\right)\right)_{f\in\phi^{-1}(F)} \in \left(\mathbb{K}^k\right)^{\phi^{-1}(F)} \text{ is linearly independent}\right) \\
& \left(\begin{array}{c}\text{since } \phi : \{1,2,\ldots,|E|\} \to E \text{ is a bijection, and thus the} \\ \text{family } \left(\pi_k\left(v_{\phi(f)}\right)\right)_{f\in\phi^{-1}(F)} \text{ is just a reindexing of the family } (\pi_k(v_e))_{e\in F}\end{array}\right) \\
\Longleftrightarrow\ & \left(\text{the family } (f\text{-th column of } X)_{f\in\phi^{-1}(F)} \in \left(\mathbb{K}^k\right)^{\phi^{-1}(F)} \text{ is linearly independent}\right) \\
& \left(\text{since (49) yields } (f\text{-th column of } X)_{f\in\phi^{-1}(F)} = \left(\pi_k\left(v_{\phi(f)}\right)\right)_{f\in\phi^{-1}(F)}\right) \\
\Longleftrightarrow\ & \left(\phi^{-1}(F) \text{ is an independent set of the vector matroid of } X\right) \\
& (\text{by the definition of a vector matroid}) \\
\Longleftrightarrow\ & \left(\phi^{-1}(F) \text{ is an independent set of } \mathcal{V}\right) \qquad (50) \\
& (\text{since } \mathcal{V} \text{ is the vector matroid of } X).
\end{aligned}$$

Forget that we fixed F. We thus have proved the equivalence (50) for each k-element subset F of E.

Recall that $\mathcal{G}_k$ is nonempty. In other words, there exists some $B \in \mathcal{G}_k$. Consider this B. Thus, B is a k-element set in $\mathcal{G}$ (since $\mathcal{G}_k$ is the set of all k-element sets in $\mathcal{G}$). Hence, the equivalence (50) (applied to $F = B$) yields the equivalence

$$(B \in \mathcal{G}_k) \iff \left(\phi^{-1}(B) \text{ is an independent set of } \mathcal{V}\right).$$

Thus, $\phi^{-1}(B)$ is an independent set of $\mathcal{V}$ (since $B \in \mathcal{G}_k$). Since ϕ is a bijection, we have $\left|\phi^{-1}(B)\right| = |B| = k$ (since B is a k-element set). In other words, $\phi^{-1}(B)$ is a k-element set. Thus, the matroid $\mathcal{V}$ has a k-element independent set (namely, $\phi^{-1}(B)$). Therefore, the rank of $\mathcal{V}$ is at least k. Since we also know that the rank of $\mathcal{V}$ is at most k, we thus conclude that the rank of $\mathcal{V}$ is exactly k. Hence, the rank of $\mathcal{V}'$ is exactly k as well (since the matroid $\mathcal{V}'$ is isomorphic to the matroid $\mathcal{V}$). Therefore, the bases of $\mathcal{V}'$ are the k-element independent sets of $\mathcal{V}'$. Thus, a k-element set is an independent set of $\mathcal{V}'$ if and only if it is a basis of $\mathcal{V}'$.

Now, let F again be a k-element subset of E. Thus, F is an independent set of $\mathcal{V}'$ if and only if F is a basis of $\mathcal{V}'$ (since a k-element set is an independent set of $\mathcal{V}'$ if and only if it is a basis of $\mathcal{V}'$). Also, $\phi^{-1}(F)$ is an independent set of $\mathcal{V}$ if and only if F is an independent set of $\mathcal{V}'$ (since the independent sets of the matroid $\mathcal{V}'$ are the subsets I of E for which $\phi^{-1}(I)$ is an independent set of $\mathcal{V}$).

Now, we have the following chain of logical equivalences:

$$\begin{aligned}
(F \in \mathcal{G}_k) &\iff \left(\phi^{-1}(F) \text{ is an independent set of } \mathcal{V}\right) \qquad \text{(by (50))} \\
&\iff \left(F \text{ is an independent set of } \mathcal{V}'\right) \\
&\qquad \left(\begin{array}{c}\text{since } \phi^{-1}(F) \text{ is an independent set of } \mathcal{V} \\ \text{if and only if } F \text{ is an independent set of } \mathcal{V}'\end{array}\right) \\
&\iff \left(F \text{ is a basis of } \mathcal{V}'\right)
\end{aligned}$$

(since F is an independent set of $\mathcal{V}'$ if and only if F is a basis of $\mathcal{V}'$).

Forget that we fixed F. We thus have proved that, for each k-element subset F of E, we have the equivalence

$$(F \in \mathcal{G}_k) \iff \left(F \text{ is a basis of } \mathcal{V}'\right).$$

Hence, $\mathcal{G}_k$ is the collection of all bases of $\mathcal{V}'$ (since all bases of $\mathcal{V}'$ are k-element subsets of E (because $\mathcal{V}'$ is a matroid of rank k)). Hence, $\mathcal{G}_k$ is the collection of bases of a representable matroid on the ground set E (since $\mathcal{V}'$ is a representable matroid on the ground set E). This proves Proposition 1.7. □

14. Appendix: Proofs of some properties of $\mathbb{L}$

We shall next prove four lemmas that were left unproven in Section 6: namely, Lemma 6.4, Lemma 6.8, Lemma 6.11 and Lemma 6.12.

Proof of Lemma 6.4. **(a)** Let $a \in \mathbb{L}$ be a nonzero element. We must prove that the element a belongs to $\mathbb{L}_+$ if and only if its order $\operatorname{ord} a$ is nonnegative (i.e., we have $\operatorname{ord} a \geqslant 0$). In other words, we must prove that $a \in \mathbb{L}_+$ if and only if $\operatorname{ord} a \geqslant 0$.

Recall that $\operatorname{ord} a$ is defined as the smallest $\beta \in \mathbb{V}$ such that $\left[t_\beta\right] a \neq 0$. Thus, $\operatorname{ord} a$ is a $\beta \in \mathbb{V}$ such that $\left[t_\beta\right] a \neq 0$. In other words, $\operatorname{ord} a \in \mathbb{V}$ and $\left[t_{\operatorname{ord} a}\right] a \neq 0$.

We shall now prove the implication

$$(a \in \mathbb{L}_+) \implies (\operatorname{ord} a \geqslant 0). \tag{51}$$

[*Proof of (51):* Assume that $a \in \mathbb{L}_+$. Then, $a \in \mathbb{L}_+ = \mathbb{K}\left[\mathbb{V}_{\geqslant 0}\right]$. In other words, a is a $\mathbb{K}$-linear combination of the family $(t_\alpha)_{\alpha \in \mathbb{V}_{\geqslant 0}}$. Thus, every $\alpha \in \mathbb{V}$ that satisfies $[t_\alpha]\, a \neq 0$ is an element of $\mathbb{V}_{\geqslant 0}$. Applying this to $\alpha = \operatorname{ord} a$, we conclude that $\operatorname{ord} a$ is an element of $\mathbb{V}_{\geqslant 0}$ (since $\left[t_{\operatorname{ord} a}\right] a \neq 0$). In other words, $\operatorname{ord} a \geqslant 0$. Thus, the implication (51) is proved.]

Next, let us prove the implication

$$(\operatorname{ord} a \geqslant 0) \implies (a \in \mathbb{L}_+). \tag{52}$$

[*Proof of (52):* Assume that $\operatorname{ord} a \geqslant 0$. Now, let $\alpha \in \mathbb{V}$ be such that $\alpha < 0$. We shall show that $[t_\alpha] a = 0$.

Indeed, assume the contrary. Thus, $[t_\alpha] a \neq 0$. But recall that $\operatorname{ord} a$ is the smallest $\beta \in \mathbb{V}$ such that $[t_\beta] a \neq 0$. Hence, for any $\beta \in \mathbb{V}$ satisfying $[t_\beta] a \neq 0$, we must have $\beta \geqslant \operatorname{ord} a$. Applying this to $\beta = \alpha$, we obtain $\alpha \geqslant \operatorname{ord} a \geqslant 0$. This contradicts $\alpha < 0$.

This contradiction shows that our assumption was wrong. Hence, we have proved that $[t_\alpha] a = 0$.

Forget that we fixed α. We thus have proved that

$$[t_\alpha] a = 0 \qquad \text{for each } \alpha \in \mathbb{V} \text{ satisfying } \alpha < 0. \tag{53}$$

But $a \in \mathbb{L} = \mathbb{K}[\mathbb{V}]$. Thus,

$$\begin{aligned} a &= \sum_{\alpha \in \mathbb{V}} ([t_\alpha] a) \cdot t_\alpha = \underbrace{\sum_{\substack{\alpha \in \mathbb{V}; \\ \alpha \geqslant 0}}}_{= \sum_{\alpha \in \mathbb{V}_{\geqslant 0}}} ([t_\alpha] a) \cdot t_\alpha + \sum_{\substack{\alpha \in \mathbb{V}; \\ \alpha < 0}} \underbrace{([t_\alpha] a)}_{\substack{=0 \\ \text{(by (53))}}} \cdot t_\alpha \\ &\qquad (\text{since each } \alpha \in \mathbb{V} \text{ satisfies either } \alpha \geqslant 0 \text{ or } \alpha < 0 \text{ (but not both)}) \\ &= \sum_{\alpha \in \mathbb{V}_{\geqslant 0}} ([t_\alpha] a) \cdot t_\alpha + \underbrace{\sum_{\substack{\alpha \in \mathbb{V}; \\ \alpha < 0}} 0 t_\alpha}_{=0} = \sum_{\alpha \in \mathbb{V}_{\geqslant 0}} ([t_\alpha] a) \cdot t_\alpha. \end{aligned}$$

Thus, a is a $\mathbb{K}$-linear combination of the family $(t_\alpha)_{\alpha \in \mathbb{V}_{\geqslant 0}}$. In other words, $a \in \mathbb{K}[\mathbb{V}_{\geqslant 0}]$. In other words, $a \in \mathbb{L}_+$ (since $\mathbb{L}_+ = \mathbb{K}[\mathbb{V}_{\geqslant 0}]$). This proves the implication (52).]

Combining the two implications (51) and (52), we obtain the equivalence

$$(a \in \mathbb{L}_+) \iff (\operatorname{ord} a \geqslant 0).$$

In other words, $a \in \mathbb{L}_+$ if and only if $\operatorname{ord} a \geqslant 0$. This proves Lemma 6.4 **(a)**.

(b) Let $a \in \mathbb{L}$ be nonzero. The definition of $\operatorname{ord} a$ shows that

$$\operatorname{ord} a = (\text{the smallest } \beta \in \mathbb{V} \text{ such that } [t_\beta] a \neq 0).$$

The same argument (applied to $-a$ instead of a) yields

$$\begin{aligned} \operatorname{ord}(-a) &= \left(\text{the smallest } \beta \in \mathbb{V} \text{ such that } \underbrace{[t_\beta](-a)}_{=-[t_\beta] a} \neq 0 \right) \\ &= (\text{the smallest } \beta \in \mathbb{V} \text{ such that } -[t_\beta] a \neq 0) \\ &= (\text{the smallest } \beta \in \mathbb{V} \text{ such that } [t_\beta] a \neq 0) \\ &\qquad (\text{since the condition ``} -[t_\beta] a \neq 0\text{'' is equivalent to ``} [t_\beta] a \neq 0\text{''}). \end{aligned}$$

Comparing these two equalities, we obtain $\operatorname{ord}(-a) = \operatorname{ord} a$. This proves Lemma 6.4 **(b)**.

(c) This is analogous to the standard fact that any two nonzero polynomials $f, g \in \mathbb{K}[X]$ satisfy $fg \neq 0$ and $\deg(fg) = \deg f + \deg g$. For the sake of completeness, let us nevertheless present the proof.

Recall that $\operatorname{ord} a$ is defined as the smallest $\beta \in \mathbb{V}$ such that $[t_\beta]\, a \neq 0$. Thus, $\operatorname{ord} a$ is a $\beta \in \mathbb{V}$ such that $[t_\beta]\, a \neq 0$. In other words, $\operatorname{ord} a \in \mathbb{V}$ and $[t_{\operatorname{ord} a}]\, a \neq 0$. The same argument (applied to b instead of a) yields $\operatorname{ord} b \in \mathbb{V}$ and $[t_{\operatorname{ord} b}]\, b \neq 0$. From $[t_{\operatorname{ord} a}]\, a \neq 0$ and $[t_{\operatorname{ord} b}]\, b \neq 0$, we obtain $([t_{\operatorname{ord} a}]\, a) \cdot ([t_{\operatorname{ord} b}]\, b) \neq 0$ (since $[t_{\operatorname{ord} a}]\, a$ and $[t_{\operatorname{ord} b}]\, b$ belong to the integral domain $\mathbb{K}$).

Moreover,

$$[t_\beta]\, a = 0 \qquad \text{for each } \beta \in \mathbb{V} \text{ satisfying } \beta < \operatorname{ord} a \tag{54}$$

(since $\operatorname{ord} a$ is the **smallest** $\beta \in \mathbb{V}$ such that $[t_\beta]\, a \neq 0$). Similarly,

$$[t_\beta]\, b = 0 \qquad \text{for each } \beta \in \mathbb{V} \text{ satisfying } \beta < \operatorname{ord} b. \tag{55}$$

For any $\alpha \in \mathbb{V}$, we have

$$\begin{aligned}
[t_\alpha]\,(ab) &= \sum_{\beta \in \mathbb{V}} \left([t_\beta]\, a\right) \cdot \left([t_{\alpha-\beta}]\, b\right) \qquad \left(\begin{array}{c}\text{by the definition of the multiplication}\\ \text{in the group algebra } \mathbb{L} = \mathbb{K}[\mathbb{V}]\end{array}\right)\\
&= \sum_{\substack{\beta \in \mathbb{V};\\ \beta < \operatorname{ord} a}} \underbrace{\left([t_\beta]\, a\right)}_{\substack{=0\\ \text{(by (54))}}} \cdot \left([t_{\alpha-\beta}]\, b\right) + \sum_{\substack{\beta \in \mathbb{V};\\ \beta \geqslant \operatorname{ord} a}} \left([t_\beta]\, a\right) \cdot \left([t_{\alpha-\beta}]\, b\right)\\
&\qquad \left(\text{since each } \beta \in \mathbb{V} \text{ satisfies either } \beta < \operatorname{ord} a \text{ or } \beta \geqslant \operatorname{ord} a \text{ (but not both)}\right)\\
&= \underbrace{\sum_{\substack{\beta \in \mathbb{V};\\ \beta < \operatorname{ord} a}} 0 \cdot \left([t_{\alpha-\beta}]\, b\right)}_{=0} + \sum_{\substack{\beta \in \mathbb{V};\\ \beta \geqslant \operatorname{ord} a}} \left([t_\beta]\, a\right) \cdot \left([t_{\alpha-\beta}]\, b\right)\\
&= \sum_{\substack{\beta \in \mathbb{V};\\ \beta \geqslant \operatorname{ord} a}} \left([t_\beta]\, a\right) \cdot \left([t_{\alpha-\beta}]\, b\right).
\end{aligned} \tag{56}$$

Now, let $\alpha \in \mathbb{V}$ be such that $\alpha < \operatorname{ord} a + \operatorname{ord} b$. Then, for every $\beta \in \mathbb{V}$ satisfying $\beta \geqslant \operatorname{ord} a$, we have

$$\underbrace{\alpha}_{< \operatorname{ord} a + \operatorname{ord} b} - \underbrace{\beta}_{\geqslant \operatorname{ord} a} < (\operatorname{ord} a + \operatorname{ord} b) - \operatorname{ord} a = \operatorname{ord} b$$

and thus

$$[t_{\alpha-\beta}]\, b = 0 \tag{57}$$

(by (55), applied to $\alpha - \beta$ instead of β). Hence, (56) becomes

$$[t_\alpha]\,(ab) = \sum_{\substack{\beta \in \mathbb{V};\\ \beta \geqslant \operatorname{ord} a}} \left([t_\beta]\, a\right) \cdot \underbrace{\left([t_{\alpha-\beta}]\, b\right)}_{\substack{=0\\ \text{(by (57))}}} = \sum_{\substack{\beta \in \mathbb{V};\\ \beta \geqslant \operatorname{ord} a}} \left([t_\beta]\, a\right) \cdot 0 = 0.$$

Forget that we fixed α. We thus have proved that

$$[t_\alpha](ab) = 0 \qquad \text{for any } \alpha \in \mathbb{V} \text{ satisfying } \alpha < \operatorname{ord} a + \operatorname{ord} b. \tag{58}$$

On the other hand, for every $\beta \in \mathbb{V}$ satisfying $\beta > \operatorname{ord} a$, we have

$$(\operatorname{ord} a + \operatorname{ord} b) - \underbrace{\beta}_{>\operatorname{ord} a} < (\operatorname{ord} a + \operatorname{ord} b) - \operatorname{ord} a = \operatorname{ord} b$$

and thus

$$\left[t_{(\operatorname{ord} a + \operatorname{ord} b) - \beta}\right] b = 0 \tag{59}$$

(by (55), applied to $(\operatorname{ord} a + \operatorname{ord} b) - \beta$ instead of β). Hence, (56) (applied to $\alpha = \operatorname{ord} a + \operatorname{ord} b$) yields

$$\begin{aligned}
&[t_{\operatorname{ord} a + \operatorname{ord} b}](ab) \\
&= \sum_{\substack{\beta \in \mathbb{V};\\ \beta \geqslant \operatorname{ord} a}} ([t_\beta] a) \cdot \left(\left[t_{(\operatorname{ord} a + \operatorname{ord} b) - \beta}\right] b\right) \\
&= ([t_{\operatorname{ord} a}] a) \cdot \underbrace{\left(\left[t_{(\operatorname{ord} a + \operatorname{ord} b) - \operatorname{ord} a}\right] b\right)}_{\substack{=[t_{\operatorname{ord} b}]b \\ (\text{since } (\operatorname{ord} a + \operatorname{ord} b) - \operatorname{ord} a = \operatorname{ord} b)}} + \sum_{\substack{\beta \in \mathbb{V};\\ \beta > \operatorname{ord} a}} ([t_\beta] a) \cdot \underbrace{\left(\left[t_{(\operatorname{ord} a + \operatorname{ord} b) - \beta}\right] b\right)}_{\substack{=0 \\ (\text{by } (59))}} \\
&\qquad (\text{here, we have split off the addend for } \beta = \operatorname{ord} a \text{ from the sum}) \\
&= ([t_{\operatorname{ord} a}] a) \cdot ([t_{\operatorname{ord} b}] b) + \underbrace{\sum_{\substack{\beta \in \mathbb{V};\\ \beta > \operatorname{ord} a}} ([t_\beta] a) \cdot 0}_{=0} = ([t_{\operatorname{ord} a}] a) \cdot ([t_{\operatorname{ord} b}] b) \neq 0.
\end{aligned}$$

Thus, $\operatorname{ord} a + \operatorname{ord} b$ is a $\beta \in \mathbb{V}$ such that $[t_\beta](ab) \neq 0$. In view of (58), we conclude that $\operatorname{ord} a + \operatorname{ord} b$ is the **smallest** such β.

From $[t_{\operatorname{ord} a + \operatorname{ord} b}](ab) \neq 0$, we obtain $ab \neq 0$. Thus, ab is a nonzero element of $\mathbb{L}$. Hence, $\operatorname{ord}(ab)$ is defined as the smallest $\beta \in \mathbb{V}$ such that $[t_\beta](ab) \neq 0$. But we already know that $\operatorname{ord} a + \operatorname{ord} b$ is the smallest such β. Comparing these two results, we conclude that $\operatorname{ord}(ab) = \operatorname{ord} a + \operatorname{ord} b$. This proves Lemma 6.4 **(c)**.

(d) Assume the contrary. Thus, $\operatorname{ord}(a+b) < \min\{\operatorname{ord} a, \operatorname{ord} b\} \leqslant \operatorname{ord} a$. Similarly, $\operatorname{ord}(a+b) < \operatorname{ord} b$.

Recall that $\operatorname{ord} a$ is defined as the smallest $\beta \in \mathbb{V}$ such that $[t_\beta] a \neq 0$. Hence, in particular, no such β is smaller than $\operatorname{ord} a$. In other words,

$$[t_\beta] a = 0 \qquad \text{for each } \beta \in \mathbb{V} \text{ satisfying } \beta < \operatorname{ord} a.$$

Applying this to $\beta = \operatorname{ord}(a+b)$, we find $\left[t_{\operatorname{ord}(a+b)}\right] a = 0$. Similarly, $\left[t_{\operatorname{ord}(a+b)}\right] b = 0$.

But $\operatorname{ord}(a+b)$ is defined as the smallest $\beta \in \mathbb{V}$ such that $[t_\beta](a+b) \neq 0$. Thus, $\operatorname{ord}(a+b)$ is a $\beta \in \mathbb{V}$ such that $[t_\beta](a+b) \neq 0$. In other words, $\operatorname{ord}(a+b) \in \mathbb{V}$ and $\left[t_{\operatorname{ord}(a+b)}\right](a+b) \neq 0$. But $\left[t_{\operatorname{ord}(a+b)}\right](a+b) \neq 0$ contradicts

$$\left[t_{\operatorname{ord}(a+b)}\right](a+b) = \underbrace{\left[t_{\operatorname{ord}(a+b)}\right]a}_{=0} + \underbrace{\left[t_{\operatorname{ord}(a+b)}\right]b}_{=0} = 0+0 = 0.$$

This contradiction shows that our assumption was wrong. This proves Lemma 6.4 **(d)**. □

Proof of Lemma 6.8. First, let us observe that the distance function $d : E \underline{\times} E \to \mathbb{V}$ in Definition 6.7 is well-defined.

[*Proof:* We must show that $-\operatorname{ord}(a-b) \in \mathbb{V}$ is well-defined for each $(a,b) \in E \underline{\times} E$.

So let us fix $(a,b) \in E \underline{\times} E$. Thus, a and b are two distinct elements of E (by the definition of $E \underline{\times} E$). Since a and b are distinct, we have $a-b \neq 0$. Hence, the element $a-b$ of $\mathbb{L}$ is nonzero, and therefore $\operatorname{ord}(a-b) \in \mathbb{V}$ is well-defined. Thus, $-\operatorname{ord}(a-b) \in \mathbb{V}$ is well-defined.

Forget that we fixed (a,b). We thus have showed that $-\operatorname{ord}(a-b) \in \mathbb{V}$ is well-defined for each $(a,b) \in E \underline{\times} E$. This completes our proof.]

Now, consider the distance function $d : E \underline{\times} E \to \mathbb{V}$ in Definition 6.7. Our goal is to prove that (E,w,d) is a $\mathbb{V}$-ultra triple whenever $w : E \to \mathbb{V}$ is a function. According to the definition of a "$\mathbb{V}$-ultra triple", this boils down to proving the following two statements:

> *Statement 1:* We have $d(a,b) = d(b,a)$ for any two distinct elements a and b of E.
>
> *Statement 2:* We have $d(a,b) \leqslant \max\{d(a,c), d(b,c)\}$ for any three distinct elements a, b and c of E.

Let us prove these two statements.

[*Proof of Statement 1:* Let a and b be two distinct elements of E. Thus, $a-b \neq 0$ (since a and b are distinct), so that $a-b$ is a nonzero element of $\mathbb{L}$. Hence, Lemma 6.4 **(b)** (applied to $a-b$ instead of a) yields $\operatorname{ord}(-(a-b)) = \operatorname{ord}(a-b)$. But the definition of d yields $d(a,b) = \operatorname{ord}(a-b)$ and $d(b,a) = \operatorname{ord}\left(\underbrace{b-a}_{=-(a-b)}\right) = \operatorname{ord}(-(a-b)) = \operatorname{ord}(a-b)$. Comparing these two equalities, we find $d(a,b) = d(b,a)$. This proves Statement 1.]

[*Proof of Statement 2:* Let a, b and c be three distinct elements of E. Then, Statement 1 (applied to c instead of a) yields $d(c,b) = d(b,c)$.

We have $a-b \neq 0$ (since a and b are distinct), so that $a-b$ is a nonzero element of $\mathbb{L}$. Likewise, $a-c$ and $c-b$ are nonzero elements of $\mathbb{L}$. Now, the nonzero elements $a-c$ and $c-b$ of $\mathbb{L}$ have the property that their sum $(a-c)+(c-b)$ is nonzero (since $(a-c)+(c-b) = a-b$ is nonzero). Thus, Lemma 6.4 **(d)** (applied to $a-c$ and $c-b$ instead of a and b) yields $\operatorname{ord}((a-c)+(c-b)) \geqslant \min\{\operatorname{ord}(a-c), \operatorname{ord}(c-b)\}$. In view of $(a-c)+(c-b) = a-b$, this rewrites as

$$\operatorname{ord}(a-b) \geqslant \min\{\operatorname{ord}(a-c), \operatorname{ord}(c-b)\}.$$

Hence,

$$\begin{aligned} -\underbrace{\operatorname{ord}(a-b)}_{\geqslant \min\{\operatorname{ord}(a-c),\operatorname{ord}(c-b)\}} &\leqslant -\min\{\operatorname{ord}(a-c), \operatorname{ord}(c-b)\} \\ &= \max\{-\operatorname{ord}(a-c), -\operatorname{ord}(c-b)\}. \end{aligned} \tag{60}$$

But the definition of d yields

$$\begin{aligned} d(a,b) &= -\operatorname{ord}(a-b) && \text{and} \\ d(a,c) &= -\operatorname{ord}(a-c) && \text{and} \\ d(c,b) &= -\operatorname{ord}(c-b). \end{aligned}$$

In view of these equalities, we can rewrite (60) as $d(a,b) \leqslant \max\{d(a,c), d(c,b)\}$. In view of $d(c,b) = d(b,c)$, this further rewrites as $d(a,b) \leqslant \max\{d(a,c), d(b,c)\}$. This proves Statement 2.]

We thus have proved both Statements 1 and 2. Therefore, (E, w, d) is a $\mathbb{V}$-ultra triple whenever $w : E \to \mathbb{V}$ is a function (by the definition of a "$\mathbb{V}$-ultra triple"). This proves Lemma 6.8. □

Proof of Lemma 6.11. The unity of the $\mathbb{K}$-algebra $\mathbb{L}_+ = \mathbb{K}[\mathbb{V}_{\geqslant 0}]$ is $1_{\mathbb{L}_+} = t_0$. Hence, the map π sends this unity to $\pi(1_{\mathbb{L}_+}) = \pi(t_0) = [t_0]\, t_0 = 1$. Also, the map π is clearly $\mathbb{K}$-linear (since any $x, y \in \mathbb{L}_+$ satisfy $[t_0](x+y) = ([t_0]\, x) + ([t_0]\, y)$, and since any $x \in \mathbb{L}_+$ and $\lambda \in \mathbb{K}$ satisfy $[t_0](\lambda x) = \lambda [t_0]\, x$).

Let $x, y \in \mathbb{L}_+$. We shall prove that $\pi(xy) = \pi(x) \cdot \pi(y)$.

We have $x \in \mathbb{L}_+ = \mathbb{K}[\mathbb{V}_{\geqslant 0}]$. In other words, x is a $\mathbb{K}$-linear combination of the family $(t_\alpha)_{\alpha \in \mathbb{V}_{\geqslant 0}}$. In other words, we can write x in the form $x = \sum_{\alpha \in \mathbb{V}_{\geqslant 0}} \lambda_\alpha t_\alpha$ for some family $(\lambda_\alpha)_{\alpha \in \mathbb{V}_{\geqslant 0}} \in \mathbb{K}^{\mathbb{V}_{\geqslant 0}}$ of coefficients λ_α (such that all but finitely many $\alpha \in \mathbb{V}_{\geqslant 0}$ satisfy $\lambda_\alpha = 0$). Consider this family $(\lambda_\alpha)_{\alpha \in \mathbb{V}_{\geqslant 0}}$. From $x = \sum_{\alpha \in \mathbb{V}_{\geqslant 0}} \lambda_\alpha t_\alpha$, we obtain $[t_0]\, x = \lambda_0$. Thus, the definition of π yields

$$\pi(x) = [t_0]\, x = \lambda_0. \tag{61}$$

We have $y \in \mathbb{L}_+ = \mathbb{K}[\mathbb{V}_{\geqslant 0}]$. In other words, y is a $\mathbb{K}$-linear combination of the family $(t_\alpha)_{\alpha \in \mathbb{V}_{\geqslant 0}} = (t_\beta)_{\beta \in \mathbb{V}_{\geqslant 0}}$. In other words, we can write y in the form $y = \sum_{\beta \in \mathbb{V}_{\geqslant 0}} \mu_\beta t_\beta$ for some family $(\mu_\beta)_{\beta \in \mathbb{V}_{\geqslant 0}} \in \mathbb{K}^{\mathbb{V}_{\geqslant 0}}$ of coefficients μ_β (such that all but finitely many $\beta \in \mathbb{V}_{\geqslant 0}$ satisfy $\mu_\beta = 0$). Consider this family $\left(\mu_\beta\right)_{\beta \in \mathbb{V}_{\geqslant 0}}$. From $y = \sum\limits_{\beta \in \mathbb{V}_{\geqslant 0}} \mu_\beta t_\beta$, we obtain $[t_0]\, y = \mu_0$. Thus, the definition of π yields

$$\pi\left(y\right) = [t_0]\, y = \mu_0. \tag{62}$$

For any pair $(\alpha, \beta) \in \mathbb{V}_{\geqslant 0} \times \mathbb{V}_{\geqslant 0}$ satisfying $(\alpha, \beta) \neq (0, 0)$, we have

$$[t_0]\left(t_{\alpha+\beta}\right) = 0 \tag{63}$$

[27].

Now, multiplying the equalities $x = \sum\limits_{\alpha \in \mathbb{V}_{\geqslant 0}} \lambda_\alpha t_\alpha$ and $y = \sum\limits_{\beta \in \mathbb{V}_{\geqslant 0}} \mu_\beta t_\beta$, we obtain

$$\begin{aligned} xy &= \left(\sum_{\alpha \in \mathbb{V}_{\geqslant 0}} \lambda_\alpha t_\alpha\right)\left(\sum_{\beta \in \mathbb{V}_{\geqslant 0}} \mu_\beta t_\beta\right) = \underbrace{\sum_{\alpha \in \mathbb{V}_{\geqslant 0}} \sum_{\beta \in \mathbb{V}_{\geqslant 0}}}_{= \sum_{(\alpha,\beta) \in \mathbb{V}_{\geqslant 0} \times \mathbb{V}_{\geqslant 0}}} \lambda_\alpha \mu_\beta \underbrace{t_\alpha t_\beta}_{= t_{\alpha+\beta}} \\ &= \sum_{(\alpha,\beta) \in \mathbb{V}_{\geqslant 0} \times \mathbb{V}_{\geqslant 0}} \lambda_\alpha \mu_\beta t_{\alpha+\beta}. \end{aligned}$$

Applying the map π to both sides of this equality, we obtain

$$\begin{aligned} \pi\left(xy\right) &= \pi\left(\sum_{(\alpha,\beta) \in \mathbb{V}_{\geqslant 0} \times \mathbb{V}_{\geqslant 0}} \lambda_\alpha \mu_\beta t_{\alpha+\beta}\right) = [t_0]\left(\sum_{(\alpha,\beta) \in \mathbb{V}_{\geqslant 0} \times \mathbb{V}_{\geqslant 0}} \lambda_\alpha \mu_\beta t_{\alpha+\beta}\right) \\ &\qquad \left(\text{by the definition of } \pi\right) \\ &= \sum_{(\alpha,\beta) \in \mathbb{V}_{\geqslant 0} \times \mathbb{V}_{\geqslant 0}} \lambda_\alpha \mu_\beta \,[t_0]\left(t_{\alpha+\beta}\right) \\ &= \underbrace{\lambda_0}_{\substack{= \pi(x) \\ \text{(by (61))}}} \underbrace{\mu_0}_{\substack{= \pi(y) \\ \text{(by (62))}}} \underbrace{[t_0]\left(t_{0+0}\right)}_{\substack{= [t_0](t_0) \\ = 1}} + \sum_{\substack{(\alpha,\beta) \in \mathbb{V}_{\geqslant 0} \times \mathbb{V}_{\geqslant 0}; \\ (\alpha,\beta) \neq (0,0)}} \lambda_\alpha \mu_\beta \underbrace{[t_0]\left(t_{\alpha+\beta}\right)}_{\substack{= 0 \\ \text{(by (63))}}} \\ &\qquad \left(\text{here, we have split off the addend for } (\alpha,\beta) = (0,0) \text{ from the sum}\right) \\ &= \pi\left(x\right)\pi\left(y\right) + \underbrace{\sum_{\substack{(\alpha,\beta) \in \mathbb{V}_{\geqslant 0} \times \mathbb{V}_{\geqslant 0}; \\ (\alpha,\beta) \neq (0,0)}} \lambda_\alpha \mu_\beta 0}_{= 0} = \pi\left(x\right)\pi\left(y\right). \end{aligned}$$

[27] *Proof of (63):* Let $(\alpha, \beta) \in \mathbb{V}_{\geqslant 0} \times \mathbb{V}_{\geqslant 0}$ be a pair satisfying $(\alpha, \beta) \neq (0, 0)$. From $(\alpha, \beta) \in \mathbb{V}_{\geqslant 0} \times \mathbb{V}_{\geqslant 0}$, we obtain $\alpha \in \mathbb{V}_{\geqslant 0}$. In other words, $\alpha \geqslant 0$ (by the definition of $\mathbb{V}_{\geqslant 0}$). Similarly, $\beta \geqslant 0$. At least one of the two elements α and β is $\neq 0$ (since $(\alpha, \beta) \neq (0, 0)$). In other words, we have $\alpha \neq 0$ or $\beta \neq 0$. Since α and β play symmetric roles in our setting, we can WLOG assume that $\alpha \neq 0$ (since otherwise, we can achieve this by swapping α with β). Combining $\alpha \neq 0$ with $\alpha \geqslant 0$, we obtain $\alpha > 0$. Adding this inequality to $\beta \geqslant 0$, we obtain $\alpha + \beta > 0 + 0 = 0$. Hence, $\alpha + \beta \neq 0$, so that $[t_0]\left(t_{\alpha+\beta}\right) = 0$. This proves (63).

Forget that we fixed x and y. We thus have showed that $\pi(xy) = \pi(x) \cdot \pi(y)$ for all $x, y \in \mathbb{L}_+$. Hence, π is a $\mathbb{K}$-algebra homomorphism (since π is $\mathbb{K}$-linear and $\pi(1_{\mathbb{L}_+}) = 1$). This proves Lemma 6.11. □

Proof of Lemma 6.12. Lemma 6.4 **(a)** shows that a belongs to $\mathbb{L}_+$ if and only if its order $\operatorname{ord} a$ is nonnegative. Hence, $\operatorname{ord} a$ is nonnegative (since a belongs to $\mathbb{L}_+$). In other words, $\operatorname{ord} a \geqslant 0$.

Recall that $\operatorname{ord} a$ is defined as the smallest $\beta \in \mathbb{V}$ such that $[t_\beta] a \neq 0$. Thus, $\operatorname{ord} a$ is a $\beta \in \mathbb{V}$ such that $[t_\beta] a \neq 0$. In other words, $\operatorname{ord} a \in \mathbb{V}$ and $[t_{\operatorname{ord} a}] a \neq 0$.

We shall first prove the implication

$$(\operatorname{ord} a = 0) \implies (\pi(a) \neq 0). \tag{64}$$

[*Proof of (64):* Assume that $\operatorname{ord} a = 0$. Then, $0 = \operatorname{ord} a$, so that $[t_0] a = [t_{\operatorname{ord} a}] a \neq 0$. But the definition of π yields $\pi(a) = [t_0] a \neq 0$. Thus, the implication (64) is proved.]

Next, let us prove the implication

$$(\pi(a) \neq 0) \implies (\operatorname{ord} a = 0). \tag{65}$$

[*Proof of (65):* Assume that $\pi(a) \neq 0$. But the definition of π yields $\pi(a) = [t_0] a$. Hence, $[t_0] a = \pi(a) \neq 0$.

Now, recall that $\operatorname{ord} a$ is the **smallest** $\beta \in \mathbb{V}$ such that $[t_\beta] a \neq 0$. Hence, if $\beta \in \mathbb{V}$ satisfies $[t_\beta] a \neq 0$, then $\beta \geqslant \operatorname{ord} a$. Applying this to $\beta = 0$, we obtain $0 \geqslant \operatorname{ord} a$ (since $[t_0] a \neq 0$). Combining this with $\operatorname{ord} a \geqslant 0$, we find $\operatorname{ord} a = 0$. This proves the implication (65).]

Combining the two implications (65) and (64), we obtain the logical equivalence $(\pi(a) \neq 0) \iff (\operatorname{ord} a = 0)$. In other words, $\pi(a) \neq 0$ holds if and only if $\operatorname{ord} a = 0$. This proves Lemma 6.12. □

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