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The q-deformed random-to-random family in the Hecke algebra

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Abstract. We generalize Reiner–Saliola–Welker's well-known but mysterious family of *k-random-to-random shuffles* from Markov chains on symmetric groups to Markov chains on the Type-A Iwahori–Hecke algebras. We prove that the family of operators pairwise commutes and has eigenvalues that are polynomials in q with non-negative integer coefficients. Our work generalizes work of Reiner–Saliola–Welker and Lafrenière for the symmetric group, and simplifies all known proofs in this case.

Keywords: Iwahori–Hecke algebra, symmetric group algebra, random-to-random shuffle, card shuffling, Young–Jucys–Murphy elements, discrete Markov chains

1 Introduction

In this extended abstract, we study and generalize a well-known family of Markov chains that emerge at the intersection of probability theory, algebraic combinatorics, and representation theory. This family, called the *k-random-to-random shuffles* $\mathcal{R}_{n,k}(1)$, models a method of shuffling cards, wherein one removes k cards (uniformly) at random from a deck of n cards, and then reinserts them in new (uniform) random positions. The case that $k = 1$, also known as *random-to-random shuffling*, was introduced in the early 1990s by Diaconis and Saloff-Coste; the general family was defined by Reiner–Saliola–Welker [16].

It is the spectral properties of the k-random-to-random shuffles that have made them an enduring figure in combinatorics, and the subject of three FPSAC talks (2009, 2019, 2025). In particular, the $\mathcal{R}_{n,k}(1)$ were shown to pairwise commute in [16], and conjectured in [17, $k = 1$] and [16, $k > 1$] to have eigenvalues in $\mathbb{Z}_{\geq 0}$ (after suitable normalization). These conjectures were affirmed in work of Dieker–Saliola [8, $k = 1$] and Lafrenière

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[11, $k > 1$]. Yet many mysteries endured: for example, Lafrenière conjectured a formula for the second-largest eigenvalue of $\mathcal{R}_{n,k}(1)$, and the proofs of both commutativity and spectral formulas were highly technical and specific to the symmetric group.

In this extended abstract, we define a deformation of the k-random-to-random shuffles to the *Type A Iwahori-Hecke algebra* $\mathcal{H}_n := \mathcal{H}_n(q)$, where they induce Markov chains $\mathcal{R}_{n,k}$ on $\mathcal{H}_n$. This deformation is motivated in part by recent interest in Markov chains on Hecke algebras; for example, Bufetov showed that many interacting particle systems arising in statistical mechanics can be understood as random walks on $\mathcal{H}_n$ [7]. His work implies that questions of convergence of such Markov chains are well-posed; from this perspective, it is of significant interest to understand the eigenvalues of these processes. In this setting, if q is real and ≥ 1, we can think of q^{-1} as a probability.

On the algebra-combinatorial side, our study of the family $\{\mathcal{R}_{n,k}\}$ will uncover new properties of the Hecke algebra, while also shedding light on the original mysterious family $\{\mathcal{R}_{n,k}(1)\}$ in $\mathbb{R}[\mathfrak{S}_n]$. In particular, we generalize all known results about the $\mathcal{R}_{n,k}(1)$ in [8, 11, 16, 17] to $\mathcal{R}_{n,k}$, and resolve (and lift to positive q) Lafrenière's conjectural second-largest eigenvalue formula. Our work provides simpler proofs than all previously published proofs of the $q = 1$ case. The main results are summarized below:

Theorem 1. *Let $\mathcal{R}_{n,k}$ be the k-random-to-random shuffle in $\mathcal{H}_n := \mathcal{H}_n(q)$ (defined in (4.2)).*

1. *For a fixed n, the $\mathcal{R}_{n,k}$ pairwise commute.*
2. *The eigenvalues of $\mathcal{R}_{n,k}$ are in $\mathbb{Z}_{\geq 0}[q]$.*
3. *When $q \in \mathbb{R}_{>0}$, the $\mathcal{R}_{n,k}$ are diagonalizable, and the second largest eigenvalue is*

$$[k]!_q \begin{bmatrix} n-2 \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q .$$

This is an extended abstract of the paper [3], and proceeds as follows. In Section 2 we define the $\mathcal{R}_{n,k}(1)$ as Markov chains on the symmetric group. We then define the Hecke algebra $\mathcal{H}_n$ in Section 3, and the shuffles $\mathcal{R}_{n,k}$ in Section 4. Our main results are elaborated on in Section 5, and Section 6 discusses the primary techniques used.

2 k-random-to-random in the symmetric group

Our story begins with one of the most notorious Markov chains on the symmetric group, called *random-to-bottom*, given by right multiplication in $\mathbb{Z}[\mathfrak{S}_n]$ by the element

$$\mathcal{B}^*_n(1) = \sum_{i=1}^{n} (i, i+1, \ldots, n) \qquad \text{(in cycle notation)} .$$

In probabilistic language, if we view permutations in one-line notation as decks of n cards labeled $1, 2, \ldots, n$, then $\mathcal{B}_n^*(1)$ corresponds to picking a card at random and moving it to the bottom of the deck, where the bottom card of the deck is the n-th position.

The *random-to-random shuffling element* is obtained by composing this element $\mathcal{B}_n^*(1)$ with the *bottom-to-random shuffling element* $\mathcal{B}_n(1)$:

$$\mathcal{R}_n(1) = \mathcal{B}_n^*(1)\,\mathcal{B}_n(1) \qquad \text{and} \qquad \mathcal{B}_n(1) = \sum_{i=1}^{n} (n, n-1, \ldots, i+1, i)\,.$$

Reiner, Saliola and Welker introduced in their monograph [16] a family of elements they called *k-random-to-random shuffling elements* $\mathcal{R}_{n,k}(1) \in \mathbb{Z}[\mathfrak{S}_n]$. This family included $\mathcal{R}_n(1) =: \mathcal{R}_{n,1}(1)$, and each element $\mathcal{R}_{n,k}(1)$ can be understood as a (rescaled) shuffling process as follows. Again, associate a deck of n cards with a word w of length n with letters $1, 2, \ldots, n$. Right multiplication by $\mathcal{R}_{n,k}(1)$ acts on w by randomly selecting k distinct letters $w_{i_1}, \ldots, w_{i_k}$ and placing them in new random positions. More precisely,

$$\mathcal{R}_{n,k}(1) = \tfrac{1}{k!}\,\mathcal{B}_{n,k}^*(1)\,\mathcal{B}_{n,k}(1) \in \mathbb{Z}\left[\mathfrak{S}_n\right],$$

where the actions of $\mathcal{B}_{n,k}^*(1)$ and $\mathcal{B}_{n,k}(1)$ are defined as follows:

1. $\mathcal{B}_{n,k}^*(1)$ selects k letters from a word w and moves them to the end of the word;
2. $\mathcal{B}_{n,k}(1)$ moves the last k letters in a word u to new, uniformly random positions.

Example 2. Consider $n = 4$ and $k = 2$. Using one-line notation, $\mathcal{B}_{4,2}^*(1)$ is

$$\begin{aligned} 1234 \cdot \mathcal{B}_{4,2}^*(1) = {} & 3412 + 3421 + 2413 + 2431 + 2314 + 2341 \\ & + 1423 + 1432 + 1324 + 1342 + 1234 + 1243, \end{aligned}$$

which is the sum of all permutations that can be obtained from 1234 by moving two letters to the end of the word; the red color indicates the moved letters. Similarly,

$$\begin{aligned} 1234 \cdot \mathcal{B}_{4,2}(1) = {} & 1234 + 1243 + 1324 + 1342 + 1423 + 1432 \\ & + 3124 + 3142 + 3412 + 4123 + 4132 + 4312 \end{aligned}$$

is the sum of all permutations obtained from 1234 by moving its last two letters.

Suppose we are interested in the coefficient of 2314 under the image of $\mathcal{R}_{4,2}(1)$ acting on 1234. There are eight ways (shown below) to obtain 2314 by first moving two letters from 1234 to the end of the word, and then moving those letters to new positions:

$$\begin{array}{lll} 1234 \to 2314 \to 2314 & \qquad & 1234 \to 2341 \to 2314 \\ 1234 \to 2413 \to 2314 & & 1234 \to 2431 \to 2314 \\ 1234 \to 3412 \to 2314 & & 1234 \to 3421 \to 2314 \\ 1234 \to 1423 \to 2314 & & 1234 \to 1432 \to 2314 \end{array}$$

Hence the coefficient of 2314 in $1234 \cdot \mathcal{R}_{4,2}(1)$ is $8/2! = 4$.

As a probabilist, one is interested in the spectra of the $\mathcal{R}_{n,k}(1)$—but this question is of interest to combinatorialists because of the following remarkable properties.

Theorem 3. *Let $0 \leq k \leq n$.*

1. *(Dieker–Saliola [8, $k = 1$], Lafrenière [11, $k > 1$]) The eigenvalues of $\mathcal{R}_{n,k}(1)$ are in $\mathbb{Z}_{\geq 0}$.*
2. *(Reiner–Saliola–Welker [16], Lafrenière [11]) For a fixed n, the $\mathcal{R}_{n,k}(1)$ pairwise commute.*

Theorem 3 (1) was conjectured as early as 2002 by Uyemura-Reyes for $k = 1$ [17]; this conjecture was extended in 2014 [16] by Reiner–Saliola–Welker. Its subsequent proof used the representation theory of the symmetric group.

In fact, the $\mathcal{B}_{n,k}(1)$ are examples of a more general family of shuffles introduced by Bidigare, Hanlon and Rockmore in their seminal work [2], which studies the theory of random walks on hyperplane arrangements. Such random walks are thus known as *BHR random walks*. This study was developed further by Brown and Diaconis [5, 6].

The $\mathcal{R}_{n,k}(1)$, too, are instances of a more general construction introduced by Reiner, Saliola and Welker called *symmetrized shuffling operators*. These operators are obtained by composing a BHR random walk with its transpose. Reiner–Saliola–Welker introduced one such operator for every integer partition $\lambda \vdash n$, each corresponding to a Markov chain on the symmetric group $\mathfrak{S}_n$. Despite the monograph devoted to their study [16], many questions about symmetrized shuffling operators remain.

For instance, [16] identified just *two* families of partitions giving rise to pairwise commuting elements in $\mathbb{R}[\mathfrak{S}_n]$. The first family is precisely the k-random-to-random shuffling elements $\mathcal{R}_{n,k}(1)$ which are the focus of our study—and generalization—here.

Even within the family $\mathcal{R}_{n,k}(1)$, mysteries remained. For instance, the spectral formulas in [8, 11] utilized the *content* of a partition, hinting at a link to the *Young–Jucys–Murphy elements* for the symmetric group. This connection was elucidated in recent work [1], which introduced a q-deformed version of $\mathcal{R}_{n,1}(1)$ in the Hecke algebra $\mathcal{H}_n$ and proved that its eigenvalues are in $\mathbb{Z}_{\geq 0}[q]$. Importantly, the proof established the connection to the Young–Jucys–Murphy elements of $\mathcal{H}_n$, which opened the door to a variety of techniques from representation theory. We will solidify this connection and point of view for the full family $\mathcal{R}_{n,k}$ (see Theorem 13).

3 Going quantum

We now move to the quantum deformation, where we enrich the "classical" objects we have been working with in Section 2 by an extra parameter q. The deformation of the integers will be the *q-integers*, which are Laurent polynomials defined for any $m \in \mathbb{Z}$:

$$[m]_q := \frac{1-q^m}{1-q} = \begin{cases} 1 + q + q^2 + \cdots + q^{m-1}, & \text{if } m \geq 0; \\ -q^{-1} - q^{-2} - \cdots - q^m, & \text{if } m \leq 0. \end{cases}$$

We shall also use the *q-factorials* and *q-binomials*

$$[k]!_q := [1]_q [2]_q \cdots [k]_q , \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[n-k]!_q [k]!_q}.$$

Algebraically, our deformation will move us from the symmetric group algebra to the *Iwahori–Hecke algebra*. We fix a commutative ring $\mathbf{k}$, a nonnegative integer n, and a scalar $q \in \mathbf{k}$. We let $\mathfrak{S}_n$ denote the symmetric group on the set $[n] := \{1, 2, \ldots, n\}$. This group is generated by the simple transpositions $s_i := (i, i+1)$ for all $i \in [n-1]$.

The n-th *Hecke algebra* (or *Iwahori–Hecke algebra*) is the (associative unital) $\mathbf{k}$-algebra $\mathcal{H}_n := \mathcal{H}_n(q)$ with generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$\begin{aligned} T_i^2 &= (q-1)\, T_i + q && \text{for all } i \in [n-1]\,; \\ T_i T_j &= T_j T_i && \text{whenever } |i-j| > 1; \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \text{for all } i \in [n-2]\,. \end{aligned}$$

For $q = 1$, this is the group algebra $\mathbf{k}[\mathfrak{S}_n]$ (upon identifying each T_i with $s_i = (i, i+1)$).

The Hecke algebras for all $n \geq 0$ fit into a chain of canonical embeddings $\mathbf{k} = \mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \cdots$, which we regard as inclusions.

For general q, the Hecke algebra $\mathcal{H}_n$ still shares many of the properties of $\mathbf{k}[\mathfrak{S}_n]$, albeit in subtler and more complicated ways (see [9, 10, 14, 15, 13] for details). In particular, $\mathcal{H}_n$ is still a free $\mathbf{k}$-module of rank $n!$, with a basis $(T_w)_{w \in \mathfrak{S}_n}$ indexed by permutations $w \in \mathfrak{S}_n$. The basis vectors are defined by

$$T_w := T_{i_1} T_{i_2} \cdots T_{i_k}, \text{ where } s_{i_1} s_{i_2} \cdots s_{i_k} \text{ is a reduced expression for } w.$$

For $q = 1$, this T_w is just w. When $\mathbf{k}$ is an infinite field and q is generic (actually, it suffices that $q \neq 0$ and q is not a nontrivial root of unity), the Hecke algebra $\mathcal{H}_n$ is still semisimple and has "the same representation theory" as $\mathfrak{S}_n$, in the sense that its simple modules are q-deformed versions of the classical Specht modules indexed by partitions $\lambda \vdash n$. Consequently, we denote these modules by S^λ.

Finally, there is a unique $\mathbf{k}$-algebra anti-automorphism $\mathcal{H}_n \to \mathcal{H}_n$ sending each T_w to $T_{w^{-1}}$, which we will use. We shall denote the image of an element $a \in \mathcal{H}_n$ under this anti-automorphism by a^*; it is called the *antipode* of a. By construction, $(a^*)^* = a$.

Remark 4. In what follows, we will be interested in the eigenvalues of elements inside of $\mathcal{H}_n$. We clarify what this means: if V is a right $\mathcal{H}_n$-module, and if $a \in \mathcal{H}_n$ is any element, then a acts on V as a $\mathbf{k}$-module endomorphism. The eigenvalues of this endomorphism will be called the *eigenvalues of a on V*. When V is the right regular representation of $\mathcal{H}_n$ (that is, $\mathcal{H}_n$ itself), then these eigenvalues will just be called the *eigenvalues of a*. When we talk about multiplicities of eigenvalues, we shall always mean their algebraic multiplicities by default, and we shall tacitly understand that the multiplicities of coinciding eigenvalues must be summed together. Thus, when we say that an endomorphism ϕ of a free $\mathbf{k}$-module V has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $a_1, a_2, \ldots, a_k$, we mean that its characteristic polynomial $\det(xI - \phi)$ factors as $\prod_{i=1}^{k} (x - \lambda_i)^{a_i}$.

Remark 5. When q is a prime power, $\mathcal{H}_n$ can be realized as a subalgebra of the group algebra $\mathbf{k}\left[\mathrm{GL}_n\left(\mathbb{F}_q\right)\right]$, and can be identified with the $\mathrm{GL}_n\left(\mathbb{F}_q\right)$-endomorphism ring of $\mathbf{k}[G/B]$ for $G = \mathrm{GL}_n\left(\mathbb{F}_q\right)$ and B a Borel subgroup. This makes the study of $\mathcal{H}_n$ useful in the representation theory of $\mathrm{GL}_n\left(\mathbb{F}_q\right)$, but also allows for a "reverse flow" of information from $\mathrm{GL}_n\left(\mathbb{F}_q\right)$ to $\mathcal{H}_n$ (the restriction to the "q is a prime power" case can be lifted using polynomial identity arguments). Such reasoning has found use in our proofs, though we omit those details here. In fact, all of the operators studied in this abstract can be realized as Markov chains on the space $\mathbf{k}[G/B]$ with the same eigenvalues.

4 The q-deformed random-to-random shuffles

We are now ready to define the main operators of interest. For each $k \in \{0, 1, \ldots, n\}$, we define the *$q$-deformed k-bottom-to-random shuffle* and *k-random-to-bottom shuffle*

$$\mathcal{B}_{n,k} := \sum_{\substack{\sigma \in \mathfrak{S}_n; \\ \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n-k)}} T_\sigma, \qquad \mathcal{B}^*_{n,k} := \sum_{\substack{\sigma \in \mathfrak{S}_n; \\ \sigma(1) < \sigma(2) < \cdots < \sigma(n-k)}} T_\sigma. \tag{4.1}$$

That is, $\mathcal{B}_{n,k}$ is the sum of all T_σ with σ ranging over the minimum-length right coset representatives for the subgroup $\mathfrak{S}_{n-k}$ in $\mathfrak{S}_n$. Similarly, $\mathcal{B}^*_{n,k}$ is the sum over minimum left coset representatives for $\mathfrak{S}_{n-k}$. Note that $(\mathcal{B}_{n,k})^* = \mathcal{B}^*_{n,k}$, and that setting $q = 1$ turns $\mathcal{B}_{n,k}$ to $\mathcal{B}_{n,k}(1)$ and likewise for $\mathcal{B}^*_{n,k}$. The following naturally emerges from (4.1):

Proposition 6. *We have $\mathcal{B}_n := \mathcal{B}_{n,1} = \sum_{i=1}^{n} T_{(n,n-1,\ldots,i)}$. Furthermore, for $k \geq 0$, we have*

$$\mathcal{B}_{n,k} = \mathcal{B}_{n-(k-1)} \cdots \mathcal{B}_{n-1} \mathcal{B}_n, \qquad \text{where we set } \mathcal{B}_m := 0 \text{ for negative } m.$$

Note that $\mathcal{B}^*_{n,1}$ was studied by Lusztig [12] under the name $\mathbf{t}$, as well as in [1, 4].

Next, for each $k \geq 0$, we define the *q-deformed k-random-to-random shuffle*

$$\mathcal{R}_{n,k} := \frac{1}{[k]!_q} \mathcal{B}^*_{n,k} \mathcal{B}_{n,k} \in \mathcal{H}_n. \tag{4.2}$$

On its face, $\mathcal{R}_{n,k}$ is only defined when $[k]!_q$ is invertible; however, this restriction can be lifted, since the coefficients of $\mathcal{R}_{n,k}$ are actually in $\mathbb{Z}[q]$ (see [3, Proposition 2.9]).

Example 7. For $n = 3$, we have (using cycle notation for permutations)

$\mathcal{B}_{3,1} = \mathcal{B}_3 = 1 + T_2 + T_2 T_1 = T_{\mathrm{id}} + T_{(3,2)} + T_{(3,2,1)}$;

$\mathcal{B}^*_{3,1} = \mathcal{B}^*_3 = 1 + T_2 + T_1 T_2 = T_{\mathrm{id}} + T_{(2,3)} + T_{(1,2,3)}$;

$$\mathcal{R}_{3,1} = \frac{1}{[1]!_q} \mathcal{B}^*_{3,1} \mathcal{B}_{3,1} = [3]_q T_{\mathrm{id}} + q[2]_q T_{(1,2)} + [2]_q T_{(2,3)} + q T_{(1,2,3)} + q T_{(1,3,2)} + (q-1) T_{(1,3)}.$$

Note: $\mathcal{R}_{3,1}(1)$ has coefficients in $\mathbb{Z}_{\geq 0}$, but the coefficients of $\mathcal{R}_{3,1}$ are *not* in $\mathbb{Z}_{\geq 0}[q]$.

Remark 8. To view $\mathcal{R}_{n,k}$ as a Markov chain on $\mathcal{H}_n$ (for a real $q \geq 1$), one uses a basis of $\mathcal{H}_n$ given by $\widetilde{T}_w := q^{-\ell(w)} T_w$. Writing $\mathcal{R}_{n,k}$ in this basis and normalizing by $[k]!_q \left[{n \atop k}\right]_q^2$ gives a stochastic matrix when one views $q^{-1} \in (0, 1]$ as a probability.

5 Main Results

As stated in Theorem 1 (1) and (2), we generalize the main results of [8, 11, 16]. In particular, we show that (1) the elements $\mathcal{R}_{n,k}$ for all $k \geq 0$ and fixed n commute, and (2) the eigenvalues of $\mathcal{R}_{n,k}$ are in $\mathbb{Z}_{\geq 0}[q]$ for all $n, k \geq 0$.

The proof of Theorem 1 (2) computes explicit formulas for the eigenvalues of every $\mathcal{R}_{n,k}$. We describe this process and our formula in Section 6. For now, we describe the combinatorial parameters that determine these eigenvalues, which rely on two features of Young diagrams and tableaux. Write $\mathsf{SYT}(\lambda)$ and $\mathsf{SYT}(n)$ for the set of standard Young tableaux of shape λ and of size n (of any shape), respectively.

First, a *horizontal strip* $\lambda \setminus \mu$ is a pair of two partitions λ and μ that satisfy $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all i. In terms of Young diagrams, this is saying that the skew diagram of $\lambda \setminus \mu$ has no two boxes in the same column.

The second combinatorial input is the notion of *desarrangement tableaux*. Specifically, if $\mathfrak{t} \in \mathsf{SYT}(m)$, then $\mathrm{Des}(\mathfrak{t})$ denotes the set of all integers $i \in [m-1]$ such that $i+1$ appears in a row further south than i (in English notation). Then a *desarrangement tableau* is a tableau $\mathfrak{t} \in \mathsf{SYT}(m)$ such that the smallest element of $[m] \setminus \mathrm{Des}(\mathfrak{t})$ is even. Given a partition μ, we let d^μ be the number of desarrangement tableaux of shape μ.

Example 9. We have $d^{(3,2)} = 2$; the two desarrangement tableaux are:

1	3	4
2	5	

and

1	3	5
2	4	

.

Theorem 10. *Every eigenvalue of $\mathcal{R}_{n,k}$ has the form of a certain polynomial $\mathcal{E}_{\lambda \setminus \mu}(k) \in \mathbb{Z}[q]$, indexed by a horizontal strip $\lambda \setminus \mu$ for $\lambda \vdash n$. Further:*

(i) The multiplicity of $\mathcal{E}_{\lambda \setminus \mu}(k)$ is $d^\mu f^\lambda$ where $f^\lambda = |\mathsf{SYT}(\lambda)|$.

(ii) We have both recursive and explicit formulas for the $\mathcal{E}_{\lambda \setminus \mu}(k)$ (see Section 6.3).

One subtlety about Theorem 10 is that even if $\lambda \setminus \mu$ is a horizontal strip, the eigenvalue $\mathcal{E}_{\lambda \setminus \mu}(k)$ will appear with multiplicity zero whenever $d^\mu = 0$.

We will say more about the eigenvalues in Section 6.3 (see also Table 1). For now, we highlight product formulas for several eigenvalues of $\mathcal{R}_{n,k}$, including the two largest.

In fact, the second-largest eigenvalue of $\mathcal{R}_{n,k}(1)$ was conjectured by Lafrenière in [11, Conjecture 92]; Theorem 11 affirms her Conjecture, and proves it for arbitrary q.

Theorem 11. *For $0 \leq j, \ell, k \leq n$, the eigenvalue $\mathcal{E}_{\lambda\backslash\mu}(k)$ for $\lambda = (n-\ell, 1^\ell)$ and $\mu = (j, 1^\ell)$ is given by*

$$\mathcal{E}_{(n-\ell,1^\ell)\backslash(j,1^\ell)}(k) = [k]!_q \begin{bmatrix} n-j-\ell \\ k \end{bmatrix}_q \begin{bmatrix} n+j \\ k \end{bmatrix}_q . \tag{5.1}$$

When $q \in \mathbb{R}_{>0}$, the case $\ell = j = 0$ yields the largest eigenvalue of $\mathcal{R}_{n,k}$:

$$\mathcal{E}_{(n)\backslash\varnothing}(k) = [k]!_q \begin{bmatrix} n \\ k \end{bmatrix}_q^2 . \tag{5.2}$$

When $q \in \mathbb{R}_{>0}$, the case $\ell = j = 1$ yields the second-largest eigenvalue of $\mathcal{R}_{n,k}$:

$$\mathcal{E}_{(n-1,1)\backslash(1,1)}(k) = [k]!_q \begin{bmatrix} n-2 \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q .$$

Recall that the Specht modules of $\mathcal{H}_n$ are written S^λ for $\lambda \vdash n$. In fact, we show that the case $\ell = 1$ and $j \in [n-1]$ above describes all of the eigenvalues of $\mathcal{R}_{n,k}$ acting on the Specht module $S^{(n-1,1)}$. The case $\ell = j = 0$ yields the unique eigenvalue from the $\mathcal{R}_{n,k}$ action on $S^{(n)}$. More generally, the formula in (5.1) describes some of the eigenvalues obtained from the $\mathcal{R}_{n,k}$ action on $S^{(n-\ell,1^\ell)}$; this is explained more explicitly in Section 6.

Shape $\lambda \setminus \mu$	Multiplicity of $\mathcal{E}_{\lambda\backslash\mu}(k)$	$\mathcal{E}_{\lambda\backslash\mu}(1)$	$\mathcal{E}_{\lambda\backslash\mu}(2)$	$\mathcal{E}_{\lambda\backslash\mu}(3)$
$(3) \setminus \varnothing$	1	$[3]_q^2$	$[2]!_q \cdot [3]_q^2$	$[3]!_q$
$(2,1) \setminus (2,1)$	2	0	0	0
$(2,1) \setminus (1,1)$	2	$[4]_q$	0	0
$(1,1,1) \setminus (1,1)$	1	$[1]_q$	0	0

Table 1: The eigenvalues of $\mathcal{R}_{n,k}$ when $n = 3$ for $k \in \{1,2,3\}$.

When $q \in \mathbb{R}_{>0}$, we can also construct a joint eigenbasis for our operators, and describe the relationships between their images and kernels. We will use this eigenbasis to compute the eigenvalues of $\mathcal{R}_{n,k}$ for arbitrary q using polynomial identity arguments.

Theorem 12. *Let $n, k \geq 0$ and $q \in \mathbb{R}_{>0}$. Then, the elements $\mathcal{R}_{n,k}$ (acting on $\mathcal{H}_n$) are diagonalizable with a common eigenbasis, and there is a containment of images and kernels:*

$$\begin{aligned} \mathcal{H}_n &= \operatorname{im}(\mathcal{R}_{n,0}) \supseteq \operatorname{im}(\mathcal{R}_{n,1}) \supseteq \operatorname{im}(\mathcal{R}_{n,2}) \supseteq \cdots \supseteq \operatorname{im}(\mathcal{R}_{n,n+1}) = \{0\}; \\ \mathcal{H}_n &= \ker(\mathcal{R}_{n,n+1}) \subseteq \ker(\mathcal{R}_{n,n}) \subseteq \cdots \subseteq \ker(\mathcal{R}_{n,0}) = \{0\}. \end{aligned}$$

6 Techniques

6.1 The key recursion

The crucial tool for proving our main results is a new and remarkable recursion relating the elements $\mathcal{R}_{n,k}$, $\mathcal{R}_{n-1,k}$, and $\mathcal{R}_{n-1,k-1}$ to the *Young–Jucys–Murphy (YJM) elements*. Assume without loss of generality that $q \in \mathbf{k}^\times$. The YJM elements $J_1, J_2, \ldots, J_n \in \mathcal{H}_n$ are

$$J_k := \sum_{i=1}^{k-1} q^{i-k} T_{(i,k)} \in \mathcal{H}_n.$$

Setting $q = 1$ recovers the classical YJM elements of $\mathbf{k}[\mathfrak{S}_n]$. Note that $J_1 = 0$ and $J_k^* = J_k$ for each $k \in [n]$. It is well-known that the elements $J_1, J_2, \ldots, J_n$ commute, and that J_n commutes with all elements of $\mathcal{H}_{n-1}$. Our proof needs the following straightforward recursions, coming from the definitions of $\mathcal{R}_{n,k}$ and $\mathcal{B}_{n,k}$:

$$B_{n,k} = \mathcal{B}_{n-1,k-1}\,\mathcal{B}_n = \mathcal{B}_{n-k+1}\,\mathcal{B}_{n,k-1}, \qquad [k]_q\,\mathcal{R}_{n,k} = \mathcal{B}_n^*\,\mathcal{R}_{n-1,k-1}\,\mathcal{B}_n \qquad \text{for } k \geq 1.$$

Our key recursion is as follows, see also [3, Theorem 3.6]:

Theorem 13. *For any $1 \leq k \leq n$, we have*

$$\mathcal{B}_n\,\mathcal{R}_{n,k} = \left(q^k\,\mathcal{R}_{n-1,k} + \left([n+1-k]_q + q^{n+1-k} J_n\right)\mathcal{R}_{n-1,k-1}\right)\mathcal{B}_n\,.$$

Rough structure of the proof. Our proof involves some elementary but intricate computations (details in [3, §3]). At the origin is an identity proved in [1, (5.2)]:

$$\mathcal{B}_n\,\mathcal{B}_n^* = [n]_q + q^n J_n + \mathcal{B}_{n-1}^*\,T_{n-1}\,\mathcal{B}_{n-1}\,. \tag{6.1}$$

Since $\mathcal{B}_n^*$ is the first factor in $\mathcal{B}_{n,k}^*$ (for $k > 0$) and thus also in $\mathcal{R}_{n,k}$, this is a first step towards the proof, but there is a long way to go. We set

$$\lambda_{n,i} := T_{n-1} \cdots T_{n-i+1}\left([n-i+1]_q + q^{n-i+1} J_{n-i+1}\right) \qquad \text{for each } i \in [n].$$

Then, we set $\gamma_{n,k} := \sum_{i=1}^{k} \lambda_{n,i}$ and $\Lambda_{n,k} := \mathcal{B}_{n-1,k-1}^*\,\gamma_{n,k}$ for all $0 \leq k \leq n$. It is easy to see that every $k \in [n]$ satisfies $\gamma_{n,k} = [n]_q + q^n J_n + T_{n-1}\gamma_{n-1,k-1}$. The first major step is to show (by induction on k, using (6.1)) that each $k \in [n]$ satisfies

$$\mathcal{B}_n\,\mathcal{B}_{n,k}^* = \mathcal{B}_{n-1,k}^*\,(T_{n-1}T_{n-2}\cdots T_{n-k})\,\mathcal{B}_{n-k} + \Lambda_{n,k}, \tag{6.2}$$

where $T_0 := 0$. Another induction on k shows that each $k \in [n]$ satisfies

$$\gamma_{n,k} = [k]_q\left([n-k+1]_q + q^{n-k+1} J_n\right) + I_{n,k} \text{ for some } I_{n,k} \in \mathcal{H}_n \text{ such that } I_{n,k}\,\mathcal{B}_{n,k} = 0.$$

Multiplying this by $\mathcal{B}_{n-1,k-1}^*$ from the left and by $\mathcal{B}_{n,k}$ from the right, we find

$$\Lambda_{n,k}\,\mathcal{B}_{n,k} = [k]_q\left([n+1-k]_q + q^{n+1-k} J_n\right)\mathcal{B}_{n-1,k-1}^*\,\mathcal{B}_{n,k}\,.$$

This allows us to simplify the $\Lambda_{n,k}$ term when multiplying (6.2) by $\mathcal{B}_{n,k}$ from the right. After some simplifications, Theorem 13 emerges. □

6.2 Proving Theorem 1 (1) (pairwise commutativity) from Theorem 13

We prove that $\mathcal{R}_{n,j}\,\mathcal{R}_{n,i} = \mathcal{R}_{n,i}\,\mathcal{R}_{n,j}$ using a fairly straightforward argument by induction on n. Our main tools are the key recursion (Theorem 13), the simpler recursion $[k]_q\,\mathcal{R}_{n,k} = \mathcal{B}_n^*\,\mathcal{R}_{n-1,k-1}\,\mathcal{B}_n$, and clever applications of the $*$-map. See [3, §4] for details.

6.3 Proving Theorem 1 (2) (the eigenvalues) from Theorem 13

6.3.1 Restricting to individual Specht modules

Proving Theorem 10 requires another ingredient: the representation theory of $\mathcal{H}_n$. Recall that when $\mathcal{H}_n$ is semisimple, the irreducible representations of $\mathcal{H}_n$ can be described as Specht modules S^λ for λ an integer partition of n. Writing S^λ as a right $\mathcal{H}_n$-module and $(S^\lambda)^*$ as a left $\mathcal{H}_n$-module, there is a bimodule decomposition

$$\mathcal{H}_n \cong \bigoplus_{\lambda \vdash n} (S^\lambda)^* \otimes S^\lambda. \tag{6.3}$$

Since $\mathcal{R}_{n,k}$ acts by right multiplication, its corresponding eigenspaces are left $\mathcal{H}_n$-modules.

Hence, to find the eigenvalues of $\mathcal{R}_{n,k}$ on all of $\mathcal{H}_n$, it suffices to find its eigenvalues on each S^λ and then use (6.3). Theorem 10 (i) thus reduces to showing that the eigenvalues of $\mathcal{R}_{n,k}$ on a given S^λ are the $\mathcal{E}_{\lambda\setminus\mu}(k)$ for horizontal strips $\lambda \setminus \mu$ with multiplicity d^μ.

6.3.2 Lifting eigenvectors from smaller Specht modules

Fix $\lambda \vdash n$ and let $\lambda' \vdash n-1$ be obtained by removing a box from λ, written $\lambda' \lessdot \lambda$.

To obtain the eigenvalues on S^λ, we "lift" an eigenvector for both $\mathcal{R}_{n-1,k-1}$ and $\mathcal{R}_{n-1,k}$ from $S^{\lambda'}$ to an eigenvector of $\mathcal{R}_{n,k}$ on S^λ. Proposition 14 below shows that this lifting process changes the eigenvalue in a predictable way (see [1, Proposition 5.2] for explicit details). Let $c_{\lambda\setminus\lambda'} = i - j$ be the *content* of the only box in the skew shape $\lambda \setminus \lambda'$ appearing in row i and column j of λ (in English notation).

Proposition 14. *For $1 \le k \le n$, fix $\lambda' \vdash n-1$ and $\lambda \vdash n$ with $\lambda' \lessdot \lambda$. Suppose $v \in S^{\lambda'}$ is*

- *an eigenvector of $\mathcal{R}_{n-1,k}$ with eigenvalue ε_k, and*
- *an eigenvector of $\mathcal{R}_{n-1,k-1}$ with eigenvalue ε_{k-1}.*

Then one can construct $u \in S^\lambda$ such that u is 0 or *is an eigenvector for $\mathcal{R}_{n,k}$ with eigenvalue*

$$q^k \varepsilon_k + [n+1-k+c_{\lambda\setminus\lambda'}]_q\,\varepsilon_{k-1}. \tag{6.4}$$

Proposition 14 relies on the recursion in Theorem 13 and joint eigenbasis in Theorem 12. Specifically, the colors in Theorem 13 indicate the corresponding effect those terms have on the eigenvalues in Proposition 14. The lifting process involves multiplying v by $\mathcal{B}_n$ and projecting onto the appropriate S^λ. We can iterate the process in Proposition 14 to lift eigenvectors from S^μ to S^λ for any $\mu \subset \lambda$ (see [3, Lemma 5.8]).

6.3.3 An $\mathcal{R}_{n,k}$-eigenbasis for S^λ

The lifting process in Proposition 14 gives a method of constructing eigenvectors, but does not yet say (1) when this process yields 0 and (2) if all eigenvectors of $\mathcal{R}_{n,k}$ can be constructed in this manner. Here, we utilize Theorem 12, which implies that for $q \in \mathbb{R}_{>0}$, the eigenbasis for $\mathcal{R}_{n,1}$ described in [1, Theorem 7.2] is a basis for every $\mathcal{R}_{n,k}$.

This eigenbasis is constructed by lifting $\mathcal{R}_{j,1}$-kernel eigenvectors in S^μ to the Specht module S^λ. A vector $u \in S^\lambda$ obtained from this process will be non-zero if and only if $\lambda \setminus \mu$ is a horizontal strip. In this way, we can conclude that all eigenvectors of $\mathcal{R}_{n,k}$ are obtained by the lifting process in Proposition 14, and in particular the eigenvalue $\mathcal{E}_{\lambda\setminus\mu}(k)$ comes from an eigenvector in S^λ lifted from a $\mathcal{R}_{j,1}$-kernel eigenvector in S^μ. By [1], the kernel of $\mathcal{R}_{j,1}$ acting on S^μ has dimension d^μ, which, along with (6.3), can be used to explain the multiplicity formula $d^\mu f^\lambda$ for $\mathcal{E}_{\lambda\setminus\mu}(k)$ in Theorem 10 (i).

To state the explicit eigenvalue formulas, we set some additional notation. Let $\mathsf{t}^{\lambda\setminus\mu}$ be the skew tableau of shape $\lambda \setminus \mu$ given by filling the boxes with $j+1, j+2, \ldots, n$ left-to-right within rows and row-by-row starting from the top, and $\mathsf{c}_{\mathsf{t}^{\lambda\setminus\mu},\ell}$ the content (i.e. column minus row) of the box containing ℓ in $\mathsf{t}^{\lambda\setminus\mu}$. We set $\mathfrak{d}_{\ell,i,\lambda\setminus\mu} := [\ell + 1 - i + \mathsf{c}_{\mathsf{t}^{\lambda\setminus\mu},\ell}]_q$.

Proposition 15. *Let $\lambda \vdash n$ and $\mu \vdash j \leq n$ such that $d^\mu \neq 0$ and $\lambda \setminus \mu$ is a horizontal strip. Then the eigenvalue $\mathcal{E}_{\lambda\setminus\mu}(k)$ from Theorem 10 has the form*

$$\mathcal{E}_{\lambda\setminus\mu}(k) = q^{nk-\binom{k}{2}} \sum_{j<(\ell_1<\ell_2<\cdots<\ell_k)\leq n} \prod_{m=1}^{k} q^{-\ell_m} \mathfrak{d}_{\ell_m,m,\lambda\setminus\mu} \in \mathbb{Z}_{\geq 0}[q]. \tag{6.5}$$

The polynomial $\mathcal{E}_{\lambda\setminus\mu}(k) \in \mathbb{Z}_{\geq 0}[q]$ is identically 0 for $k > |\lambda \setminus \mu|$, and otherwise is monic.

One subtlety here compared to $k = 1$ is that the kernels of $\mathcal{R}_{n,k}$ are more complicated when $k > 1$, so describing when ε_k from Proposition 14 gives zero adds significant complexity to the formula in (6.5). We show below the simplest example of Proposition 15.

Example 16. *Let $\lambda \setminus \mu = (3) \setminus \emptyset$. Then, $j = 0$, $n = 3$, and $\mathcal{E}_{\lambda\setminus\mu}(2)$ is*

$$q^5\Bigg(\underbrace{q^{-3}[1]_q[2]_q}_{\ell_1,\ell_2=1,2} + \underbrace{q^{-4}[1]_q[4]_q}_{\ell_1,\ell_2=1,3} + \underbrace{q^{-5}[3]_q[4]_q}_{\ell_1,\ell_2=2,3}\Bigg) = [2]!_q \cdot [3]_q^2.$$

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