Approche Duale des représentations du groupe symétrique
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Errata and addenda by Darij Grinberg

I will refer to the results appearing in the book "Approche Duale des représentations du groupe symétrique" by the numbers under which they appear in this book (specifically, in its version of 29 August 2016).

Most of my corrections are written in English, as I don't speak French well enough.

9. Errata

- **Page 2, Exemple 1.5:** The words "permutation de $\mathcal{S}_n$" are ambiguous: do you mean a permutation **in** the set $\mathcal{S}_n$ (of which there are $n!$), or a permutation **of** the set $\mathcal{S}_n$ (of which there are $(n!)!$) ? I know you mean the former, but maybe not every reader does.

- **Pages 1–13:** Somewhere here you should introduce a few more notations that you are using. Namely:

  – If (V, ρ) is a representation of a group G, and if u is any element of the group ring $\mathbb{C}[G]$, then the endomorphism $\rho(u)$ of V is defined as follows: Write u in the form $u = \sum_{g \in G} \alpha_g e_g$ (with $\alpha_g \in \mathbb{C}$), and set $\rho(u) = \sum_{g \in G} \alpha_g \rho(g)$. (You are using this notation, e.g., in the formulas (a) and (b) on page 13.) Notice that $\rho(u)$ depends $\mathbb{C}$-linearly on u, and that we have $\rho(e_g) = \rho(g)$ for each $g \in G$. Thus, the values $\rho(u)$ for $u \in \mathbb{C}[G]$ encode exactly the same information as the original homomorphism $\rho : G \to \mathrm{GL}(V)$.

  – If G is a group, if $\chi : G \to \mathbb{C}$ is any map, and if u is any element of the group ring $\mathbb{C}[G]$, then the complex number $\chi(u)$ is defined as follows: Write u in the form $u = \sum_{g \in G} \alpha_g e_g$ (with $\alpha_g \in \mathbb{C}$), and set $\chi(u) = \sum_{g \in G} \alpha_g \chi(g)$. (You are using this notation, e.g., in Lemme 3.4 on page 39.) Notice that $\chi(u)$ depends $\mathbb{C}$-linearly on u, and that we have $\chi(e_g) = \chi(g)$ for each $g \in G$. Thus, the values $\chi(u)$ for $u \in \mathbb{C}[G]$ encode exactly the same information as the original map $\chi : G \to \mathbb{C}$.

- **Page 3, Définition 1.7:** Replace "Une représentation est dite **irréductible**" by "Une représentation (V, ρ) est dite **irréductible**" (since you later refer to both V and ρ in this sentence).

- **Page 3, Définition 1.7:** Both in the definition of "indécomposable" and in the definition of "irréductible", a requirement that $V \neq 0$ should be added.
- **Page 3, Remarque 1.8:** You probably want to replace the word "irréductibles" by "indécomposables". (They are, of course, equivalent, but only the version with "indécomposables" really follows directly from the definitions).
- **Page 3, Théorème 1.9:** When you say "un nombre fini", it might be useful to point out that you are counting irreducible representations **up to isomorphism**.
- **Page 4, Démonstration de Corollaire 1.15:** Replace "χ_{V_N}" by "χ^{V_N}".
- **Page 5, §1.2:** When you define the "**type cyclique**", it would be worthwhile pointing out that cycles of size 1 (that is, cycles corresponding to fixed points of σ) are included in the decomposition. (Many authors, particularly those of introductory algebra texts, tend to omit them.)
- **Page 6:** In "et donc de représentations irréductibles de S_n", replace "S_n" by "$\mathcal{S}_n$".
- **Page 8, Démonstration de Lemme 1.23:** In "d'un seul paramètre α_{id}", replace "id" by "Id".
- **Page 8, Démonstration de Lemme 1.23:** You write: "Il n'est pas difficile de vérifier la CNS suivante".

  From a sufficiently experienced point of view, this CNS is really not hard to verify. However, I don't think that omitting this proof is appropriate when you are targeting an undergraduate readership. The proof is somewhat similar to the proof of Lemme 1.31, and more difficult if anything; it is strange that you present the latter in all its detail while leaving the former entirely to the reader. At the very least, it's reasonable to give some references for the proof of your CNS (more precisely: of the fact that $\pi \in \mathcal{S}_n$ can be written in the form $\sigma\tau$ with $\sigma \in \mathrm{RS}(T)$ and $\tau \in \mathrm{CS}(T)$ if and only if the condition (P) is satisfied):

  - The fact that you are claiming (or, more precisely, the sufficiency of your condition (P)) is implicit in the proof of Lemma 4.40 in:

    Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, Elena Yudovina, *Introduction to representation theory*, arXiv:0901.0827v5.
  - The same fact is a particular case of Lemma 3 (b) in:

    Graham Gill, *Representation theory of the symmetric group: basic elements*, 16 January 2006.

  - It is also a consequence of the Lemma in §2.3 of:

    Claudio Procesi, *Chapter 9: Tensor symmetry*, 28 November 2005.

  - It also follows from Lemma (2.6) in:

    Stuart Martin, *Part III Representation Theory, Lent Term 2013 (notes by Clive Newstead)*, 26 April 2013

    (applied to $\mu = \lambda$, $t_\lambda = \pi^{-1}(T)$ and $t_\mu = T$).

  - It also follows from Lemma 2 in §2 of:

    William Crawley-Boevey, *Lectures on representation theory and invariant theory*, 1999

    (or, rather, is proven by a similar argument).

  (I have only listed references freely available online; of course, there are many others as well.)

- **Page 9, Démonstration de Lemme 1.23:** In "ne vérifiant pas (P)", replace the "(P)" by a textmode "(P)".

- **Page 10, Exemple 1.26:** On the last line of this example, replace "$\operatorname{Vect}((u_1 + \cdots + u_n))$" by "$\operatorname{Vect}((e_1 + \cdots + e_n))$".

- **Page 11, Démonstration de Proposition 1.27:** Let me suggest a different proof of Proposition 1.27 – one that does not apply the (somewhat obscure) Lemme 1.19, but instead makes use of Proposition 1.28. (This is not circular reasoning, because the proof of Proposition 1.28 does not rely on Proposition 1.27 anywhere.)

  Second proof of Proposition 1.27. We want to prove that V_λ is irreducible. Since "irreducible" is equivalent to "indecomposable" (for a representation of a finite group), it suffices to show that V_λ is indecomposable. Thus, let us prove this. Since $C_\lambda \neq 0$, we have $V_\lambda \neq \{0\}$. Thus, it remains to show that V_λ cannot be written in the form $(V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ for two nonzero subspaces V_1 and V_2.

  Assume the contrary. Thus, $V_\lambda = (V_1 \oplus V_2, \rho_1 \oplus \rho_2)$ for two nonzero subspaces V_1 and V_2. Thus, both subspaces V_1 and V_2 are stable under the action of $\mathcal{S}_n$. We have $V_\lambda = V_1 \oplus V_2$ and thus $\dim(V_\lambda) = \dim(V_1 \oplus V_2) = \dim(V_1) + \underbrace{\dim(V_2)}_{\substack{>0 \\ \text{(since } V_2 \text{ is nonzero)}}} > \dim(V_1)$. Hence, $V_\lambda \not\subseteq V_1$.

  Let n_1 and n_2 be as in Proposition 1.28. Then, Proposition 1.28 yields $n_1 n_2 \mid n!$, so that $n_1 n_2 \neq 0$ and thus $n_2 \neq 0$. Now, $C_\lambda^2 = n_2 C_\lambda \neq 0$ (since $n_2 \neq 0$ and $C_\lambda \neq 0$).

  We have $C_\lambda \in \mathbb{C}[\mathcal{S}_n] C_\lambda = V_\lambda = V_1 \oplus V_2$. Hence, there exist some $v_1 \in V_1$ and $v_2 \in V_2$ such that $C_\lambda = v_1 + v_2$. Consider these v_1 and v_2.

If we had $C_\lambda \in V_1$, then we would have $V_\lambda = \mathbb{C}\left[\mathcal{S}_n\right] \underbrace{C_\lambda}_{\in V_1} \subseteq \mathbb{C}\left[\mathcal{S}_n\right] V_1 \subseteq V_1$ (since V_λ is stable under the action of $\mathcal{S}_n$), which would contradict $V_\lambda \not\subseteq V_1$. Hence, we cannot have $C_\lambda \in V_1$. In other words, we have $C_\lambda \notin V_1$.

We have $v_1 \in V_1 \subseteq V_\lambda = \mathbb{C}\left[\mathcal{S}_n\right] C_\lambda$. In other words, there exists some $w_1 \in \mathbb{C}\left[\mathcal{S}_n\right]$ satisfying $v_1 = w_1 C_\lambda$. Consider this w_1.

The element $x = C_\lambda w_1 C_\lambda$ satisfies the equalities (1) and (2) from the proof of Lemme 1.23 (because of (1.3) and (1.4)). Hence, x is a scalar multiple of C_λ (as follows from Lemme 1.23). In other words, $C_\lambda w_1 C_\lambda = \alpha C_\lambda$ for some $\alpha \in \mathbb{C}$. Consider this α. We have $\alpha C_\lambda = C_\lambda \underbrace{w_1 C_\lambda}_{= v_1 \in V_1} \in C_\lambda V_1 \subseteq V_1$ (since V_λ is stable under the action of $\mathcal{S}_n$). If α was nonzero, then we would have $C_\lambda = \dfrac{1}{\alpha} \underbrace{C_\lambda}_{\in V_1} \in \dfrac{1}{\alpha} V_1 \subseteq V_1$, which would contradict $C_\lambda \notin V_1$. Hence, α cannot be nonzero. Thus, $\alpha = 0$. Hence, $C_\lambda w_1 C_\lambda = \underbrace{\alpha}_{=0} C_\lambda = 0$. Thus, $C_\lambda \underbrace{v_1}_{= w_1 C_\lambda} = C_\lambda w_1 C_\lambda = 0$. Similarly, $C_\lambda v_2 = 0$. Now,

$$C_\lambda^2 = C_\lambda \underbrace{C_\lambda}_{= v_1 + v_2} = C_\lambda \left(v_1 + v_2\right) = \underbrace{C_\lambda v_1}_{=0} + \underbrace{C_\lambda v_2}_{=0} = 0.$$

This contradicts $C_\lambda^2 \neq 0$. This contradiction shows that our assumption was false. Hence, V_λ cannot be written in the form $\left(V_1 \oplus V_2, \rho_1 \oplus \rho_2\right)$ for two nonzero subspaces V_1 and V_2. Thus, the representation V_λ is indecomposable (since $V_\lambda \neq \{0\}$), and therefore irreducible (since Maschke's theorem tells us that "irreducible" and "indecomposable" is the same thing). This proves Proposition 1.27 again. $\square$

- **Page 12, Démonstration de Proposition 1.28:** Replace "id" by "Id" twice in this proof. (Maybe it is easier to just root out these errors by a find-replace? Or do you use Id and id for two different things at some point?)

- **Page 18, Démonstration de Lemma 2.2:** On the second line of this proof, the words "dans ce cas" sound somewhat inappropriate to me (you aren't really focussing on a case). Maybe something like "for this reason" would be better.

- **Page 18:** The last equality sign in the equality

$$\begin{aligned} \chi^\lambda\left(\pi\right) &:= \operatorname{Tr}\left(\rho\left(\pi\right)\right) = \operatorname{Tr}\left(\varphi_{\lambda,\pi}\right) \\ &= \frac{\dim\left(V_\lambda\right)}{n!} \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma \in \operatorname{RS}(T)}} \sum_{\substack{\tau \in \mathcal{S}_n \\ \tau \in \operatorname{CS}(T)}} \varepsilon\left(\tau\right) \operatorname{Tr}\left(x \mapsto e_\pi \cdot x \cdot e_\sigma \cdot e_\tau\right) \end{aligned}$$

requires some work to verify; maybe it's better to add some explanation such as "because the definition of $\varphi_{\lambda,\pi}$ becomes

$$\begin{aligned}\varphi_{\lambda,\pi} &= \frac{\dim(V_\lambda)}{n!}\left(x \mapsto e_\pi \cdot x \cdot C_\lambda\right) \qquad \text{(by the definition of } p_\lambda\text{)} \\ &= \frac{\dim(V_\lambda)}{n!} \sum_{\substack{\sigma\in\mathcal{S}_n \\ \sigma\in\mathrm{RS}(T)}} \sum_{\substack{\tau\in\mathcal{S}_n \\ \tau\in\mathrm{CS}(T)}} \varepsilon(\tau)\left(x \mapsto e_\pi \cdot x \cdot e_\sigma \cdot e_\tau\right) \\ &\qquad \text{(by the definition of } C_\lambda\text{)}.\end{aligned}$$

".

- **Page 18:** Before "Nous allons manipuler", I'd suggest adding something like "Therefore,

  $$\begin{aligned}\frac{n!\chi^\lambda(\pi)}{\dim(V_\lambda)} &= \sum_{\substack{\sigma\in\mathcal{S}_n \\ \sigma\in\mathrm{RS}(T)}} \sum_{\substack{\tau\in\mathcal{S}_n \\ \tau\in\mathrm{CS}(T)}} \varepsilon(\tau)\operatorname{Card}\left\{g\in\mathcal{S}_n \mid g = \pi\cdot g\cdot\sigma\cdot\tau\right\} \\ &= \sum_{\substack{\sigma\in\mathcal{S}_n \\ \sigma\in\mathrm{RS}(T)}} \sum_{\substack{\tau\in\mathcal{S}_n \\ \tau\in\mathrm{CS}(T)}} \varepsilon(\tau) \sum_{g\in\mathcal{S}_n} \delta_{g,\pi\cdot g\cdot\sigma\cdot\tau} = \sum_{g\in\mathcal{S}_n} \sum_{\substack{\sigma\in\mathrm{RS}(T) \\ \tau\in\mathrm{CS}(T)}} \varepsilon(\tau)\,\delta_{g,\pi\cdot g\cdot\sigma\cdot\tau}.\end{aligned}$$

  ". This would help bridge the gap to the computations on page 19 (which otherwise seem to come out of the blue).

- **Page 19:** Replace "$\delta_{\pi=\tau'\sigma'}$" by "$\delta_{\pi,\tau'\sigma'}$".

- **Page 19:** Replace "$\delta_{\pi=\tau\sigma}$" by "$\delta_{\pi,\tau\sigma}$".

- **Page 19:** Replace "$CS(T)$" by "$\mathrm{CS}(T)$" (twice on this page).

  Also, replace "$RS(T)$" by "$\mathrm{RS}(T)$".

  (Again, this kind of mistake might be susceptible to an automated search.)

- **Page 20, Remarque 2.6:** Replace "$CS(T)$" by "$\mathrm{CS}(T)$".

  Also, replace "$RS(T)$" by "$\mathrm{RS}(T)$".

- **Page 21, Démonstration de Proposition 2.7:** Replace "de la définition (2.5)" by "de la Définition 2.5". (You are citing a definition, not a formula.)

- **Page 22, Démonstration de Proposition 2.7:** Replace "sont dans la même case" by "sont la même case".

- **Page 22, Démonstration de Proposition 2.7:** In the last equation in this proof, replace the "$\varepsilon(\tau)$" (on the right hand side) by an "$\varepsilon(\check{\tau})$" (and maybe add a justification, such as "because $\varepsilon\left(\tau_{\mid\{1,\ldots,k\}}\right) = \varepsilon(\tau)$").

- **Page 22, Exemple 2.9:** Replace "$\in S_2$" by "$\in \mathcal{S}_2$".
- **Page 23, Exemple 2.9:** Replace "Le théorème nous donne" by "Proposition 2.7 nous donne".
- **Page 24, Démonstration de Théorème 2.13:** In this proof, you are using the notation $\delta_{\mathcal{A}}$ for $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$ when $\mathcal{A}$ is any logical statement. It might be worth defining this notation.
- **Page 26, Corollaire 2.16:** Replace "$\dim \left(V^{p\times q}\right)$" by "$\dim \left(V_{p\times q}\right)$".
- **Page 26, Remarque 2.17:** Replace "$\dim \left(V^{p\times q}\right)$" by "$\dim \left(V_{p\times q}\right)$".
- **Page 27, Corollaire 2.22:** I'd clarify here that λ is considered fixed.
- **Page 27, Exemple 2.24:** These aren't all the possible graphs; you are missing the ones where one of $V_\circ(G)$ and $V_\bullet(G)$ is empty :)
- **Page 27, Exemple 2.24:** I don't think you have ever defined the notation $\ell(\lambda)$ for the length of a partition λ.
- **Page 27, Remarque 2.25:** I know it looks totally obvious, but it wouldn't hurt to explain what the "sommets blancs" of a bipartite graph G are (namely, the elements of $V_\circ(G)$), and what the "sommets noirs" of a bipartite graph G are (namely, the elements of $V_\bullet(G)$).
- **Page 29, Proposition 2.31:** The sum on the right hand side has many undefined addends. You should either replace "$\varphi : V_G \to \mathbb{N}^*$" by "$\varphi : V_G \to \{1, 2, \ldots, m\}$" under the summation sign (saying that m is the size of $\mathbf{p}$), or define the values p_i and q_i to be 0 for $i > m$.
- **Page 29, Démonstration de Proposition 2.31:** You write: "On part du lemme 2.19", but the setting of Proposition 2.31 is more general than the setting of Lemme 2.19 (for example, the bipartite graph G in Proposition 2.31 may have isolated vertices, which can never happen for a bipartite graph of the form $G_{\sigma,\tau}$). You probably want to refer to Définition 2.23 instead. Furthermore, whenever you refer to "$(\star)_h$", you mean the third condition in Définition 2.23.
- **Page 29, Démonstration de Proposition 2.31:** Replace "$\varphi(v) = j$ si $q_1 + \cdots + q_{j-1} < h_c(v) \leq q_1 + \cdots + q_j$" by "$\varphi(v) = j$ si $q_{j+1} + \cdots + q_m < h_c(v) \leq q_j + \cdots + q_m$".
- **Page 31, Proposition 2.39:** The graphs in this bijection should not be allowed to have isolated vertices.

- **Page 35, Notes et références:** You seem to write (my bad grasp of French might be misleading me here) that Young symmetrizers have (surprisingly) not been previously used to compute characters of $\mathcal{S}_n$. This doesn't seem true to me. In §36–§38 of Daniel Edwin Rutherford's *Substitutional Analysis*, these characters are computed using the material previously studied in the book, which includes both Young symmetrizers and Young's seminormal form; this is more complicated than your argument, but still doesn't use symmetric functions.

- **Page 37:** After "et l'appellerons le **caractère normalisé**", add a period.

- **Page 38, §2.1:** You say that "Λ est une algèbre", and that this follows from Proposition 3.3. Maybe it is worth pointing out one more step in this argument: Namely, the function $\mathrm{Ch}_\varnothing$ (corresponding to the partition $\varnothing$ of 0) sends every partition λ to

$$\underbrace{|\lambda|(|\lambda|-1)\cdots(|\lambda|-0+1)}_{=(\text{empty product})=1} \underbrace{\widehat{\chi}^\lambda(\mathrm{Id})}_{=\dfrac{\chi^\lambda(\mathrm{Id})}{\dim(V_\lambda)}} = \frac{\chi^\lambda(\mathrm{Id})}{\dim(V_\lambda)} = \frac{1}{\dim(V_\lambda)} \underbrace{\chi^\lambda(\mathrm{Id})}_{\substack{=\mathrm{Tr}(\rho^\lambda(\mathrm{id})) \\ =\dim(V_\lambda)}}$$
$$= \frac{1}{\dim(V_\lambda)} \dim(V_\lambda) = 1.$$

  Hence, $\mathrm{Ch}_\varnothing$ is the unity of the algebra of all functions on the set of all Young diagrams. Therefore, the latter unity belongs to Λ. Hence, in order to prove that Λ is an algebra, it suffices to prove Proposition 3.3.

- **Page 39, Lemme 3.4:** It is not necessary to relegate the proof of Lemme 3.4 to [Sag 01]. Here is a simple proof using nothing but the results shown in Chapter 1 of your book:

  Proof of Lemme 3.4. Assume that $x \in \mathbb{C}[\mathcal{S}_n]$ is central.

  The element x of $\mathbb{C}[\mathcal{S}_n]$ is central. In other words, x lies in the center of the ring $\mathbb{C}[\mathcal{S}_n]$. Hence, $xC_\lambda = C_\lambda x$. Hence, every $\sigma \in \mathrm{RS}(T)$ satisfies

$$e_\sigma \cdot \underbrace{xC_\lambda}_{=C_\lambda x} = \underbrace{e_\sigma \cdot C_\lambda}_{\substack{=C_\lambda \\ (\text{by } (1.3))}} x = C_\lambda x = xC_\lambda. \tag{1}$$

  Moreover, every $\tau \in \mathrm{CS}(T)$ satisfies

$$x \underbrace{C_\lambda \cdot e_\tau}_{\substack{=\varepsilon(\tau)C_\lambda \\ (\text{by } (1.4))}} = \varepsilon(\tau)\, xC_\lambda. \tag{2}$$

  But Lemme 1.23 shows that if some element $y \in \mathbb{C}[\mathcal{S}_n]$ satisfies

$$(e_\sigma \cdot y = y \qquad \text{for every } \sigma \in \mathrm{RS}(T))$$

and
$$(y \cdot e_\tau = \varepsilon(\tau) y \qquad \text{for every } \tau \in \mathrm{CS}(T)),$$
then y is a scalar multiple of C_λ. Applying this to $y = xC_\lambda$, we conclude that xC_λ is a scalar multiple of C_λ (since we have (1) for every $\sigma \in \mathrm{RS}(T)$, and since we have (2) for every $\tau \in \mathrm{CS}(T)$). In other words, there exists some scalar $\gamma \in \mathbb{C}$ such that $xC_\lambda = \gamma C_\lambda$. Consider this γ.

Now, let $w \in V_\lambda$. Then, $w \in V_\lambda = \mathbb{C}[\mathcal{S}_n] C_\lambda$. In other words, there exists some $u \in \mathbb{C}[\mathcal{S}_n]$ such that $w = uC_\lambda$. Consider this u. We have $xu = ux$ (since x lies in the center of the ring $\mathbb{C}[\mathcal{S}_n]$). Now,
$$\left(\rho^\lambda(x)\right)(w) = x \underbrace{w}_{=uC_\lambda} = \underbrace{xu}_{=ux} C_\lambda = u \underbrace{xC_\lambda}_{=\gamma C_\lambda} = \gamma \underbrace{uC_\lambda}_{=w} = \gamma w = \left(\gamma \operatorname{Id}_{V_\lambda}\right)(w).$$

Now, forget that we fixed w. We thus have shown that $\left(\rho^\lambda(x)\right)(w) = \left(\gamma \operatorname{Id}_{V_\lambda}\right)(w)$ for each $w \in V_\lambda$. In other words, $\rho^\lambda(x) = \gamma \operatorname{Id}_{V_\lambda}$. But the definition of the character χ^λ yields
$$\chi^\lambda(x) = \operatorname{Tr}\left(\underbrace{\rho^\lambda(x)}_{=\gamma \operatorname{Id}_{V_\lambda}}\right) = \operatorname{Tr}\left(\gamma \operatorname{Id}_{V_\lambda}\right) = \gamma \underbrace{\operatorname{Tr}\left(\operatorname{Id}_{V_\lambda}\right)}_{=\dim(V_\lambda)} = \gamma \dim(V_\lambda).$$

Now, the definition of $\widehat{\chi}^\lambda$ yields $\widehat{\chi}^\lambda(x) = \dfrac{\chi^\lambda(x)}{\dim(V_\lambda)} = \gamma$ (since $\chi^\lambda(x) = \gamma \dim(V_\lambda)$). Now,
$$\rho^\lambda(x) = \underbrace{\gamma}_{=\widehat{\chi}^\lambda(x)} \operatorname{Id}_{V_\lambda} = \widehat{\chi}^\lambda(x) \operatorname{Id}_{V_\lambda}.$$

This proves Lemme 3.4. □

- **Page 40, Démonstration de Proposition 3.3:** After the word "Ainsi", replace "$\mathrm{Ch}_{(2,1)}(\lambda)$" by "$\mathrm{Ch}_{(2)}(\lambda) \cdot \mathrm{Ch}_{(2)}(\lambda)$".
- **Page 41:** How exactly do you mean the word "planaire" in "$\{M \in \mathcal{M}_k,\ M \text{ planaire}\}$"? Do you mean that the underlying graph is planar, or does there have to be a planar embedding under which the "système de rotation" goes counterclockwise around each vertex?

  (This asked, you don't seem to use this characterization of $\mathcal{T}_k$ anywhere; thus, maybe it is better left to some footnote, where it is less likely to trip up readers?)
- **Page 41:** Replace "Cette famille correspond aux factorisations $\sigma\tau = \zeta_k$" by "Cette famille correspond aux factorisations $\tau\sigma = \zeta_k$".

- **Page 41, Définition 3.8:** Remove the first option ("soit un sommet noir seul •,"). Indeed, a node with no children is already covered by the second option (since the "liste ordonnée d'arbres plans enracinés" can be an empty list). That said, of course, it's worth pointing out that the list is allowed to be empty, which allows you to construct "the first rooted plane tree" out of nothing.

- **Pages 41–42, Démonstration de Lemme 3.10:** I find this somewhat too brief. Maybe it is worth at least pointing out why the map constructed is unicellular (intuitively, walking along the edges starting from the chosen edge performs something like a left-to-right depth-first traversal of the tree; I am not sure about the details myself).

- **Page 42:** Maybe replace "en restreignant dans $\mathrm{Ch}_{(l-1)}$" by "en restreignant dans $\mathrm{Ch}_{(l-1)} = (-1)^{l-1} \sum_{M \in \mathcal{M}_{l-1}} (-1)^{|V_\bullet(M)|} N_{G(M)}$". (Otherwise it is not clear where there is a sum in $\mathrm{Ch}_{(l-1)}$; after all, the definition $\mathrm{Ch}_{(l-1)}$ involves no summations.)

- **Page 42, Définition 3.12:** I would replace "$\sum_{\substack{T \text{ arbre plan enraciné} \\ \text{à } l \text{ sommets} \\ (l-1 \text{ arêtes})}}$" by "$\sum_{T \in \mathcal{T}_{l-1}}$". (In writing "$\sum_{\substack{T \text{ arbre plan enraciné} \\ \text{à } l \text{ sommets} \\ (l-1 \text{ arêtes})}} (-1)^{|V_\bullet(T)|} N_{G(T)}$", you are tacitly using the bijection from Lemme 3.10 to identify plane binary trees with elements of $\mathcal{T}_k$. I don't think this is a good idea, seeing that your proof of Lemme 3.10 is rather sketchy and that the identification creates fertile ground for confusion, as you point out yourself in Remarque 3.11. My suggested replacement avoids this identification. As far as I understand, you don't really use the "arbres plans enracinés" anywhere outside Lemme 3.10; therefore it seems unreasonable to invoke them in Définition 3.12.)

- **Page 42, Proposition 3.14:** I would replace "vect (Ch_μ)" by "vect $(\mathrm{Ch}_\mu) = \Lambda$", so that the reader won't have to wonder why you are suddenly circumscribing Λ.

- **Page 42, Proposition 3.14:** Is this obvious, or are you intending to prove this later, or are you omitting the proof? Maybe a few words could clear this up.

- **Page 61, Lemme 4.17:** Instead of "τ_i et σ_i permutent les mêmes éléments que π_i", I would suggest "τ_i et σ_i sont des permutations de l'ensemble des éléments du cycle π_i" (or whatever is the closest correct French sentence to this). Your formulation falsely suggests that τ_i and σ_i actually have to nontrivially permute the elements of π_i (i.e., cannot leave any of them fixed).

- **Page 61, Remarque 4.18:** I would add the extra observation (between the first observation and the second observation) that

$$C(\sigma) = \bigsqcup_{i=1}^{s} C(\sigma_i) \qquad \text{and} \qquad C(\tau) = \bigsqcup_{i=1}^{s} C(\tau_i)$$

(since the permutations $\sigma_1, \sigma_2, \ldots, \sigma_s$ act on disjoint sets, and so do the permutations $\tau_1, \tau_2, \ldots, \tau_s$). This is a useful step towards the second observation you make (about decomposing the graph $G_{\sigma,\tau}$ as a disjoint union).

10. An approach to Proposition 3.38

Here is something more speculative. I am just as vexed as you seem to be about the fact that no elementary proof of Proposition 3.38 is known. To me, it says that we are missing some insights about matchings in bipartite graphs that perhaps could shed a new light on matching theory. (You don't actually mention matchings, but transport polytopes are closely related to them, and the "expanseur" condition has a well-known interpretation in terms of existence of matchings at least in the particular case where each $v \in V_\circ$ satisfies $h(\circ) = 2$.) I have been trying for a while to find the missing insights (independently from your [DFŚ 10] paper because I don't speak the language of polytopes), and I have something that looks like a partial result. Maybe you see a way to complete it?

I call a decorated graph (G, h) a *semi-expander* if each subset V of $V_\circ$ satisfies $\left|\overline{\mathcal{V}(V)}\right| \geq \sum_{\circ \in V} h(\circ)$. Thus, expanders[1] are always semi-expanders, but not vice versa.

If $V_\circ$ and $V_\bullet$ are two finite sets, and if k is a nonnegative integer, then I introduce the following notations:

- A map $\varphi : V_\circ \sqcup V_\bullet \to \mathbb{N}^*$ is *k-packed* if $\varphi(V_\circ) = \varphi(V_\bullet) = \{1, 2, \ldots, k\}$. (Notice that a k-packed map $\varphi : V_\circ \sqcup V_\bullet \to \mathbb{N}^*$ can exist only if $|V_\circ| \geq k$ and $|V_\bullet| \geq k$.)
- Now, let $h : V_\circ \to \{2, 3, 4, \ldots\}$ be a map. A map $\varphi : V_\circ \sqcup V_\bullet \to \mathbb{N}^*$ is said to be *h-equitable* if each $i \in \mathbb{N}^*$ satisfies $\left|\varphi^{-1}(i)\right| = \sum_{\substack{\circ \in V_\circ; \\ \varphi(\circ) = i}} h(\circ)$.

I can prove the following fact:

[1] I have taken the liberty to translate your "graph expanseur" as "expander". I don't know whether this is reasonable...

Proposition 10.1. Let (G, h) be a decorated graph that is a semi-expander. Then,

$$\sum_{k \geq 0} (-1)^k \sum_{\substack{\varphi : V_\circ \sqcup V_\bullet \to \mathbb{N}^* \text{ is } k\text{-packed} \\ \text{and } h\text{-equitable}}} [\varphi \text{ is increasing on } G]$$
$$= (-1)^{\operatorname{Con}(G)} [(G, h) \text{ is an expander}] .$$

(Here, as usual, the Iverson bracket notation $[\mathcal{A}]$ for the truth value of a statement $\mathcal{A}$ is used; you seem to call it $\delta_{\mathcal{A}}$.)

How does this proposition help?

It lets us deduce your Proposition 3.38 easily from a variant of your Proposition 3.19[2] **under the extra assumption that all the $(G \setminus E', h)$ involved are semi-expanders**. This obnoxious assumption is what stands between this approach and a self-contained proof of your Proposition 3.38.[3]

My questions thus are the following:

- **Question 1:** Can we find a formula similar to Proposition 10.1 but without requiring (G, h) to be a semi-expander? This formula would represent $(-1)^{\operatorname{Con}(G)} [(G, h) \text{ is an expander}]$ as a linear combination of terms of the form $[\varphi \text{ is increasing on } G]$ for varying $\varphi : V_\circ \sqcup V_\bullet \to \mathbb{N}^*$ (with coefficients dependent on h but independent of G). Thus, roughly speaking, we want to tell if (G, h) is an expander (and whether it has an even or an odd number of connected components) by probing lots of maps $\varphi : V_\circ \sqcup V_\bullet \to \mathbb{N}^*$ for their increasingness on G.

- **Question 2:** My proof of Proposition 10.1 is not as simple as I'd prefer it to be. It starts by showing that the set of all subsets V of $V_\circ$ satisfying $\left|\overline{\mathcal{V}(V)}\right| = \sum_{\circ \in V} h(\circ)$ is a sublattice of the Boolean lattice of all subsets of $V_\circ$

[2]Here is the variant I am talking about:

Let us use the notation of your Proposition 3.19. Let $\varphi : V_\circ \sqcup V_\bullet \to \mathbb{N}^*$ be any map. Then,

$$\sum_{E' \subseteq E_{\overrightarrow{C}}} (-1)^{|E'|} [\varphi \text{ is increasing on } G \setminus E'] = 0.$$

(Of course, the proof of this is similar to the proof of your Proposition 3.19.)

[3]Moreover, if we can somehow get this assumption lifted, then we might also be able to simplify your argument: Your approach is to prove Proposition 3.38 and then use it to define a linear functional I_ν (on the space of all polynomials in the variables $p_1, p_2, p_3, \ldots, q_1, q_2, q_3, \ldots$) that satisfies

$$I_\nu (N_G) = I_\nu (G) \qquad \text{for each bipartite graph } G. \tag{3}$$

I believe that this detour is unnecessary: We should instead be able to define the linear functional I_ν directly, and then use it to prove Proposition 3.38. Thus, the problem is shifted to defining I_ν. Again, this could be done if not for the "semi-expander" assumption in Proposition 10.1.

[4]. Then, it proceeds by observing that this sublattice is a Boolean lattice of rank $\operatorname{Con}(G)$ if (G, h) is an expander, and otherwise has some behavior[5] which quickly dooms it to having Möbius function 0; therefore, its Möbius function is always $(-1)^{\operatorname{Con}(G)} [(G, h) \text{ is an expander}]$. Finally, Proposition 10.1 is obtained from this by comparing this with another evaluation of the Möbius function, viz. Philip Hall's theorem (Proposition 3.8.5 in Stanley's *Enumerative Combinatorics volume 1*), and re-encoding it via decreasing maps.

I suspect that there should be an easier proof, using a sign-reversing involution with few fixed points (the way such identities are usually proven in combinatorics). The involution should probably act on the set

$$\bigsqcup_{k\geq 0} \{h\text{-equitable } k\text{-packed maps } \varphi : V_\circ \sqcup V_\bullet \to \mathbb{N}^* \text{ that are increasing on } G\}.$$

I furthermore would not be surprised if the involution uses what Benedetti and Sagan (arXiv:1410.5023v4) call the "split-merge technique": i.e.,

- under some conditions (the "splitting case"), the involution maps a k-packed map φ to a $(k+1)$-packed map φ' given by

$$\varphi'(v) = \begin{cases} \varphi(v), & \text{if } \varphi(v) < i; \\ i, & \text{if } \varphi(v) = i \text{ and } \mathcal{C}(v) \text{ holds;} \\ i+1, & \text{if } \varphi(v) = i \text{ and } \mathcal{C}(v) \text{ does not hold;} \\ \varphi(v)+1, & \text{if } \varphi(v) > i \end{cases} \qquad \text{for all } v \in V_\circ \sqcup V_\bullet$$

  where i is some number and $\mathcal{C}$ is some predicate;

- under some other conditions (the "merging case"), the involution maps a k-packed map φ to a $(k-1)$-packed map φ' given by

$$\varphi'(v) = \begin{cases} \varphi(v), & \text{if } \varphi(v) < i; \\ i, & \text{if } \varphi(v) \in \{i, i+1\}; \\ \varphi(v)-1, & \text{if } \varphi(v) > i+1 \end{cases} \qquad \text{for all } v \in V_\circ \sqcup V_\bullet$$

  where i is some number;

- in the remaining (rare) cases, the involution sends φ to φ.

If we are lucky, then this involution might generalize to an answer to Question 1 as well.

[4]This is actually a particular case of a known fact about cuts in networks: See, e.g., Exercise 7 on UMN Math 5707 Spring 2017 homework set #5.

[5]Specifically: It is a bounded lattice that is not complemented.