A left ideal Gelfand model for the symmetric group

Sarah Brauner, Patricia Commins,
Darij Grinberg, Franco Saliola

working draft, March 19, 2026

Contents

This is a working draft. It will need a lot of editing. Some of the lemmas are probably redundant or can be found in the literature. Sections marked with (*) contain ideas and to-do lists rather than actual results, and will likely be removed (if the ideas don't work out) or elaborated upon (if they do).

This paper will likely be split into two eventually.

1. Introduction

Ever since Alfred Young's legendary "On quantitative substitutional analysis" paper series [Young77], the group algebra $\mathbb{Q}[S_n]$ of the symmetric group S_n has been studied as the origin of permutational symmetries on vectorial objects such as polynomials and tensors. While the representations of the algebra (i.e., of the symmetric group) are well-understood at least in characteristic 0, there has come forth a steady stream of remarkable families of elements of $\mathbb{Q}[S_n]$ that exhibit surprising, nontrivial and often deep properties. We shall refer to these elements as *shuffles*, as many of them (viz., those with nonnegative coefficients) can be interpreted as random ways to shuffle a deck of cards. Some of these shuffles actually originated in this card-shuffling context, such as the top-to-random, bottom-to-random and Gilbert–Shannon–Reeds shuffles (see [DiaFul23] for these and many others), though the latter have independently arisen in the study of Hochschild cohomology of commutative algebras (see [GerSch91] and [DiaFul23, §5.6][1]). Others originate, e.g., in Lie theory (the Dynkin and Klyachko idempotents [Reuten93, Theorem 8.16], [BleLau92]) or in real algebraic geometry ([KarPur23, §3]).

Two particularly mysterious families of shuffles in $\mathbb{Q}[S_n]$ were constructed in the memoir [ReSaWe11] by Reiner, Saliola and Welker. The first of them – the *random-to-random shuffles*, denoted $\nu_{(k,1^{n-k})}$ in [ReSaWe11] – has since seen its main properties proved ([DieSal18], [Lafren19], [AFBCCL24], [BCGS25]). We shall here study the second, which we call the *dyadic shuffles* ($\nu_{(2^k,1^{n-2k})}$ in [ReSaWe11]). Despite these two families being constructed in parallel ways in [ReSaWe11] (as "symmetrizations" of BHR shuffles) and – to some extent – having analogous properties, we have found their similitude to be shallow, and the proofs we will give use

[1] Not all of us agree with the characterization of Hochschild cohomology as "an esoteric part of modern algebra".

methods entirely different from those that have worked for the first family. (In particular, while the first family can be q-deformed into the Hecke algebra, no such deformation has been found for the second family.)

The easiest way to define these dyadic shuffles is by

$$\mathcal{S}_{n,k} := \sum_{w \in S_n} \operatorname{incmat}_k(w)\, w \in \mathbb{Q}[S_n]$$

for all $n, k \in \mathbb{N}$, where $\operatorname{incmat}_k(w)$ denotes $\dfrac{1}{k!}$ times the number of ways to choose $2k$ distinct elements $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, n\}$ such that each $p \in \{1, 2, \ldots, k\}$ satisfies $i_p < j_p$ and $w(i_p) < w(j_p)$. As an example, using one-line notation, $\operatorname{incmat}_2(13254) = 10$. (This is a restatement of Definition 4.1.1 below, except that we shall use an arbitrary field $\mathbf{k}$ of characteristic 0 instead of the $\mathbb{Q}$ here.) Clearly, $\mathcal{S}_{n,0}$ is a known central element (the sum of all permutations in S_n), while $\mathcal{S}_{n,k}$ is just 0 when $2k > n$. It is the intermediate shuffles $\mathcal{S}_{n,1}, \mathcal{S}_{n,2}, \ldots, \mathcal{S}_{n,\lfloor n/2 \rfloor}$ that are interesting.

The major surprises – already proved in [ReSaWe11, Theorem 1.6] – are that

- **(a)** these shuffles $\mathcal{S}_{n,0}, \mathcal{S}_{n,1}, \mathcal{S}_{n,2}, \ldots$ all commute, and
- **(b)** each of them has nonnegative integer eigenvalues (when acting by left or right multiplication on $\mathbb{Q}[S_n]$ or on any representation of $\mathbb{Q}[S_n]$).

The proofs in [ReSaWe11] rely fundamentally on a *Gelfand model* of S_n: a representation of S_n that can be decomposed as a direct sum of all irreducible representations (i.e., all Specht modules $\mathcal{S}^\lambda$ with λ a partition of n), each appearing exactly once. The specific model constructed in [ReSaWe11, §5.1] is the Whitney homology of a poset, or rather a copy thereof in $\mathbb{Q}[S_n]$ identified using character computations.

In the present work, we shall simplify this by finding a much simpler Gelfand model of S_n: Namely, our Gelfand model is

$$\mathcal{G} := \operatorname{span}\left(G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k} \ \mid\ i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in [n] \text{ are distinct}\right) \subseteq \mathbb{Q}[S_n],$$

where

$$G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k} := \sum_{\substack{w \in S_n; \\ w(i_s) < w(j_s) \text{ for all } s \in \{1,2,\ldots,k\}}} w^{-1} \in \mathbb{Q}[S_n].$$

This is a left ideal of $\mathbb{Q}[S_n]$, thus a representation, and we will show that it is a Gelfand model using the basic theory of Specht modules. It is related to the classical Gelfand model of Kodiyalam–Verma [KodVer04] and Adin–Postnikov–Roichman [AdPoRo08, Theorem 1.2]; in fact, it has a filtration whose associated graded object is canonically isomorphic to the latter. More usefully for us, it contains all the dyadic shuffles $\mathcal{S}_{n,i}$, and even better, we have $\mathcal{S}_{n,i} \in \mathcal{G}^* \mathcal{G}$, where $*$ denotes the antipode map of $\mathbb{Q}[S_n]$ (the linear map sending each permutation to its inverse). This alone is sufficient to prove the above properties **(a)** and **(b)** by general properties of Gelfand models. In fact, our paper begins with a general study of Gelfand models of S_n (most of them generalizable to finite-dimensional split semisimple algebras, though we don't elaborate on this).

We then move on to proving new properties of the dyadic shuffles. Among them are the identities

$$\binom{n-2(k-1)}{2}\mathcal{S}_{n,k-1} = \mathcal{S}_{n,k}\left(\mathcal{B}_n - (n-2k)\right) = \left(\mathcal{B}_n^* - (n-2k)\right)\mathcal{S}_{n,k},$$

where $\mathcal{B}_n$ is the bottom-to-random shuffle (i.e., the sum of the cycles $(n, n-1, \ldots, j)$ for all $j \in \{1, 2, \ldots, n\}$), and

$$w_0\mathcal{S}_{n,k} = \mathcal{S}_{n,k}w_0 = \sum_{i=0}^{k}(-1)^i\binom{n-2i}{2k-2i}\frac{(2k-2i)!}{2^{k-i}(k-i)!}\mathcal{S}_{n,i},$$

where $w_0 \in S_n$ is the permutation sending $1, 2, \ldots, n$ to $n, n-1, \ldots, 1$.

Finally (TODO: this part needs to be written up), we take a bite off one of the open problems of [ReSaWe11], namely the computation of the eigenvalues of the $\mathcal{S}_{n,k}$ acting on Specht modules $\mathcal{S}^\lambda$. For general reasons, at most one of these eigenvalues is nonzero for each choice of k and λ (which means that it can also be viewed as the trace of $\mathcal{S}_{n,k}$ acting on $\mathcal{S}^\lambda$, that is, as its value under the irreducible character corresponding to λ); this nonzero value is a positive integer. But a combinatorial or even just manifestly positive formula for this value is far from known (see [ReSaWe11, Problem 5.5]). Lafrenière conjectured a formula [Lafren19, Conjecture 142] for all *hook-shaped* partitions λ as well as general recursions which hold for all λ. We prove each of her conjectures (TODO).

Plotlines: After this introduction, the paper is structured as follows:

1. In Section 2, we prove general properties of multiplicity-free representations and Gelfand models of S_n over a field $\mathbf{k}$ of characteristic 0. We suspect that these are folklore to the experts, but we could not find them explicitly stated in the literature, whence we lay out the (fairly easy) proofs here.

2. In Section 3, we construct the left ideal $\mathcal{G}$ of $\mathbf{k}[S_n]$ and prove that it is a Gelfand model of S_n. We furthermore study its canonical filtration and use it to recover the results of [KodVer04] and [AdPoRo08, Theorem 1.2].

3. In Section 4, we introduce the dyadic shuffles $\mathcal{S}_{n,k}$ and prove various identities holding for them. We reprove their commutativity and the nonnegative integrality of their eigenvalues using the Gelfand model $\mathcal{G}$.

4. In Section **??** (TODO), we examine the eigenvalues of the $\mathcal{S}_{n,k}$. First, we prove Lafrenière's conjectures on recursions satisfied by the eigenvalues of $\mathcal{S}_{n,k}$ as k grows. Then, we prove her conjecture for the eigenvalues of $\mathcal{S}_{n,k}$ on hook-shaped Specht modules.

Acknowledgements: We thank Nadia Lafrenière for prompting this research in her thesis [Lafren19]. We also thank Lafrenière and Trevor Karn for many discussions on the dyadic family. Much of this work was done at the MFO Oberwolfach in Fall 2024 (Oberwolfach Research Fellows, 2442p) and at the ICERM Providence in Fall 2025 (program "Categorification and Computation in Algebraic Combinatorics"). We thank both of these institutes for their hospitality.

Notations. We shall follow the notations of [Grinbe24] whenever possible (but not in the notations for the antipode and the dual space). We fix a field $\mathbf{k}$ of characteristic 0. Rings and algebras are associative and unital by default. We let n be a nonnegative integer, and S_n be the symmetric group of the set $[n] := \{1, 2, \ldots, n\}$. The group algebra $\mathbf{k}[S_n]$ of this group will be denoted by $\mathcal{A}$. The notation "$\lambda \vdash n$" means "λ is a partition of n". Young diagrams will be drawn in English notation. The Specht modules (i.e., the irreducible representations of S_n, constructed using polytabloids) are denoted by $\mathcal{S}^\lambda$, where $\lambda \vdash n$. Further notations will be introduced along the way.

2. General theorems on multiplicity-free left ideals

In this section, we shall study the general behavior of a multiplicity-free representation of S_n. Many of our results are likely to be implicit in the literature in some form or another, and generalize far beyond the case of S_n-representations, but we will restrict ourselves to S_n to keep our focus.

2.1. Basics and definitions

Let $\mathbf{k}$ be a field of characteristic 0. Let $n \in \mathbb{N}$. Consider the symmetric group S_n. Let $\mathcal{A}$ be the group algebra $\mathbf{k}[S_n]$. Thus, the representations of S_n over $\mathbf{k}$ are the left $\mathcal{A}$-modules. We recall that the $\mathbf{k}$-algebra $\mathcal{A}$ is semisimple; thus, each representation of S_n can be decomposed as a direct sum of irreducible representations. The irreducible representations of S_n are (up to isomorphism) the Specht modules $\mathcal{S}^\lambda$ corresponding to the partitions λ of n. Moreover, each Specht module $\mathcal{S}^\lambda$ is absolutely irreducible, i.e., satisfies

$$\operatorname{End}_{\mathcal{A}}\left(\mathcal{S}^\lambda\right) \cong \mathbf{k} \tag{1}$$

as $\mathbf{k}$-algebras. For the proofs of these facts, see [EGHetc11, §5.12–§5.13] (or combine [Grinbe25, Proposition 4.5] with the fact that a $\mathbf{k}$-algebra that is $\cong \mathbf{k}$ as a $\mathbf{k}$-module must also be $\cong \mathbf{k}$ as a $\mathbf{k}$-algebra). Note that every irreducible representation I of S_n is isomorphic to some Specht module $\mathcal{S}^\lambda$, and thus satisfies

$$\operatorname{End}_{\mathcal{A}} I \cong \mathbf{k} \tag{2}$$

as $\mathbf{k}$-algebras (by (1)).

A representation of S_n is said to be *multiplicity-free* if it can be written as a direct sum of mutually non-isomorphic irreducible representations of S_n. Well-known examples of multiplicity-free representations are the restrictions $\operatorname{Res}^{S_{n+1}}_{S_n} \mathcal{S}^\lambda$ of Specht modules $\mathcal{S}^\lambda$ of the symmetric group S_{n+1} (see, e.g., [Sagan01, Theorem 2.8.3 part 1]). We shall, however, focus on a different source of examples.

Let J be a left ideal of the group algebra $\mathcal{A}$ such that J is multiplicity-free as a representation of S_n (that is, J can be written as a direct sum of mutually non-isomorphic irreducible representations of S_n). We shall derive some properties of J from this. In Section 4, these will be used to provide an alternative proof of of [ReSaWe11, Theorem 1.6]; they also yield some stronger claims along the same lines.

2.2. Commutativity of $\operatorname{End}_{\mathcal{A}} J$ and $J[J, J] = 0$

The following is a well-known property of multiplicity-free modules over a split semisimple algebra:

Proposition 2.2.1. The endomorphism algebra $\operatorname{End}_{\mathcal{A}} J$ is commutative.

Proof. Let $J = I_1 \oplus I_2 \oplus \cdots \oplus I_k$ be a decomposition of J into irreducible subrepresentations (this exists because $\mathcal{A}$ is semisimple). Then, these irreducible addends $I_1, I_2, \ldots, I_k$ are pairwise non-isomorphic (since J is multiplicity-free), and thus their pairwise hom-spaces $\operatorname{Hom}_{\mathcal{A}}(I_a, I_b)$ with $a \neq b$ are all 0 (by Schur's lemma or by [Grinbe25, Proposition 4.5]). Hence, the canonical "block-diagonal" embedding

$$\prod_{j=1}^{k} \operatorname{End}_{\mathcal{A}}(I_j) \to \operatorname{End}_{\mathcal{A}}(I_1 \oplus I_2 \oplus \cdots \oplus I_k),$$

$$(f_1, f_2, \ldots, f_k) \mapsto \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_k \end{pmatrix}$$

is surjective (since any endomorphism $f \in \operatorname{End}_{\mathcal{A}}(I_1 \oplus I_2 \oplus \cdots \oplus I_k)$ that does not lie in its image would have a nontrivial projection on some $\operatorname{Hom}_{\mathcal{A}}(I_a, I_b)$ with $a \neq b$). Therefore, this embedding is a $\mathbf{k}$-algebra isomorphism. Thus, we have the $\mathbf{k}$-algebra isomorphism

$$\operatorname{End}_{\mathcal{A}} J = \operatorname{End}_{\mathcal{A}}(I_1 \oplus I_2 \oplus \cdots \oplus I_k) \cong \prod_{j=1}^{k} \underbrace{\operatorname{End}_{\mathcal{A}}(I_j)}_{\substack{\cong \mathbf{k} \\ \text{(by (2))}}} \cong \prod_{j=1}^{k} \mathbf{k} = \mathbf{k}^k.$$

Hence, $\operatorname{End}_{\mathcal{A}} J$ is commutative, since $\mathbf{k}^k$ is commutative. □

Recall that the *commutator* $[u,v]$ of two elements $u,v \in \mathcal{A}$ is defined to be $uv - vu \in \mathcal{A}$. Furthermore, the *commutator* $[U,V]$ of two $\mathbf{k}$-vector subspaces U and V of $\mathcal{A}$ is defined to be the span of all commutators $[u,v] = uv - vu$ with $u \in U$ and $v \in V$.

Theorem 2.2.2. We have $J\,[J,J] = 0$.

Proof. It suffices to show that $J\,[a,b] = 0$ for all $a \in J$ and $b \in J$. So let us fix $a \in J$ and $b \in J$. Consider the map

$$\rho_a : J \to J, \quad x \mapsto xa.$$

This map ρ_a is well-defined (since $a \in J$ and since J is a left ideal), and is left $\mathcal{A}$-linear. Thus, $\rho_a \in \operatorname{End}_{\mathcal{A}} J$. Similarly, we can define an analogous map $\rho_b \in \operatorname{End}_{\mathcal{A}} J$ that sends each x to xb. Since $\operatorname{End}_{\mathcal{A}} J$ is commutative (by Proposition 2.2.1), we have $\rho_a \circ \rho_b = \rho_b \circ \rho_a$. In other words, each $x \in J$ satisfies $xab = xba$. In other words, each $x \in J$ satisfies $x\,[a,b] = 0$. In other words, $J\,[a,b] = 0$. This proves Theorem 2.2.2. □

Note that the "opposite" version $[J,J]\,J = 0$ of Theorem 2.2.2 does not hold in general.

Remark 2.2.3. More generally, Proposition 2.2.1 and Theorem 2.2.2 hold if we replace $\mathcal{A}$ by any finite-dimensional semisimple $\mathbf{k}$-algebra R with the property that all irreducible left R-modules satisfy (2). Indeed, the above proofs use nothing but the conditions we just listed.

2.3. Left ideals of $\mathcal{A}$ are generated by idempotents

For our next vanishing commutator theorem, we need a general property of left ideals of group algebras:

Theorem 2.3.1. Let G be a finite group. Let I be a left ideal of the group algebra $\mathbf{k}[G]$. (We are still assuming that $\mathbf{k}$ has characteristic 0.) Then, there exists an idempotent $e \in I$ such that $I = \mathbf{k}[G] \cdot e$.

Proof. A left ideal of $\mathbf{k}[G]$ is a left G-subrepresentation of $\mathbf{k}[G]$. Thus, Maschke's theorem shows that there is a left $\mathbf{k}[G]$-linear projection $\pi : \mathbf{k}[G] \to I$. Consider this π.

Set $e := \pi(1)$ (where 1 is the unity of $\mathbf{k}[G]$). Then, $e \in I$. Thus, $\mathbf{k}[G] \cdot e \subseteq I$ (since I is a left ideal). Furthermore, each $x \in I$ satisfies

$$\begin{aligned} x &= \pi(x) && (\text{since } \pi \text{ is a projection onto } I) \\ &= \pi(x1) = x \underbrace{\pi(1)}_{=e} && (\text{since } \pi \text{ is left } \mathbf{k}[G]\text{-linear}) \\ &= xe. \end{aligned}$$

Applying this to $x = e$, we find $e = ee$. This shows that e is idempotent. Furthermore, $I \subseteq \mathbf{k}[G] \cdot e$ (since we showed that each $x \in I$ satisfies $x = xe \in \mathbf{k}[G] \cdot e$). Combined with $\mathbf{k}[G] \cdot e \subseteq I$, this yields $I = \mathbf{k}[G] \cdot e$. Thus, the proof of Theorem 2.3.1 is complete. □

Remark 2.3.2. More generally, Theorem 2.3.1 holds if we replace $\mathbf{k}[G]$ by any finite-dimensional semisimple $\mathbf{k}$-algebra (since Maschke's theorem is a general property of such algebras).

2.4. Commutativity of J^*J and of K^*J

Next, let us consider the *antipode* of the group algebra $\mathbf{k}[S_n]$. This is the $\mathbf{k}$-linear map from $\mathbf{k}[S_n]$ to $\mathbf{k}[S_n]$ that sends each $w \in S_n$ to w^{-1}. This antipode will be denoted by $x \mapsto x^*$. It is known to be an involutive $\mathbf{k}$-algebra anti-automorphism (see [Grinbe24, §3.11.4], where this map is denoted by S). We let J^* denote the image of the left ideal J under this antipode. We now claim the following:

Theorem 2.4.1. We have $[J^*J,\ J^*J] = 0$.

More generally:

Theorem 2.4.2. Let K be a further left ideal of $\mathcal{A}$ that is multiplicity-free as a representation of S_n. Then, $[K^*J,\ K^*J] = 0$.

Before we prove this, we pave our way with several well-intended lemmas. The first is a well-known basic fact about rings (see, e.g., [EGHetc11, Lemma 5.13.4]):

Lemma 2.4.3. Let R be a ring, and let M be a left R-module. Let $e \in R$ be idempotent. Then, $\operatorname{Hom}_R(Re, M) \cong eM$ as abelian groups (where Hom_R denotes the hom-space of left R-modules).

Moreover, if R is a $\mathbf{k}$-algebra, then this isomorphism is an isomorphism of $\mathbf{k}$-vector spaces.

Proof of Lemma 2.4.3 (sketched). It is easy to see that the maps

$$\begin{aligned} \operatorname{Hom}_R(Re, M) &\to eM, \\ f &\mapsto f(e) \end{aligned}$$

and

$$\begin{aligned} eM &\to \operatorname{Hom}_R(Re, M), \\ x &\mapsto (Re \to M,\ r \mapsto rx) \end{aligned}$$

are $\mathbb{Z}$-linear and mutually inverse. Thus, they are isomorphisms of abelian groups. When R is a $\mathbf{k}$-algebra, they are furthermore $\mathbf{k}$-linear and thus are isomorphisms of $\mathbf{k}$-vector spaces. □

Lemma 2.4.4. Let V and W be two left ideals of $\mathcal{A}$ that are non-isomorphic and irreducible as representations of S_n. Then, $VW = 0$.

Proof. The irreducible representations V and W are non-isomorphic. Hence, Schur's lemma shows that $\operatorname{Hom}_{\mathcal{A}}(V, W) = 0$. In particular, for any $x \in W$, the map

$$V \to W,$$
$$y \mapsto yx$$

must be 0 (since this map is left $\mathcal{A}$-linear and thus belongs to $\operatorname{Hom}_{\mathcal{A}}(V, W) = 0$). But this means that $yx = 0$ for all $y \in V$ and $x \in W$. In other words, $VW = 0$, qed. □

Lemma 2.4.5. Let $e \in \mathcal{A}$ be an idempotent such that the left $\mathcal{A}$-module $\mathcal{A}e$ is irreducible. Then, the left $\mathcal{A}$-module $\mathcal{A}e^*$ is also irreducible.

Proof. Assume the contrary. Then, the left $\mathcal{A}$-module $\mathcal{A}e^*$ is not irreducible.

First, we observe that $\mathcal{A}e \neq 0$ (since $\mathcal{A}e$ is irreducible), so that $e \neq 0$ and therefore $e^* \neq 0$ (since the antipode map $\mathcal{A} \to \mathcal{A},\ x \mapsto x^*$ is bijective). Therefore, $\mathcal{A}e^* \neq 0$.

But we assumed that $\mathcal{A}e^*$ is not irreducible. Hence, $\mathcal{A}e^*$ is not indecomposable (by the Maschke theorem). Since $\mathcal{A}e^* \neq 0$, this shows that $\mathcal{A}e^*$ is a direct sum $P \oplus Q$ of two nontrivial left $\mathcal{A}$-submodules P and Q. Consequently, the hom-space $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}e^*, \mathcal{A}e^*)$ has dimension ≥ 2 (since it contains at least the projections onto P and Q).

But Lemma 2.4.3 (applied to $R = \mathcal{A}$ and $M = \mathcal{A}e$) yields $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}e,\ \mathcal{A}e) \cong e\mathcal{A}e$. Thus,

$$e\mathcal{A}e \cong \operatorname{Hom}_{\mathcal{A}}(\mathcal{A}e,\ \mathcal{A}e) = \operatorname{End}_{\mathcal{A}}(\mathcal{A}e) \cong \mathbf{k}$$

(by (2), applied to $V = \mathcal{A}e$, since $\mathcal{A}e$ is irreducible).

Applying the antipode anti-isomorphism $\mathcal{A} \to \mathcal{A},\ x \mapsto x^*$ to $e\mathcal{A}e \cong \mathbf{k}$, we obtain $e^*\mathcal{A}e^* \cong \mathbf{k}$. Furthermore, e is idempotent, so that e^* is idempotent as well (since $x \mapsto x^*$ is an algebra anti-morphism). Thus, Lemma 2.4.3 yields $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}e^*,\ \mathcal{A}e^*) \cong e^*\mathcal{A}e^* \cong \mathbf{k}$, so that $\dim(\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}e^*,\ \mathcal{A}e^*)) = \dim \mathbf{k} = 1$. This contradicts the fact that the hom-space $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}e^*, \mathcal{A}e^*)$ has dimension ≥ 2. This contradiction shows that our assumption was false. Hence, Lemma 2.4.5 is proved. □

Lemma 2.4.6. Let V and W be two left ideals of $\mathcal{A}$ that are irreducible as representations of S_n. Then, $\dim(V^*W) \leq 1$.

Proof. We know that V is a left ideal of the algebra $\mathcal{A} = \mathbf{k}[S_n]$. Hence, Theorem 2.3.1 (applied to $G = S_n$ and $I = V$) shows that there exists an idempotent $e \in V$ such that $V = \mathbf{k}[S_n] \cdot e$. Consider this e. Then, $V = \mathbf{k}[S_n] \cdot e = \mathcal{A}e$. Hence, the left $\mathcal{A}$-module $\mathcal{A}e$ is irreducible (since V is irreducible). Thus, Lemma 2.4.5 yields that the left $\mathcal{A}$-module $\mathcal{A}e^*$ is irreducible.

The element e is idempotent. Hence, e^* is idempotent as well (since the antipode $\mathcal{A} \to \mathcal{A},\ x \mapsto x^*$ is an algebra anti-morphism). Therefore, Lemma 2.4.3 (applied to $\mathcal{A}$, W and e^* instead of R, M and e) yields $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}e^*,\ W) \cong e^*W$ as $\mathbf{k}$-vector spaces.

But all irreducible left $\mathcal{A}$-modules I satisfy $\operatorname{Hom}_{\mathcal{A}}(I, I) = \operatorname{End}_{\mathcal{A}} I \cong \mathbf{k}$ (by (2)) and thus $\dim(\operatorname{Hom}_{\mathcal{A}}(I, I)) = 1$. Hence, any two irreducible left $\mathcal{A}$-modules P and Q satisfy $\dim(\operatorname{Hom}_{\mathcal{A}}(P, Q)) \leq 1$ (indeed, $\dim(\operatorname{Hom}_{\mathcal{A}}(P, Q))$ is 1 if $P \cong Q$ and 0 otherwise). Thus, in particular, $\dim(\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}e^*,\ W)) \leq 1$ (since $\mathcal{A}e^*$ and W are two irreducible left $\mathcal{A}$-modules). In view of $\operatorname{Hom}_{\mathcal{A}}(\mathcal{A}e^*,\ W) \cong e^*W$, we can rewrite this as $\dim(e^*W) \leq 1$.

However, $\mathcal{A}W = W$ (since W is a left ideal of $\mathcal{A}$). From $V = \mathcal{A}e$, we obtain $V^* = (\mathcal{A}e)^* = e^*\mathcal{A}$, so that $V^*W = e^* \underbrace{\mathcal{A}W}_{=W} = e^*W$. Thus, $\dim(V^*W) = \dim(e^*W) \leq 1$. This proves Lemma 2.4.6. □

Proof of Theorem 2.4.2. Decompose the S_n-representation J as a direct sum of irreducible subrepresentations:

$$J = J_1 \oplus J_2 \oplus \cdots \oplus J_k.$$

These subrepresentations $J_1, J_2, \ldots, J_k$ are pairwise non-isomorphic (since J is multiplicity-free). Likewise, decompose the S_n-representation K as a direct sum of pairwise non-isomorphic irreducible subrepresentations:

$$K = K_1 \oplus K_2 \oplus \cdots \oplus K_\ell.$$

Now, we have

$$J = J_1 \oplus J_2 \oplus \cdots \oplus J_k = \sum_{i=1}^{k} J_i \tag{3}$$

and similarly

$$K = \sum_{j=1}^{\ell} K_j.$$

The latter equality entails (by applying the antipode to both sides)

$$K^* = \sum_{j=1}^{\ell} K_j^*. \tag{4}$$

Multiplying this with (3), we find

$$K^*J = \left(\sum_{j=1}^{\ell} K_j^*\right)\left(\sum_{i=1}^{k} J_i\right) = \sum_{j,i} K_j^* J_i.$$

Hence, in order to achieve our goal (of showing that $[K^*J,\ K^*J] = 0$), it shall suffice to prove that

$$\left[K_j^* J_i,\ K_v^* J_u\right] = 0 \tag{5}$$

for all $i, u \in \{1, 2, \ldots, k\}$ and $j, v \in \{1, 2, \ldots, \ell\}$. So let us prove this.

Fix $i, u \in \{1, 2, \ldots, k\}$ and $j, v \in \{1, 2, \ldots, \ell\}$. We must prove (5). Our proof depends on some cases:

- *Case 1:* Assume that $i \neq u$. Then, the left $\mathcal{A}$-modules J_i and J_u are not isomorphic (since $J_1, J_2, \ldots, J_k$ are pairwise non-isomorphic). Thus, Lemma 2.4.4 (applied to $V = J_i$ and $W = J_u$) yields $J_i J_u = 0$. The same argument (with the roles of i and u interchanged) yields $J_u J_i = 0$. Now,

$$\left[K_j^* J_i,\ K_v^* J_u\right] \subseteq K_j^* J_i \underbrace{K_v^* J_u}_{\substack{\subseteq \mathcal{A} J_u = J_u \\ \text{(since } J_u \text{ is a} \\ \text{left ideal)}}} + K_v^* J_u \underbrace{K_j^* J_i}_{\substack{\subseteq \mathcal{A} J_i = J_i \\ \text{(since } J_i \text{ is a} \\ \text{left ideal)}}} \subseteq K_j^* \underbrace{J_i J_u}_{=0} + K_v^* \underbrace{J_u J_i}_{=0} = 0.$$

  In other words, $\left[K_j^* J_i,\ K_v^* J_u\right] = 0$. Hence, (5) is proved in Case 1.

- *Case 2:* Assume that $j \neq v$. Then, the left $\mathcal{A}$-modules K_j and K_v are not isomorphic (since $K_1, K_2, \ldots, K_\ell$ are pairwise non-isomorphic). Hence, ma 2.4.4 (applied to $V = K_j$ and $W = K_v$) yields $K_j K_v = 0$. Applying the antipode (i.e., the algebra anti-automorphism $a \mapsto a^*$) to this equality, we obtain $K_j^* K_v^* = 0$ (since $\left(K_j K_v\right)^* = K_v^* K_j^*$). The same argument (with the roles of j and v interchanged) yields $K_v^* K_j^* = 0$. Note that K_j is a left ideal of $\mathcal{A}$; hence, $\mathcal{A} K_j = K_j$. Applying the antipode to this equality, we obtain $K_j^* \mathcal{A} = K_j^*$. Similarly, $K_v^* \mathcal{A} = K_v^*$. Now,

$$\left[K_j^* J_i,\ K_v^* J_u\right] \subseteq \underbrace{K_j^* J_i}_{\subseteq K_j^* \mathcal{A} = K_j^*} K_v^* J_u + \underbrace{K_v^* J_u}_{\subseteq K_v^* \mathcal{A} = K_v^*} K_j^* J_i \subseteq \underbrace{K_j^* K_v^*}_{=0} J_u + \underbrace{K_v^* K_j^*}_{=0} J_i = 0.$$

  In other words, $\left[K_j^* J_i,\ K_v^* J_u\right] = 0$. Hence, (5) is proved in Case 2.

- *Case 3:* Assume that $i = u$ and $j = v$. Lemma 2.4.6 (applied to $V = K_j$ and $W = J_i$) yields that $\dim\left(K_j^* J_i\right) \leq 1$ (since K_j and J_i are left ideals that are irreducible as representations of S_n). Thus, any two elements of $K_j^* J_i$ are scalar multiples of each other (unless one of them is 0), and therefore commute. In other words, $\left[K_j^* J_i,\ K_j^* J_i\right] = 0$. Since $i = u$ and $j = v$, we can rewrite this as $\left[K_j^* J_i,\ K_v^* J_u\right] = 0$. Hence, (5) is proved in Case 3.

We have now proved (5) in all three cases. Hence, (5) always holds. As explained above, this yields $[K^* J,\ K^* J] = 0$ and thus proves Theorem 2.4.2. □

Remark 2.4.7. More generally, Theorem 2.4.2 and the lemmas we used in its proof hold if we replace $\mathcal{A}$ by any finite-dimensional semisimple $\mathbf{k}$-algebra R with the property that all irreducible left R-modules satisfy (2) and with an algebra anti-automorphism $R \to R,\ a \mapsto a^*$.

2.5. Some 1-dimensional spaces

Recall that J is a left ideal of $\mathcal{A}$ that is multiplicity-free as an S_n-representation.

Lemma 2.5.1. Let K be an irreducible representation of S_n. Then, $\dim\left(\operatorname{Hom}_{\mathcal{A}}\left(J, K\right)\right) \leq 1$.

Proof. Decompose the S_n-representation J as a direct sum of irreducible subrepresentations:

$$J = J_1 \oplus J_2 \oplus \cdots \oplus J_k.$$

These subrepresentations $J_1, J_2, \ldots, J_k$ are pairwise non-isomorphic (since J is multiplicity-free). Hence, at most one of them is isomorphic to K. In other words, we are in one of the following two cases:

Case 1: There is a unique $i \in \{1, 2, \ldots, k\}$ such that $J_i \cong K$.

Case 2: There is no $i \in \{1, 2, \ldots, k\}$ such that $J_i \cong K$.

Let us first consider Case 1. In this case, there is a unique $i \in \{1, 2, \ldots, k\}$ such that $J_i \cong K$. Consider this i. Now, from $J = J_1 \oplus J_2 \oplus \cdots \oplus J_k$ and $K \cong J_i$, we obtain

$$\begin{aligned} \operatorname{Hom}_{\mathcal{A}}\left(J, K\right) &\cong \operatorname{Hom}_{\mathcal{A}}\left(J_1 \oplus J_2 \oplus \cdots \oplus J_k,\ J_i\right) \\ &\cong \bigoplus_{r=1}^{k} \operatorname{Hom}_{\mathcal{A}}\left(J_r, J_i\right) \cong \operatorname{Hom}_{\mathcal{A}}\left(J_i, J_i\right) \end{aligned}$$

(since the irreducible representations $J_1, J_2, \ldots, J_k$ are pairwise non-isomorphic, and thus the Schur lemma shows that $\operatorname{Hom}_{\mathcal{A}}\left(J_r, J_i\right) = 0$ for all $r \neq i$). Thus,

$$\operatorname{Hom}_{\mathcal{A}}\left(J, K\right) \cong \operatorname{Hom}_{\mathcal{A}}\left(J_i, J_i\right) \cong \operatorname{End}_{\mathcal{A}}\left(J_i\right) \cong \mathbf{k}$$

(by (2)) and therefore $\dim\left(\operatorname{Hom}_{\mathcal{A}}\left(J, K\right)\right) = \dim \mathbf{k} = 1$. Hence, Lemma 2.5.1 is proved in Case 1.

Let us now consider Case 2. In this case, there is no $i \in \{1, 2, \ldots, k\}$ such that $J_i \cong K$. Hence, each $i \in \{1, 2, \ldots, k\}$ satisfies $J_i \ncong K$ and therefore $\operatorname{Hom}_{\mathcal{A}}\left(J_i, K\right) = 0$ (by the Schur lemma, since both representations J_i and K are irreducible). Now, from $J = J_1 \oplus J_2 \oplus \cdots \oplus J_k$, we obtain

$$\operatorname{Hom}_{\mathcal{A}}\left(J, K\right) \cong \operatorname{Hom}_{\mathcal{A}}\left(J_1 \oplus J_2 \oplus \cdots \oplus J_k,\ K\right) \cong \bigoplus_{i=1}^{k} \underbrace{\operatorname{Hom}_{\mathcal{A}}\left(J_i, K\right)}_{\substack{=0 \\ \text{(as we just saw)}}} = 0.$$

Hence, $\dim\left(\operatorname{Hom}_{\mathcal{A}}\left(J, K\right)\right) = 0 \leq 1$. Thus, Lemma 2.5.1 is proved in Case 2. Thus the proof of Lemma 2.5.1 is complete. $\square$

Lemma 2.5.2. Let K be an irreducible representation of S_n. Let $a \in J$. Then, $\dim(aK) \leq 1$.

Proof. Theorem 2.3.1 (applied to $G = S_n$ and $I = J$) shows that there exists an idempotent $e \in J$ such that $J = \mathcal{A}e$. Consider this e.

Now, from $a \in J = \mathcal{A}e$, we obtain $a = be$ for some $b \in \mathcal{A}$, and thus

$$\dim(aK) = \dim(beK) \leq \dim(eK)$$

(since there is a surjective $\mathbf{k}$-linear map $eK \to beK$ sending each vector $v \in eK$ to $bv \in beK$).

But $J = \mathcal{A}e$, and thus $\operatorname{Hom}_{\mathcal{A}}(J, K) = \operatorname{Hom}_{\mathcal{A}}(\mathcal{A}e, K) \cong eK$ by Lemma 2.4.3. Hence,

$$\dim(\operatorname{Hom}_{\mathcal{A}}(J, K)) = \dim(eK). \tag{6}$$

But Lemma 2.5.1 yields $\dim(\operatorname{Hom}_{\mathcal{A}}(J, K)) \leq 1$. In view of (6), we can rewrite this as $\dim(eK) \leq 1$. Hence, $\dim(aK) \leq \dim(eK) \leq 1$. This proves Lemma 2.5.2. $\square$

Remark 2.5.3. More generally, Lemma 2.5.1 and Lemma 2.5.2 hold if we replace $\mathcal{A}$ by any finite-dimensional semisimple $\mathbf{k}$-algebra R with the property that all irreducible left R-modules satisfy (2).

2.6. Closer study of J^*J

We continue with our study of an arbitrary multiplicity-free left ideal J of $\mathcal{A}$. First, some general properties of representations of S_n need to be proved. Recall that if V and W are two representations of S_n, then a $\mathbf{k}$-bilinear map $f : V \times W \to U$ into some vector space U is said to be *S_n-invariant* if it satisfies

$$f(gv, gw) = f(v, w) \qquad \text{for all } g \in S_n \text{ and } v \in V \text{ and } w \in W.$$

We also let $V^\vee$ denote the dual space $\operatorname{Hom}(V, \mathbf{k})$ of any $\mathbf{k}$-vector space V (since the notation V^* is already taken for something else). If V is a representation of S_n, then $V^\vee$ canonically becomes a representation of S_n as well (known as the *(contragredient) dual* of V). See [Grinbe24, §5.19.3] for details (but beware that $V^\vee$ is denoted by V^* there).

Recall that if V and W are two $\mathbf{k}$-vector spaces, then any $\mathbf{k}$-bilinear form $f : V \times W \to \mathbf{k}$ gives rise to two $\mathbf{k}$-linear maps

$$\begin{aligned} f_L : V &\to W^\vee, \\ v &\mapsto (\text{the } \mathbf{k}\text{-linear map } W \to \mathbf{k},\ w \mapsto f(v, w)) \end{aligned}$$

and

$$\begin{aligned} f_R : W &\to V^\vee, \\ w &\mapsto (\text{the } \mathbf{k}\text{-linear map } V \to \mathbf{k},\ v \mapsto f(v, w)) \end{aligned}$$

(see [Grinbe24, §5.19.2]). If these two maps f_L and f_R are bijective (thus $\mathbf{k}$-vector space isomorphisms), the form f is said to be *nondegenerate*. Note that when V and W are finite-dimensional, this nondegeneracy is equivalent to saying that $\operatorname{Ker} f_L = 0$ and $\operatorname{Ker} f_R = 0$.

Lemma 2.6.1. Let V and W be two finite-dimensional representations of S_n over $\mathbf{k}$. Then, the S_n-invariant bilinear forms $f : V \times W \to \mathbf{k}$ form a vector space, which is isomorphic to $\operatorname{Hom}_{\mathcal{A}}(W, V^\vee)$.

Proof. Clearly, the S_n-invariant bilinear forms $f : V \times W \to \mathbf{k}$ form a vector space. Moreover, the map

$$\text{curry} : \{\text{bilinear forms } f : V \times W \to \mathbf{k}\} \to \operatorname{Hom}_{\mathbf{k}}\left(W, V^\vee\right), \\ f \mapsto f_R$$

(where f_R is defined as above) is easily seen to be a $\mathbf{k}$-vector space isomorphism (indeed, this is a linear-algebraic version of the standard "currying isomorphism", which turns a two-variable function into a one-variable function that outputs one-variable functions). Moreover, it sends the S_n-invariant bilinear forms $f : V \times W \to \mathbf{k}$ into the S_n-equivariant linear maps $W \to V^\vee$, that is, into the $\mathcal{A}$-linear maps $W \to V^\vee$ (since $\mathcal{A} = \mathbf{k}[S_n]$). This is an if-and-only-if statement (i.e., only S_n-invariant f's give rise to $\mathcal{A}$-linear f_R's). Thus, by restricting this isomorphism curry, we obtain a $\mathbf{k}$-vector space isomorphism

$$\{S_n\text{-invariant bilinear forms } f : V \times W \to \mathbf{k}\} \to \operatorname{Hom}_{\mathcal{A}}\left(W, V^\vee\right), \\ f \mapsto f_R.$$

This proves Lemma 2.6.1. $\square$

Lemma 2.6.2. Let V be any irreducible representation of S_n over $\mathbf{k}$. Then:

- **(a)** There is a nondegenerate symmetric S_n-invariant bilinear form $g : V \times V \to \mathbf{k}$.
- **(b)** Any S_n-invariant bilinear form $f : V \times V \to \mathbf{k}$ is symmetric.
- **(c)** Any S_n-invariant bilinear map $f : V \times V \to U$ into any vector space U is symmetric.

Proof. **(a)** We know that V is isomorphic to a Specht module $\mathcal{S}^\lambda$ for a partition λ of n. Consider this λ. Thus, we must find a nondegenerate symmetric S_n-invariant bilinear form $\mathcal{S}^\lambda \times \mathcal{S}^\lambda \to \mathbf{k}$. But the existence of such a form is a well-known fact (implicit in [Wildon18, proof of Corollary 3.5], where it is shown that the symmetric S_n-invariant form $\langle \cdot, \cdot \rangle$ on the Young module $\mathcal{M}^\lambda$ satisfies $\mathcal{S}^\lambda \cap \left(\mathcal{S}^\lambda\right)^\perp = 0$, whence the restriction of this form to the Specht module $\mathcal{S}^\lambda$ is nondegenerate[2]). Thus, part **(a)** is proved.

(b) The representation V is irreducible. Thus, $\operatorname{End}_{\mathcal{A}} V \cong \mathbf{k}$ (by (2)).

Part **(a)** shows that there is a nondegenerate symmetric S_n-invariant bilinear form $g : V \times V \to \mathbf{k}$. Consider this g. Since g is S_n-invariant and nondegenerate, we conclude that the dual space $V^\vee$ (with the contragredient representation of S_n) satisfies $V^\vee \cong V$ as S_n-representations (by the standard isomorphism induced by g). In other words, $V^\vee \cong V$ as $\mathcal{A}$-modules. Thus, $\operatorname{Hom}_{\mathcal{A}}(V,\ V^\vee) \cong \operatorname{Hom}_{\mathcal{A}}(V,\ V) = \operatorname{End}_{\mathcal{A}} V \cong \mathbf{k}$.

Lemma 2.6.1 shows that the S_n-invariant bilinear forms $f : V \times V \to \mathbf{k}$ form a $\mathbf{k}$-vector space, which is isomorphic to $\operatorname{Hom}_{\mathcal{A}}(V,\ V^\vee)$. Since $\operatorname{Hom}_{\mathcal{A}}(V,\ V^\vee) \cong \mathbf{k}$, this vector space is isomorphic to $\mathbf{k}$, thus 1-dimensional. Since there is at least one nondegenerate symmetric form in this space (by part **(a)**), we thus conclude that any form in this space is a scalar multiple of this one form, and thus is symmetric. This proves part **(b)**.

(c) This follows from part **(b)** by decomposing f into its coordinates. (In more details: Let U be any vector space. Let $f : V \times V \to U$ be an S_n-invariant bilinear map. Let $\eta : U \to \mathbf{k}$ be any $\mathbf{k}$-linear map. Then, the map $\eta \circ f : V \times V \to \mathbf{k}$ is an S_n-invariant bilinear form, and thus is symmetric (by part **(b)**). In other words, $\eta(f(v,w)) = \eta(f(w,v))$ for all $v, w \in V$. Since this holds for all $\mathbf{k}$-linear maps $\eta : U \to \mathbf{k}$, we thus conclude that $f(v,w) = f(w,v)$ for all $v, w \in V$ (since two vectors in U that become equal upon application of every $\mathbf{k}$-linear map to $\mathbf{k}$ must have been equal in the first place). In other words, f is symmetric.) □

Lemma 2.6.3. Let V and W be two non-isomorphic irreducible representations of S_n over $\mathbf{k}$. Then:

(a) Any S_n-invariant bilinear form $f : V \times W \to \mathbf{k}$ is 0.

(b) Any S_n-invariant bilinear map $f : V \times W \to U$ into any vector space U is 0.

Proof. **(a)** As in the proof of Lemma 2.6.2 **(b)**, we can see that $V^\vee \cong V$. Moreover, the S_n-invariant bilinear forms $f : V \times W \to \mathbf{k}$ form a $\mathbf{k}$-vector space that is isomorphic to $\operatorname{Hom}_{\mathcal{A}}(W,\ V^\vee)$ (by Lemma 2.6.1) and therefore to $\operatorname{Hom}_{\mathcal{A}}(W, V)$ (since $V^\vee \cong V$). But $\operatorname{Hom}_{\mathcal{A}}(W, V) = 0$, since V and W are irreducible and non-isomorphic. Thus, Lemma 2.6.3 follows.

(b) Follows from **(a)** just like Lemma 2.6.2 **(c)** follows from Lemma 2.6.2 **(b)**. □

[2]On the nose, the argument in [Wildon18, proof of Corollary 3.5] requires $\mathbf{k}$ to have characteristic 0; but the claim $\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp = 0$ holds more generally when $n!$ is invertible in $\mathbf{k}$. One way to see this is as follows: Assume that $n!$ is invertible in $\mathbf{k}$. Then, the partition λ is $\operatorname{char}\mathbf{k}$-regular. Hence, [Wildon18, Theorem 5.7] shows that $\mathcal{S}^\lambda \not\subseteq (\mathcal{S}^\lambda)^\perp$. Hence, $\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp \neq \mathcal{S}^\lambda$. Since $\mathcal{S}^\lambda$ is irreducible, this yields that $\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp = 0$, because $\mathcal{S}^\lambda \cap (\mathcal{S}^\lambda)^\perp$ is an S_n-subrepresentation of $\mathcal{S}^\lambda$.

Lemma 2.6.4. Let V and W be two left ideals of $\mathcal{A}$ that are non-isomorphic and irreducible as representations of S_n. Then, $V^*W = 0$.

Proof. The map

$$V \times W \to \mathcal{A},$$
$$(v, w) \mapsto v^*w$$

is bilinear and S_n-invariant (since $(gv)^* (gw) = v^* \underbrace{g^*}_{=g^{-1}} gw = v^* \underbrace{g^{-1}g}_{=1} w = v^*w$ for all $g \in S_n$ and all $v \in V$ and $w \in W$). Thus, it is 0 by Lemma 2.6.3 **(b)**. In other words, $v^*w = 0$ for all $v \in V$ and $w \in W$. In other words, $V^*W = 0$. □

Lemma 2.6.5. Let V be a left ideal of $\mathcal{A}$ that is irreducible as a representation of S_n. Then, $v^*w = w^*v$ for any $v, w \in V$.

Proof. The map

$$V \times V \to \mathcal{A},$$
$$(v, w) \mapsto v^*w$$

is bilinear and S_n-invariant (as we saw in the proof of Lemma 2.6.4). Hence, it is symmetric by Lemma 2.6.2 **(c)**. In other words, $v^*w = w^*v$ for any $v, w \in V$. □

Lemma 2.6.6. Let V be a left ideal of $\mathcal{A}$ that is irreducible as a representation of S_n. Then, $\dim (V^*V) = 1$.

Proof. Lemma 2.4.6 (applied to $W = V$) yields $\dim (V^*V) \leq 1$.

On the other hand, $V \neq 0$ (since V is irreducible), so that $V^* \neq 0$.

Now, let us decompose the semisimple algebra $\mathcal{A}$ itself into a direct sum of irreducible S_n-subrepresentations: $\mathcal{A} = I_1 \oplus I_2 \oplus \cdots \oplus I_\ell$. Of course, these subrepresentations $I_1, I_2, \ldots, I_\ell$ are left ideals of $\mathcal{A}$.

On the other hand, $V = \mathcal{A}V$ (since V is a left ideal of $\mathcal{A}$). Applying the antipode (i.e., the algebra anti-morphism $\mathcal{A} \to \mathcal{A}^*$, $a \mapsto a^*$) to this equality, we obtain

$$\begin{aligned} V^* &= V^*\mathcal{A} = V^* (I_1 \oplus I_2 \oplus \cdots \oplus I_\ell) \qquad (\text{since } \mathcal{A} = I_1 \oplus I_2 \oplus \cdots \oplus I_\ell) \\ &= V^*I_1 + V^*I_2 + \cdots + V^*I_\ell. \end{aligned}$$

Hence, at least one of the products $V^*I_1, V^*I_2, \ldots, V^*I_\ell$ must be nonzero (since $V^* \neq 0$). In other words, there exists some $j \in \{1, 2, \ldots, \ell\}$ such that $V^*I_j \neq 0$. Consider this j. From $V^*I_j \neq 0$, we obtain $V \cong I_j$ (otherwise, Lemma 2.6.4 would yield $V^*I_j = 0$). Hence, $V^*V \cong V^*I_j \neq 0$. Thus, $\dim (V^*V) > 0$. Combining this with $\dim (V^*V) \leq 1$, we obtain $\dim (V^*V) = 1$. This proves Lemma 2.6.6. □

Theorem 2.6.7. We have $v^*w = w^*v$ for all $v, w \in J$.

Proof. Decompose the S_n-representation J as a direct sum of irreducible subrepresentations:

$$J = J_1 \oplus J_2 \oplus \cdots \oplus J_k. \tag{7}$$

We must show that $v^*w = w^*v$ for all $v, w \in J$. This equality is linear in each of v and w. Thus, it suffices to show that $v^*w = w^*v$ for all $i, j \in \{1, 2, \ldots, k\}$ and all $v \in J_i$ and all $w \in J_j$ (by (7)). So let us fix $i, j \in \{1, 2, \ldots, k\}$. We must prove that $v^*w = w^*v$.

If $i = j$, then this follows from Lemma 2.6.5 (applied to $V = J_i = J_j$). Thus, we WLOG assume that $i \neq j$. Then, J_i and J_j are non-isomorphic as representations of S_n (since J is multiplicity-free). Hence, Lemma 2.6.4 yields $J_i^* J_j = 0$. Thus, $v^*w = 0$ (since $v \in J_i$ and $w \in J_j$). Similarly, $w^*v = 0$. Hence, $v^*w = 0 = w^*v$, qed. □

Theorem 2.6.8. Each element of J^*J is invariant under the antipode. In other words, for each $x \in J^*J$, we have $x^* = x$.

Proof. Each $x \in J^*J$ is a $\mathbf{k}$-linear combination of elements of the form w^*v with $w, v \in J$. Thus, it suffices to show that $(w^*v)^* = w^*v$ for all $v, w \in J$. But this is now easy: For any $v, w \in J$, we have $(w^*v)^* = v^* \underbrace{(w^*)^*}_{=w} = v^*w = w^*v$ (by Theorem 2.6.7). This proves Theorem 2.6.8. □

2.7. Splitness

Theorem 2.7.1. Let $a \in J$. Then, there exists a multiset Z_a of elements of $\mathbf{k}$ such that $\prod\limits_{\lambda \in Z_a} (a - \lambda) = 0$. (In other words, the minimal polynomial of a over $\mathbf{k}$ factors into linear factors.)

Proof. We say that a $\mathbf{k}$-linear endomorphism f of a finite-dimensional vector space V is *split* if there exists a multiset Z_f of elements of $\mathbf{k}$ such that $\prod\limits_{\lambda \in Z_f} (f - \lambda \operatorname{id}) = 0$ (that is, if the minimal polynomial of f factors into linear factors – or, equivalently, if all eigenvalues of f belong to $\mathbf{k}$). Clearly, if $f_1, f_2, \ldots, f_k$ are split endomorphisms of vector spaces $V_1, V_2, \ldots, V_k$, respectively, then their direct sum $f_1 \oplus f_2 \oplus \cdots \oplus f_k \in \operatorname{End}(V_1 \oplus V_2 \oplus \cdots \oplus V_k)$ is split again. Furthermore, any vector space endomorphism of rank ≤ 1 is split (indeed, if f is vector space endomorphism of rank ≤ 1, then $f \circ (f - (\operatorname{Tr} f) \operatorname{id}) = 0$).

We say that an element $b \in \mathcal{A}$ acts *splitly* on a left $\mathcal{A}$-module U if and only if there exists a multiset $Z_{b,U}$ of elements of $\mathbf{k}$ such that $\left(\prod\limits_{\lambda \in Z_{b,U}} (b - \lambda) \right) \cdot U = 0$. Thus, we must prove that a acts splitly on the left regular $\mathcal{A}$-module $\mathcal{A}$ (since this will yield $\left(\prod_{\lambda \in Z_{b,\mathcal{A}}} (a-\lambda)\right) \cdot \mathcal{A} = 0$ and thus $\prod_{\lambda \in Z_{b,\mathcal{A}}} (a-\lambda) = 0$). Note that an element $b \in \mathcal{A}$ acts splitly on a left $\mathcal{A}$-module U if and only if the action of b on U is a split endomorphism of U.

Decompose the left regular $\mathcal{A}$-module $\mathcal{A}$ as a direct sum of irreducible subrepresentations:

$$\mathcal{A} = L_1 \oplus L_2 \oplus \cdots \oplus L_k.$$

It suffices to show that a acts splitly on each of these subrepresentations $L_1, L_2, \ldots, L_k$, since the action of a on the full $\mathcal{A}$ is just the direct sum of these actions.

So let us fix $i \in \{1, 2, \ldots, k\}$. We must show that a acts splitly on L_i. It suffices to prove that the action of a on L_i has rank ≤ 1 (as a $\mathbf{k}$-linear map), since any vector space endomorphism of rank ≤ 1 is split .

So we need to show that the image of a on L_i has dimension ≤ 1. In other words, we need to show that $\dim(aL_i) \leq 1$.

But this follows immediately from Lemma 2.5.2 (applied to $K = L_i$), since the representation L_i is irreducible. □

2.8. The left ideal J as a nonunital algebra

Each left ideal of an algebra A is a nonunital subalgebra. Sometimes, this nonunital subalgebra does in fact has a unity (although this unity is usually not the unity of A). Usually, it does not. The situation for left ideals of group algebras is as follows:

Proposition 2.8.1. Let G be a finite group. Let A be its group algebra $\mathbf{k}[G]$. Let I be any left ideal of A. Then:

(a) The nonunital $\mathbf{k}$-algebra I has a right unity.

(b) The nonunital $\mathbf{k}$-algebra I has an actual (two-sided) unity if and only if I is a (two-sided) ideal of A.

Proof. Theorem 2.3.1 says that there exists an idempotent $e \in I$ such that $I = Ae$ (since $A = \mathbf{k}[G]$). Consider this e. Clearly, $ee = e$ (since e is idempotent). Each $x \in I$ satisfies $x \in I = Ae$ and thus $x = ae$ for some $a \in A$, so that $xe = a \underbrace{ee}_{=e} = ae = x$. This shows that e is a right unity of I. Thus, part **(a)** is proved.

(b) $\Longleftarrow$: Assume that I is a two-sided ideal of A. Thus, I is both a left ideal and a right ideal of A. Hence, by part **(a)**, we know that I has a right unity (since I is a left ideal). An analogous argument, with the order of factors reversed, shows that I has a left unity (since I is a right ideal). These two unities must be equal (indeed, if we call them r and ℓ, then $r = \ell r = \ell$), and thus must be a unity. So I has a unity.

$\Longrightarrow$: Assume that I has a unity. Let u be this unity. Then, $ue = u$ (since e is a right unity of I), so that $u = ue = e$ (since u is a unity of I). Hence, e is a unity of I (since u is a unity of I). Therefore, $eI = I$. In view of $I = Ae$, we can rewrite this as $eAe = Ae$. Therefore, $Ae = e \underbrace{Ae}_{\subseteq A} \subseteq eA$.

Now, let $c \in eA$. We shall show that $c = ce$. To wit, let $x := c(1-e)$. Let $y \in A$ be arbitrary. Then, $ye \in Ae = I$, so that $e \cdot ye = ye$ (since e is the unity of the algebra I). Thus, $(1-e) ye = ye - \underbrace{e \cdot ye}_{=ye} = 0$. Furthermore, we can write c as $c = ez$ for some $z \in A$ (since $c \in eA$). Consider this z. Now,

$$(xy)^2 = \underbrace{x}_{=c(1-e)} y \underbrace{x}_{=c(1-e)} y = c(1-e) y \underbrace{c}_{=ez} (1-e) y = c \underbrace{(1-e) ye}_{=0} z (1-e) y = 0.$$

Hence, the element xy of A is nilpotent, and thus the element $1 - xy$ is invertible (since 1 minus a nilpotent element is always invertible).

Forget that we fixed y. We thus have shown that $1 - xy \in A$ is invertible for each $y \in A$. In other words, x belongs to the Jacobson radical of the ring A. But A is semisimple (by Maschke's theorem, since A is the group algebra of the finite group G over the field $\mathbf{k}$ of characteristic 0). Hence, the Jacobson radical of A is 0. Therefore, $x = 0$ (since x belongs to this Jacobson radical). Hence, $0 = x = c(1-e) = c - ce$, so that $c = ce \in Ae$.

Forget that we fixed c. We thus have shown that $c \in Ae$ for each $c \in eA$. In other words, $eA \subseteq Ae$. Combining this with $Ae \subseteq eA$, we obtain $Ae = eA$.

Thus, $I = Ae = eA$. This shows that I is a right ideal of A. Since I is also a left ideal, we conclude that I is a two-sided ideal of A. □

Proposition 2.8.1 **(b)** (applied to $G = S_n$) shows that a left ideal I of $\mathcal{A} = \mathbf{k}[S_n]$ rarely has a unity. Indeed, the group algebra $\mathcal{A} = \mathbf{k}[S_n]$ has only finitely many two-sided ideals (in fact, the Artin–Wedderburn theorem shows that $\mathcal{A} \cong \prod_{\lambda \vdash n} \mathbf{k}^{f_\lambda \times f_\lambda}$, and each matrix ring $\mathbf{k}^{f_\lambda \times f_\lambda}$ is a simple $\mathbf{k}$-algebra; thus, the two-sided ideals of $\mathcal{A}$ correspond to the $2^{|\{\lambda \vdash n\}|}$ many subproducts of the direct product $\prod_{\lambda \vdash n} \mathbf{k}^{f_\lambda \times f_\lambda}$), but has infinitely many left ideals when $n \geq 3$.

Proposition 2.8.1 yields a direct (and rather disappointing) answer to the question when the nonunital $\mathbf{k}$-algebra J has a unity. The multiplicity-freeness is mostly a hindrance here. However, the multiplicity-freeness of J gives the following structural property of J:

Proposition 2.8.2. Let $J = J_1 \oplus J_2 \oplus \cdots \oplus J_k$ be a decomposition of J into irreducible S_n-subrepresentations. Then, $J \cong J_1 \times J_2 \times \cdots \times J_k$ as nonunital $\mathbf{k}$-algebras.

Proof. The subrepresentations $J_1, J_2, \ldots, J_k$ are pairwise non-isomorphic (since J is multiplicity-free). Hence, Lemma 2.4.4 shows that $J_i J_j = 0$ for all $i \neq j$. Therefore, the canonical $\mathbf{k}$-vector space isomorphism $J_1 \times J_2 \times \cdots \times J_k \to J$ is actually a nonunital $\mathbf{k}$-algebra isomorphism. $\square$

2.9. The nonunital subalgebra J^*J

The product J^*J is a nonunital subalgebra of $\mathcal{A}$ as well, since $J^* \underbrace{JJ^*J}_{\subseteq J} \subseteq J^*J$. When does it have a unity? This has a rather tricky answer, depending somewhat on $\mathbf{k}$:

Proposition 2.9.1. Let $J = J_1 \oplus J_2 \oplus \cdots \oplus J_k$ be a decomposition of J into irreducible S_n-subrepresentations. Then:

- **(a)** We have $J^*J \cong \prod_{i=1}^{k} \left(J_i^* J_i\right)$ as nonunital $\mathbf{k}$-algebras.
- **(b)** Let $\mathbf{k}_0$ be the $\mathbf{k}$-vector space $\mathbf{k}$, turned into a nonunital $\mathbf{k}$-algebra by defining the product of any two elements to be 0. Then, $J^*J \cong \mathbf{k}^r \times \mathbf{k}_0^{k-r}$ as nonunital $\mathbf{k}$-algebras for some $r \in \{0, 1, \ldots, k\}$.
- **(c)** Assume that $\mathbf{k}$ is an ordered field (for instance, $\mathbb{Q}$ or $\mathbb{R}$). Then, $J^*J \cong \mathbf{k}^k$ as nonunital $\mathbf{k}$-algebras. In particular, J^*J has a unity.

Proof. **(a)** The subrepresentations $J_1, J_2, \ldots, J_k$ are pairwise non-isomorphic (since J is multiplicity-free). Hence, Lemma 2.6.4 shows that $J_i^* J_j = 0$ for all $i \neq j$. Now, from $J = J_1 \oplus J_2 \oplus \cdots \oplus J_k = \sum_{i=1}^{k} J_i$, we obtain

$$J^*J = \left(\sum_{i=1}^{k} J_i\right)^* \sum_{i=1}^{k} J_i = \sum_{i=1}^{k} J_i^* \sum_{j=1}^{k} J_j = \sum_{i=1}^{k} \sum_{j=1}^{k} \underbrace{J_i^* J_j}_{=0 \text{ for all } i \neq j} = \sum_{i=1}^{k} J_i^* J_i.$$

Moreover, the sum on the right hand side here is a direct sum (because the addends $J_i^* J_i$ are subspaces of the respective J_i and thus are linearly disjoint). Thus,

$$J^*J = \bigoplus_{i=1}^{k} \left(J_i^* J_i\right) \qquad \text{(an internal direct sum)}.$$

Moreover, $\left(J_i^* J_i\right)\left(J_j^* J_j\right) = 0$ for all $i \neq j$ (since $J_i^* J_i \subseteq J_i^*$ and $J_j^* J_j \subseteq J_j$ but $J_i^* J_j = 0$). Thus, the canonical $\mathbf{k}$-vector space isomorphism $\prod_{i=1}^{k} \left(J_i^* J_i\right) \to J^*J$ is actually a nonunital $\mathbf{k}$-algebra isomorphism. This proves part **(a)**.

(b) This will follow from part **(a)**, once we show that each $i \in \{1, 2, \ldots, k\}$ satisfies $J_i^* J_i \cong \mathbf{k}$ or $J_i^* J_i \cong \mathbf{k}_0$. So let us show this.

Let $i \in \{1, 2, \ldots, k\}$. Then, we must prove that $J_i^* J_i \cong \mathbf{k}$ or $J_i^* J_i \cong \mathbf{k}_0$. It suffices to prove that $\dim\left(J_i^* J_i\right) = 1$, since every nonunital $\mathbf{k}$-algebra of dimension 1 is isomorphic to either $\mathbf{k}$ or $\mathbf{k}_0$ (depending on whether its product takes any nonzero value or does not). But this equality follows from Lemma 2.6.6 (applied to $V = J_i$). As explained above, this completes the proof of part **(b)**.

(c) This will follow from part **(a)**, once we show that each $i \in \{1, 2, \ldots, k\}$ satisfies $J_i^* J_i \cong \mathbf{k}$. So let us show this.

Let $i \in \{1, 2, \ldots, k\}$. Then, we must prove that $J_i^* J_i \cong \mathbf{k}$. In the proof of part **(b)**, we have already seen that $J_i^* J_i \cong \mathbf{k}$ or $J_i^* J_i \cong \mathbf{k}_0$, so we only need to rule out the case $J_i^* J_i \cong \mathbf{k}_0$.

But $\mathbf{k}$ is an ordered field. Thus, we have the following simple fact:

Star positivity trick: For any nonzero $x \in \mathcal{A}$, we have $x^* x \neq 0$.

Proof of the star positivity trick: Let $x \in \mathcal{A}$ be nonzero. Write x as $x = \sum_{g \in S_n} \xi_g g$, where $\xi_g \in \mathbf{k}$ are scalars. Then, the coefficient of the identity permutation $\operatorname{id} \in S_n$ in $x^* x$ is easily seen to be $\sum_{g \in S_n} \xi_g^2$. But $\sum_{g \in S_n} \xi_g^2 > 0$ (since $x \neq 0$ shows that at least one of the ξ_g is nonzero, but a sum of nonzero squares in an ordered field is always positive). Hence, the coefficient of the identity permutation $\operatorname{id} \in S_n$ in $x^* x$ is > 0. Thus, $x^* x \neq 0$, qed. □

Now, Lemma 2.6.6 (applied to $V = J_i$) yields $\dim\left(J_i^* J_i\right) = 1$. Hence, $J_i^* J_i \neq 0$, so that there exists some nonzero $x \in J_i^* J_i$. Consider this x. By the star positivity trick, we thus conclude that $x^* x \neq 0$. Since $x \in J_i^* J_i$, this shows that the product of the nonunital algebra $J_i^* J_i$ is not identically 0. Hence, the case $J_i^* J_i \cong \mathbf{k}_0$ is impossible. This completes the proof of part **(c)**. □

The claim of Proposition 2.9.1 **(c)** holds more generally for any field $\mathbf{k}$, as long as the left ideal J is defined over an ordered subfield of $\mathbf{k}$. In particular, it holds when J is defined over $\mathbb{Q}$, which is the case for most combinatorially meaningful J's.

2.10. Appendix: Sums of non-isomorphic irreducibles are direct

Let us finally state a basic property of representations which is surely well-known, but which we could not locate in the literature. We shall use it later on. It says that any sum of pairwise non-isomorphic irreducible representations (inside a larger representation) of $\mathcal{A}$ must be a direct sum. More generally, this holds for any finite-dimensional $\mathbf{k}$-algebra instead of $\mathcal{A}$:

Proposition 2.10.1. Let R be a finite-dimensional $\mathbf{k}$-algebra. Let V be a left R-module. Let $I_1, I_2, \ldots, I_k$ be pairwise non-isomorphic irreducible submodules of V. Then, the sum $I_1 + I_2 + \cdots + I_k$ is a direct sum.

Proof sketch. We induct on k. The *base case* ($k = 0$) is obvious, so we step to the *induction step* (from $k-1$ to k). Thus, we assume (as the induction hypothesis) that the sum $I_1 + I_2 + \cdots + I_{k-1}$ is a direct sum. Hence, there exist left R-linear projections $\pi_j : I_1 + I_2 + \cdots + I_{k-1} \to I_j$ for all $j \in [k-1]$.

Now, the intersection $I_k \cap (I_1 + I_2 + \cdots + I_{k-1})$ is a left R-submodule of I_k, and thus equals I_k or 0 (since I_k is irreducible, so that the only left R-submodules of I_k are I_k and 0). Since $I_k \cap (I_1 + I_2 + \cdots + I_{k-1}) = I_k$ is impossible (because $I_k \cap (I_1 + I_2 + \cdots + I_{k-1}) = I_k$ would yield $I_k \subseteq I_1 + I_2 + \cdots + I_{k-1}$, and thus at least one of the projections π_j would yield a nontrivial left R-module morphism

$$I_k \xrightarrow{\text{inclusion}} I_1 + I_2 + \cdots + I_{k-1} \xrightarrow{\pi_j} I_j,$$

and this would entail $I_k \cong I_j$ by Schur's lemma, contradicting the non-isomorphy of $I_1, I_2, \ldots, I_k$), we thus obtain $I_k \cap (I_1 + I_2 + \cdots + I_{k-1}) = 0$. Since the sum $I_1 + I_2 + \cdots + I_{k-1}$ is a direct sum, this yields that the sum $I_1 + I_2 + \cdots + I_k$ is a direct sum. This completes the induction step, and thus proves Proposition 2.10.1. □

3. The left ideal Gelfand model

We shall now apply the above general theory to a specific left ideal of $\mathcal{A} = \mathbf{k}[S_n]$.

As before, we fix $n \in \mathbb{N}$.

3.1. Definition

We let $[m] := \{1, 2, \ldots, m\}$ for each $m \in \mathbb{Z}$. Thus, S_n is the group of all permutations of $[n] = \{1, 2, \ldots, n\}$.

For any $i \neq j$ in $[n]$, we let $t_{i,j} \in S_n$ denote the transposition swapping i with j.

For any $2k$ distinct elements $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in [n]$, we define

$$G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k} := \sum_{\substack{w \in S_n; \\ w(i_s) < w(j_s) \text{ for all } s \in [k]}} w^{-1} \in \mathcal{A}. \tag{8}$$

Example 3.1.1. Let $[i_1 i_2 \cdots i_n]$ be shorthand for the permutation in S_n that sends $1, 2, \ldots, n$ to $i_1, i_2, \ldots, i_n$. For $n = 3$, we have

$$G_{3;\ 1} = \sum_{\substack{w \in S_3; \\ w(3) < w(1)}} w^{-1} = [231]^{-1} + [321]^{-1} + [312]^{-1} = [312] + [321] + [231].$$

For $n = 4$, we have

$$\begin{aligned} G_{1,3;\ 2,4} &= \sum_{\substack{w \in S_4; \\ w(1) < w(2); \\ w(3) < w(4)}} w^{-1} \\ &= [1234]^{-1} + [1324]^{-1} + [1423]^{-1} + [2314]^{-1} + [2413]^{-1} + [3412]^{-1} \\ &= [1234] + [1324] + [1342] + [3124] + [3142] + [3412]. \end{aligned}$$

Clearly, for any $2k$ distinct elements $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in [n]$ and any permutation $\sigma \in S_k$, we have

$$G_{i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(k)};\ j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(k)}} = G_{i_1, i_2, \ldots, i_k;\ j_1, j_2, \ldots, j_k}. \tag{9}$$

That is, $G_{i_1, i_2, \ldots, i_k;\ j_1, j_2, \ldots, j_k}$ depends only on the **set** of the pairs (i_1, j_1) , (i_2, j_2) , $\ldots$, (i_k, j_k), not on their order.

It is also easy to see that

$$wG_{i_1, i_2, \ldots, i_k;\ j_1, j_2, \ldots, j_k} = G_{w(i_1), w(i_2), \ldots, w(i_k);\ w(j_1), w(j_2), \ldots, w(j_k)}$$

for any permutation $w \in S_n$. Hence,

$$\mathcal{G} := \operatorname{span}\left(G_{i_1, i_2, \ldots, i_k;\ j_1, j_2, \ldots, j_k} \ \mid\ i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in [n] \text{ are distinct}\right)$$

is a left ideal of $\mathcal{A}$.

3.2. The main result

Most of this section will be spent proving the following description of the left $\mathcal{A}$-module structure of this ideal $\mathcal{G}$ (implicit in [ReSaWe11, §5.1]):

Theorem 3.2.1. This left ideal $\mathcal{G}$ is a Gelfand model of S_n; that is, $\mathcal{G}$ is isomorphic to the direct sum of all the Specht modules $\mathcal{S}^\lambda$ with $\lambda \vdash n$.

As a consequence of Theorem 3.2.1, it will follow that the representation $\mathcal{G}$ of S_n is multiplicity-free. Hence, Theorem 2.4.1 will yield that $[\mathcal{G}^*\mathcal{G},\ \mathcal{G}^*\mathcal{G}] = 0$. This generalizes the commutativity part of [ReSaWe11, Theorem 1.6]. Furthermore, Theorem 2.2.2 will yield $\mathcal{G}[\mathcal{G}, \mathcal{G}] = 0$.

3.3. The filtration $(\mathcal{G}_m)_{m \in \mathbb{Z}}$

Before we prove Theorem 3.2.1, we define a filtration on $\mathcal{G}$.

Namely, for each $m \in \mathbb{Z}$, we define the **k**-vector subspace

$$\mathcal{G}_m := \operatorname{span}\left(G_{i_1, i_2, \ldots, i_k;\ j_1, j_2, \ldots, j_k} \ \mid\ k \leq m\right) \qquad \text{of } \mathcal{A}.$$

This $\mathcal{G}_m$ is always a left ideal of $\mathcal{A}$ (for the same reason as why $\mathcal{G}$ is). Moreover, $\mathcal{G}_m = \mathcal{G}$ whenever $2m \geq n$. Thus, we obtain a filtration $(0 = \mathcal{G}_{-1} \subseteq \mathcal{G}_0 \subseteq \cdots \subseteq \mathcal{G}_n = \mathcal{G})$ of $\mathcal{G}$ by left $\mathcal{A}$-submodules. We can easily derive an upper bound on the dimensions of its subquotients (later we shall see that this upper bound is, in fact, the actual dimension):

Lemma 3.3.1. Let $m \in \mathbb{Z}$. An *m-matching* of $[n]$ shall mean a set of m disjoint 2-element subsets of $[n]$. Then,

$$\dim\left(\mathcal{G}_m / \mathcal{G}_{m-1}\right) \leq (\text{\# of } m\text{-matchings of } [n]) . \tag{10}$$

Proof. The definitions of $\mathcal{G}_m$ and $\mathcal{G}_{m-1}$ show that the quotient vector space $\mathcal{G}_m / \mathcal{G}_{m-1}$ is spanned by the family

$$\left(\overline{G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m}}\right)_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m \text{ are } 2m \text{ distinct elements of } [n]} \tag{11}$$

(this is an empty family if $2m > n$). Moreover, if we swap an i_p with the respective j_p in $G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m}$, then we obtain

$$\begin{aligned}
& G_{i_1,i_2,\ldots,i_{p-1},j_p,i_{p+1},\ldots i_m;\ j_1,j_2,\ldots,j_{p-1},i_p,j_{p+1},\ldots,j_m} \\
&= \sum_{\substack{w \in S_n; \\ w(i_s)<w(j_s) \text{ for all } s \neq p; \\ w(j_p)<w(i_p)}} w^{-1} = \sum_{\substack{w \in S_n; \\ w(i_s)<w(j_s) \text{ for all } s \neq p; \\ \text{but not } w(i_p)<w(j_p)}} w^{-1} \\
& \qquad \left(\begin{array}{c} \text{since } i_p \neq j_p \text{, and thus } w\left(i_p\right) \neq w\left(j_p\right) \text{ for all } w \in S_n, \\ \text{so that the condition ``} w\left(j_p\right) < w\left(i_p\right) \text{''} \\ \text{is equivalent to ``not } w\left(i_p\right) < w\left(j_p\right) \text{''} \end{array}\right) \\
&= \underbrace{\sum_{\substack{w \in S_n; \\ w(i_s)<w(j_s) \text{ for all } s \neq p}} w^{-1}}_{= G_{i_1,i_2,\ldots,i_{p-1},i_{p+1},\ldots,i_m;\ j_1,j_2,\ldots,j_{p-1},j_{p+1},\ldots,j_m}} - \underbrace{\sum_{\substack{w \in S_n; \\ w(i_s)<w(j_s) \text{ for all } s}} w^{-1}}_{= G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m}} \\
&= \underbrace{G_{i_1,i_2,\ldots,i_{p-1},i_{p+1},\ldots,i_m;\ j_1,j_2,\ldots,j_{p-1},j_{p+1},\ldots,j_m}}_{\in \mathcal{G}_{m-1}} - G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m} \\
&\equiv -G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m} \bmod \mathcal{G}_{m-1},
\end{aligned}$$

so that

$$\begin{aligned}
& \overline{G_{i_1,i_2,\ldots,i_{p-1},j_p,i_{p+1},\ldots i_m;\ j_1,j_2,\ldots,j_{p-1},i_p,j_{p+1},\ldots,j_m}} \\
& = -\overline{G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m}} \qquad \text{in } \mathcal{G}_m / \mathcal{G}_{m-1}.
\end{aligned} \tag{12}$$

Thus, in the quotient $\mathbf{k}$-vector space $\mathcal{G}_m / \mathcal{G}_{m-1}$, the residue class $\overline{G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m}}$ merely flips its sign when we swap some i_p with the corresponding j_p. Hence, up to sign, this class depends only on the m disjoint **sets** $\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_m, j_m\}$, not on the ordered pairs $(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)$ (and, as we know from (9), it also does not depend on the **order** in which these m sets are listed). This shows that the family (11) is highly redundant, and can be reduced to a smaller family indexed by the m-matchings of $[n]$ (just take one choice of $\overline{G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m}}$ for each m-matching) that still spans $\mathcal{G}_m / \mathcal{G}_{m-1}$. Consequently,

$$\dim\left(\mathcal{G}_m / \mathcal{G}_{m-1}\right) \leq (\text{\# of } m\text{-matchings of } [n]) .$$

This proves Lemma 3.3.1. □

3.4. Proving the Gelfandness half-combinatorially

We continue on our way towards the proof of Theorem 3.2.1.

The key tool will be a proposition (Proposition 3.4.1) showing that for each Specht module $\mathcal{S}^\lambda$, there is a nonzero element $G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}\mathbf{e}_T \in \mathcal{G}^*\mathcal{S}^\lambda$ (where $\mathbf{e}_T$ is a polytabloid corresponding some n-tableau T of shape λ, and where $G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$ is one of the generators of $\mathcal{G}$). This proposition will yield that $\mathcal{G}$ has a submodule isomorphic to $\mathcal{S}^\lambda$. Thus, $\mathcal{G}$ will contain a submodule isomorphic to $\bigoplus_{\lambda\vdash n}\mathcal{S}^\lambda$ (since the Specht modules $\mathcal{S}^\lambda$ are simple and non-isomorphic). On the other hand, we will show that the dimension of $\mathcal{G}$ agrees with the dimension of $\bigoplus_{\lambda\vdash n}\mathcal{S}^\lambda$, so this submodule must be in fact the whole $\mathcal{G}$. Thus, Theorem 3.2.1 will follow.

We consider this to be a half-combinatorial proof. Proposition 3.4.1 will be proved combinatorially, and the dimensions will be computed combinatorially, but the deduction of $\bigoplus_{\lambda\vdash n}\mathcal{S}^\lambda = \mathcal{G}$ from $\bigoplus_{\lambda\vdash n}\mathcal{S}^\lambda \subseteq \mathcal{G}$ and $\dim\left(\bigoplus_{\lambda\vdash n}\mathcal{S}^\lambda\right) = \dim\mathcal{G}$ is an intrusion of linear algebra.

3.4.1. The nonvanishing: statement

The crux of our proof is the following fact, which will take us a while to prove:

Proposition 3.4.1. Let λ be a partition of n. Let $k = \sum_{i\geq 1}\left\lfloor \frac{\lambda_i^t}{2}\right\rfloor$, where λ^t is the conjugate partition of λ. Let T be any n-tableau of shape λ (that is, any tableau of shape λ with entries $1,2,\ldots,n$ in some order). Then, there exist some $2k$ distinct elements $i_1,i_2,\ldots,i_k,j_1,j_2,\ldots,j_k \in [n]$ such that

$$G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}\mathbf{e}_T \neq 0,$$

where $\mathbf{e}_T \in \mathcal{S}^\lambda$ is the polytabloid corresponding to T.

Note that the k in Proposition 3.4.1 can also be described as $(n-\ddot{o})/2$, where $\ddot{o}$ is the number of odd-length columns of the Young diagram $Y(\lambda)$.

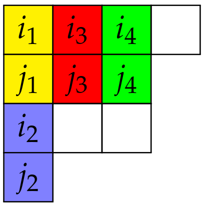
3.4.2. (*) The nonvanishing: bad proof

We shall give two proofs of Proposition 3.4.1: one bad, one good. The bad one is easier to explain, but has some disadvantages, the main of which is that it really requires $\mathbf{k}$ to have characteristic 0, whereas the proposition only needs $n!$ to be invertible in $\mathbf{k}$.

Bad proof of Proposition 3.4.1. Tile each column of T with vertical dominoes, except possibly for the bottommost cell of this column (which cannot be tiled if the column has odd length). Thus, we have altogether put k disjoint dominoes into T. Label these dominoes as $D_1, D_2, \ldots, D_k$. For each $m \in \{1, 2, \ldots, k\}$, let i_m and j_m be the entries of T in domino D_m from top to bottom. We claim that these i_m's and j_m's satisfy

$$G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k} \mathbf{e}_T \neq 0.$$

Here is an example:



Indeed, we recall the Specht–Vandermonde avatar of Young and Specht modules ([Grinbe24, Theorem 5.6.1]): There is an injective morphism β of S_n-representations from the Young module $\mathcal{M}^\lambda$ to the polynomial ring $\mathcal{P} := \mathbf{k}[x_1, x_2, \ldots, x_n]$ that sends each n-tabloid $\overline{T}$ to the monomial $\prod\limits_{i \in [n]} x_i^{r_T(i)-1}$, where $r_T(i)$ is the number of the row of T in which i lies. Restricting this β to the Specht module $\mathcal{S}^\lambda$, we see that

$$\beta(\mathbf{e}_T) = \prod_{j \geq 1} V\left(x_{T(1,j)}, x_{T(2,j)}, \ldots, x_{T\left(\lambda^t_j, j\right)}\right),$$

where $V(y_1, y_2, \ldots, y_k)$ denotes the Vandermonde determinant $\det\left(y_j^{k-i}\right)_{i,j \in [k]} = \prod\limits_{i>j} (y_i - y_j)$ for any $y_1, y_2, \ldots, y_k$.

Thus,

$$\begin{aligned}
&\beta\left(G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k} \mathbf{e}_T\right) \\
&= \underbrace{G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}}_{= \sum\limits_{\substack{w \in S_n; \\ w(i_m) < w(j_m) \text{ for each } m}} w} \underbrace{\beta(\mathbf{e}_T)}_{= \prod\limits_{j \geq 1} V\left(x_{T(1,j)}, x_{T(2,j)}, \ldots, x_{T\left(\lambda^t_j, j\right)}\right)} \\
&= \sum_{\substack{w \in S_n; \\ w(i_m) < w(j_m) \text{ for each } m}} w \cdot \prod_{j \geq 1} V\left(x_{T(1,j)}, x_{T(2,j)}, \ldots, x_{T\left(\lambda^t_j, j\right)}\right) \\
&= \sum_{\substack{w \in S_n; \\ w(i_m) < w(j_m) \text{ for each } m}} \prod_{j \geq 1} V\left(x_{w(T(1,j))}, x_{w(T(2,j))}, \ldots, x_{w\left(T\left(\lambda^t_j, j\right)\right)}\right) \\
&= \sum_{\substack{S \text{ is an } n\text{-tableau of shape } \lambda; \\ S \text{ increases on each domino } D_m}} \prod_{j \geq 1} V\left(x_{S(1,j)}, x_{S(2,j)}, \ldots, x_{S\left(\lambda^t_j, j\right)}\right) \\
&\qquad \left(\begin{array}{c} \text{where "} S \text{ increases on } D_m \text{" means that the two} \\ \text{values of } S \text{ on } D_m \text{ increase from top to bottom} \end{array}\right).
\end{aligned}$$

Clearly, it will suffice to show that this sum is $\neq 0$ (because $\beta(0) = 0$). Being a polynomial in $x_1, x_2, \ldots, x_n$, it will of course be $\neq 0$ if we can show that it gives a positive real number when we evaluate it at $(x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n)$ for any strictly increasing n-tuple $a_1 < a_2 < \cdots < a_n$ of reals. So let us do this.

Fix a strictly increasing n-tuple $a_1 < a_2 < \cdots < a_n$ of reals. We must show that

$$\sum_{\substack{S \text{ is an } n\text{-tableau of shape } \lambda; \\ S \text{ increases on each domino } D_m}} \prod_{j \geq 1} V\left(a_{S(1,j)}, a_{S(2,j)}, \ldots, a_{S\left(\lambda_j^t, j\right)}\right) > 0.$$

We can break the sum up according to the column-tabloid $\widetilde{S}$ (that is, we bunch column-equivalent tableaux S together), and then the product can be factored out of the sum, and we can deal with each column separately. We thus are left with proving the following fact: For any $p \in \mathbb{N}$ and any p reals $b_1 < b_2 < \cdots < b_p$, we have

$$\sum_{\substack{\sigma \in S_p; \\ \sigma(1) < \sigma(2); \\ \sigma(3) < \sigma(4); \\ \ldots}} V\left(b_{\sigma(1)}, b_{\sigma(2)}, \ldots, b_{\sigma(p)}\right) > 0$$

(where the relations under the summation sign go up to $\sigma(p)$ or $\sigma(p-1)$ depending on the parity of p). Since $V\left(b_{\sigma(1)}, b_{\sigma(2)}, \ldots, b_{\sigma(p)}\right) = (-1)^{\sigma} V\left(b_1, b_2, \ldots, b_p\right)$ and $V\left(b_1, b_2, \ldots, b_p\right) > 0$, this boils down to showing that

$$\sum_{\substack{\sigma \in S_p; \\ \sigma(1) < \sigma(2); \\ \sigma(3) < \sigma(4); \\ \ldots}} (-1)^{\sigma} > 0.$$

But this sum is actually $\lfloor p/2 \rfloor !$, as we can show as follows:

- If p is even, then

$$\sum_{\substack{\sigma \in S_p; \\ \sigma(1) < \sigma(2); \\ \sigma(3) < \sigma(4); \\ \ldots}} (-1)^{\sigma} = \lfloor p/2 \rfloor ! \cdot \sum_{\substack{\sigma \in S_p; \\ \sigma(1) < \sigma(2); \\ \sigma(3) < \sigma(4); \\ \ldots; \\ \sigma(1) \leq \sigma(3) \leq \cdots \leq \sigma(p-1)}} (-1)^{\sigma} \qquad \text{(by symmetry)}$$
$$= \lfloor p/2 \rfloor !,$$

  according to the formula $\sum_{\substack{\sigma \in S_p; \\ \sigma(1) < \sigma(2); \\ \sigma(3) < \sigma(4); \\ \ldots; \\ \sigma(1) \leq \sigma(3) \leq \cdots \leq \sigma(p-1)}} (-1)^{\sigma} = 1$ that Gjergji Zaimi proved in `https://math.stackexchange.com/a/58941/` (brief outline of the proof: define a sign-reversing involution Ω on the set of all $\sigma \in S_p$ that satisfy

$\sigma(1) < \sigma(3)$ and $\sigma(2) < \sigma(4)$ and $\cdots$ and $\sigma(1) \leq \sigma(3) \leq \cdots \leq \sigma(p-1)$ but are not the identity; this involution Ω shall pick the smallest k satisfying $\sigma(2k-1) = \sigma(2k+1) - 1$ and swap the values of σ at $2k$ and $2k+2$).

- If p is odd, then $\sigma(p)$ is not constrained at all in our sum, and so we can cycle it through all possible values (i.e., replace σ by $\sigma \circ \operatorname{cyc}_{i,i+1,\ldots,p}$ for all $i \in [p]$); this gives us a $1 + (-1) + 1 + (-1) + \cdots + 1 = 1$ factor that we can factor out and are left with the corresponding sum for $p-1$ instead of p. Since $p-1$ is even, this is done in the previous bullet point.

Altogether, we obtain

$$\sum_{\substack{\sigma \in S_p; \\ \sigma(1) < \sigma(2); \\ \sigma(3) < \sigma(4); \\ \cdots}} (-1)^{\sigma} = \lfloor p/2 \rfloor! > 0,$$

qed. □

3.4.3. The nonvanishing: good proof

Let us now give the good proof of Proposition 3.4.1. This needs a bit of preparation.

For the rest of Subsubsection 3.4.3, we fix the following notations: Let λ be a partition of n. Let

$$k = \sum_{i \geq 1} \lfloor \lambda_i^t / 2 \rfloor \qquad \text{and} \qquad p = \prod_{i \geq 1} \lfloor \lambda_i^t / 2 \rfloor !,$$

where λ^t is the conjugate partition of λ. Tile each column of the Young diagram $Y(\lambda)$ with vertical dominoes, except possibly for the bottommost cell of this column (which cannot be tiled if the column has odd length). Thus, we have altogether put k disjoint dominoes into $Y(\lambda)$. Label these dominoes as $D_1, D_2, \ldots, D_k$ in arbitrary order. For each $m \in [k]$, let c_m and d_m be the cells in domino D_m from top to bottom. Note that c_m lies in an odd row of $Y(\lambda)$, while d_m is the southern neighbor of c_m and thus lies in an even row. For instance, for $\lambda = (4,3,3,1)$, we have $k = 2+1+1+0 = 4$ and (choosing to label the dominoes from left to right and from top to bottom in each column) $D_1 = \{(1,1), (2,1)\}$ and $D_2 = \{(3,1), (4,1)\}$ and $D_3 = \{(1,2), (2,2)\}$ and $D_4 = \{(1,3), (2,3)\}$, and the cells $c_1, d_1, \ldots, c_4, d_4$ look as follows:

c_1	c_3	c_4	
d_1	d_3	d_4	
c_2			
d_2			

.

Let $\operatorname{Tab}(\lambda)$ be the set of all n-tableaux of shape λ. Note that these n-tableaux are defined as the bijections $Y(\lambda) \to [n]$; thus, there are $n!$ of them. For any n-tableau

$S \in \operatorname{Tab}(\lambda)$, we write $\mathbf{e}_S$ for the corresponding polytabloid in the Specht module $\mathcal{S}^\lambda$ (see [Grinbe24, Definition 5.4.1 **(a)**]).

An n-tableau $S \in \operatorname{Tab}(\lambda)$ will be called *domino-standard* if it satisfies $S(c_m) < S(d_m)$ for each $m \in [k]$. An n-tableau $S \in \operatorname{Tab}(\lambda)$ will be called *column-standard* if its entries increase down each column. Clearly, each column-standard tableau is domino-standard, but not conversely.

Now, we shall show that any n-tableau $T \in \operatorname{Tab}(\lambda)$ satisfies

$$G^*_{T(c_1),\ T(c_2),\ \ldots,\ T(c_k);\ T(d_1),\ T(d_2),\ \ldots,\ T(d_k)} \mathbf{e}_T \neq 0.$$

This will clearly prove Proposition 3.4.1. First, however, we need some lemmas.

If a and b are two integers, then $[a, b]$ shall denote the integer interval

$$\{x \in \mathbb{Z} \mid a \leq x \leq b\} = \{a, a+1, \ldots, b\}.$$

Lemma 3.4.2. Let $t_1, t_2, \ldots, t_m$ be m distinct integers. Assume that

$$t_2 < t_4 < t_6 < \cdots \tag{13}$$

(the last element of this chain of inequalities is either t_m or t_{m-1}, depending on whether m is even or odd). Assume furthermore that

$$t_{i-1} < t_i \qquad \text{for each even } i \in [m]. \tag{14}$$

Finally, assume that there exist no $a, b \in [m]$ satisfying $a \equiv b \bmod 2$ and $t_a < t_b$ with the property that none of the numbers $h \in \{t_1, t_2, \ldots, t_m\}$ satisfy $t_a < h < t_b$. Then,

$$t_1 < t_2 < \cdots < t_m.$$

Proof. We let $\{1, 3, 5, \ldots\}_{\leq m}$ denote the set of all odd integers in $[m]$. Likewise, $\{2, 4, 6, \ldots\}_{\leq m}$ shall denote the set of all even integers in $[m]$.

The m integers $t_1, t_2, \ldots, t_m$ are distinct, but we only care about their relative order, not about their exact values. Thus, we can WLOG assume that they are $1, 2, \ldots, m$ in some order (otherwise, just relabel them as $1, 2, \ldots, m$ preserving their relative order). Assume this. Then, the assumption "there exist no $a, b \in [m]$ satisfying $a \equiv b \bmod 2$ and $t_a < t_b$ with the property that none of the numbers $h \in \{t_1, t_2, \ldots, t_m\}$ satisfy $t_a < h < t_b$" can be rewritten in the simpler form "there exist no $a, b \in [m]$ satisfying $a \equiv b \bmod 2$ and $t_a < t_b$ with the property that $t_b - t_a = 1$". In other words, there exist no $a, b \in [m]$ satisfying $a \equiv b \bmod 2$ such that t_a and t_b are two consecutive integers. That is, no two of the integers $t_1, t_3, t_5, \ldots$ (the list continues until running out of subscripts) are consecutive, and no two of the integers $t_2, t_4, t_6, \ldots$ are consecutive. In other words, the two sets $\{t_1, t_3, t_5, \ldots\}$ and $\{t_2, t_4, t_6, \ldots\}$ are lacunar[3]. Of course, these two sets are disjoint, and their union is $\{t_1, t_2, \ldots, t_m\} = [m]$ (since we assumed that the integers $t_1, t_2, \ldots, t_m$ are

[3] A subset of $\mathbb{Z}$ is said to be *lacunar* if it contains no two consecutive integers.

$1, 2, \ldots, m$ in some order). But it is not hard to see that the only ways to write the set $[m]$ as a union $[m] = L \cup M$ of two disjoint lacunar sets L and M are

$$[m] = \{1, 3, 5, \ldots\}_{\leq m} \cup \{2, 4, 6, \ldots\}_{\leq m} \qquad \text{and}$$
$$[m] = \{2, 4, 6, \ldots\}_{\leq m} \cup \{1, 3, 5, \ldots\}_{\leq m}$$

(because if, say, $1 \in L$, then $2 \in M$ by the lacunarity of L, therefore $3 \in L$ by the lacunarity of M, therefore $4 \in M$ by the lacunarity of L, and so on, eventually resulting in $L = \{1, 3, 5, \ldots\}_{\leq m}$ and $M = \{2, 4, 6, \ldots\}_{\leq m}$; the case $1 \in M$ is analogous). Hence, we are in one of the following two cases:

Case 1: We have $\{t_1, t_3, t_5, \ldots\} = \{1, 3, 5, \ldots\}_{\leq m}$ and $\{t_2, t_4, t_6, \ldots\} = \{2, 4, 6, \ldots\}_{\leq m}$.

Case 2: We have $\{t_1, t_3, t_5, \ldots\} = \{2, 4, 6, \ldots\}_{\leq m}$ and $\{t_2, t_4, t_6, \ldots\} = \{1, 3, 5, \ldots\}_{\leq m}$.

Consider Case 1. In this case, $\{t_1, t_3, t_5, \ldots\} = \{1, 3, 5, \ldots\}_{\leq m}$ and $\{t_2, t_4, t_6, \ldots\} = \{2, 4, 6, \ldots\}_{\leq m}$. The latter equality, combined with (13), yields $t_2 = 2$ and $t_4 = 4$ and $t_6 = 6$ and so on. Hence, using (14), we easily see that $t_1 = 1$ (since (14) yields $t_1 < t_2 = 2$) and $t_3 = 3$ (since (14) yields $t_3 < t_4 = 4$, but the values 1 and 2 are already taken by t_1 and t_2) and $t_5 = 5$ (since (14) yields $t_5 < t_6 = 6$, but the values $1, 2, 3, 4$ are already taken by t_1, t_2, t_3, t_4) and so on (except that we don't get $t_m = m$ in this way if m is odd, since there is no t_{m+1}). Altogether, we thus have shown that $t_i = i$ for each $i \in [m-1]$ (whether even or odd). This entails that $t_m = m$ holds as well (since the numbers $t_1, t_2, \ldots, t_m$ are $1, 2, \ldots, m$ up to order, so one of them must be m), and thus we have $t_i = i$ for each $i \in [m]$. Thus, $t_1 < t_2 < \cdots < t_m$. This proves Lemma 3.4.2 in Case 1.

Now consider Case 2. In this case, $\{t_1, t_3, t_5, \ldots\} = \{2, 4, 6, \ldots\}_{\leq m}$ and $\{t_2, t_4, t_6, \ldots\} = \{1, 3, 5, \ldots\}_{\leq m}$. The latter equality, combined with (13), yields $t_2 = 1$ and $t_4 = 3$ and $t_6 = 5$ and so on. But (14) yields $t_1 < t_2 = 1$, which is absurd. Thus, Case 2 cannot happen, and so the proof of Lemma 3.4.2 is already complete. □

Lemma 3.4.3. Let $T \in \operatorname{Tab}(\lambda)$. Then, in $\mathcal{S}^\lambda$, we have

$$G^*_{T(c_1),\ T(c_2),\ \ldots,\ T(c_k);\ T(d_1),\ T(d_2),\ \ldots,\ T(d_k)} \mathbf{e}_T = \sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is domino-standard}}} \mathbf{e}_S.$$

Proof. Each n-tableau $S \in \operatorname{Tab}(\lambda)$ can be written as wT for a unique permutation $w \in S_n$. Moreover, the former n-tableau S is domino-standard if and only if it satisfies

$$S(c_s) < S(d_s) \qquad \text{for all } s \in [k],$$

that is, if the latter permutation $w \in S_n$ satisfies

$$w(T(c_s)) < w(T(d_s)) \qquad \text{for all } s \in [k]$$

(because $S = wT$ shows that $S(c_s) = w(T(c_s))$ and $S(d_s) = w(T(d_s))$ for all $s \in [k]$). Hence, there is a bijection

$$\{w \in S_n \mid w(T(c_s)) < w(T(d_s)) \text{ for all } s\} \to \{\text{domino-standard } S \in \operatorname{Tab}(\lambda)\},$$
$$w \mapsto wT.$$

Hence, we can substitute S for wT in the sum $\sum\limits_{\substack{S\in\operatorname{Tab}(\lambda)\\ \text{is domino-standard}}} \mathbf{e}_S$, and thus obtain

$$\sum_{\substack{S\in\operatorname{Tab}(\lambda)\\ \text{is domino-standard}}} \mathbf{e}_S = \sum_{\substack{w\in S_n;\\ w(T(c_s))<w(T(d_s)) \text{ for all } s}} \mathbf{e}_{wT}. \tag{15}$$

On the other hand, the definition of $G_{T(c_1),\ T(c_2),\ \ldots,\ T(c_k);\ T(d_1),\ T(d_2),\ \ldots,\ T(d_k)}$ yields

$$G_{T(c_1),\ T(c_2),\ \ldots,\ T(c_k);\ T(d_1),\ T(d_2),\ \ldots,\ T(d_k)} = \sum_{\substack{w\in S_n;\\ w(T(c_s))<w(T(d_s)) \text{ for all } s}} w^{-1}.$$

Applying the linear map $x \mapsto x^*$ to this equality (which sends each w^{-1} to w), we obtain

$$G^*_{T(c_1),\ T(c_2),\ \ldots,\ T(c_k);\ T(d_1),\ T(d_2),\ \ldots,\ T(d_k)} = \sum_{\substack{w\in S_n;\\ w(T(c_s))<w(T(d_s)) \text{ for all } s}} w.$$

Hence,

$$\begin{aligned}
&G^*_{T(c_1),\ T(c_2),\ \ldots,\ T(c_k);\ T(d_1),\ T(d_2),\ \ldots,\ T(d_k)} \mathbf{e}_T \\
&= \sum_{\substack{w\in S_n;\\ w(T(c_s))<w(T(d_s)) \text{ for all } s}} \underbrace{w\mathbf{e}_T}_{\substack{=\mathbf{e}_{wT}\\ \text{(by [Grinbe24, Lemma 5.4.6 (a)])}}} \\
&= \sum_{\substack{w\in S_n;\\ w(T(c_s))<w(T(d_s)) \text{ for all } s}} \mathbf{e}_{wT} \\
&= \sum_{\substack{S\in\operatorname{Tab}(\lambda)\\ \text{is domino-standard}}} \mathbf{e}_S \qquad (\text{by (15)}).
\end{aligned}$$

This proves Lemma 3.4.3. $\square$

Lemma 3.4.4. In $\mathcal{S}^\lambda$, we have

$$\sum_{\substack{S\in\operatorname{Tab}(\lambda)\\ \text{is domino-standard}}} \mathbf{e}_S = p \cdot \sum_{\substack{S\in\operatorname{Tab}(\lambda)\\ \text{is column-standard}}} \mathbf{e}_S.$$

Proof of Lemma 3.4.4. For any n-tableau $S \in \operatorname{Tab}(\lambda)$, let $\phi(S)$ be the column-standard tableau obtained from S by sorting the entries within each column S so that they increase from top to bottom. Thus, we have defined a map

$$\phi : \operatorname{Tab}(\lambda) \to \{\text{column-standard } Q \in \operatorname{Tab}(\lambda)\}.$$

For instance,

$$\phi : \begin{array}{|c|c|c|} \hline 3 & 2 & 5 \\ \hline 8 & 7 \\ \cline{1-2} 1 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 & 7 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array} .$$

We call ϕ the *column-sorting map*, since it sorts each column. Note that if S_1 and S_2 are two column-equivalent n-tableaux, then $\phi(S_1) = \phi(S_2)$.

Now, we claim a combinatorial equality:

Claim 1: Let $Q \in \operatorname{Tab}(\lambda)$ be column-standard. Then,

$$\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is domino-standard;} \\ \phi(S) = Q}} (-1)^{Q \circ S^{-1}} = p. \tag{16}$$

(Recall that S and Q are bijections from $Y(\lambda)$ to $[n]$, so that $Q \circ S^{-1} : [n] \to [n]$ is a well-defined permutation of $[n]$ and thus has a sign $(-1)^{Q \circ S^{-1}}$.)

Proof of Claim 1. The below proof is secretly a retelling of Gjergji Zaimi's sign-reversing involution in `https://math.stackexchange.com/a/58941/`, although it would take us even further afield to explain the precise correspondence.

The n-tableau Q is column-standard, thus domino-standard, and clearly satisfies $\phi(Q) = Q$. Hence, the sum on the left hand side of (16) contains at least the addend for $S = Q$, and this addend is $(-1)^{Q \circ Q^{-1}} = (-1)^{\text{id}} = 1$.

If a cell $c \in Y(\lambda)$ belongs to one of the dominoes $D_1, D_2, \ldots, D_k$, then the *partner* of c shall mean the unique cell of this domino that is not c. A cell that doesn't belong to any of $D_1, D_2, \ldots, D_k$ has no partner. Thus, for each $m \in [k]$, the cells c_m and d_m are each other's partners.

For any n-tableau $S \in \operatorname{Tab}(\lambda)$ and any $i \in [n]$, we let $r_S(i)$ denote the number of the row in which S contains the entry i. A *swappable pair* of an n-tableau $S \in \operatorname{Tab}(\lambda)$ shall mean a pair (i, j) of elements of $[n]$ such that

1. we have $i < j$;

2. the entries i and j lie in the same column of S;

3. we have $r_S(i) \equiv r_S(j) \bmod 2$ (that is, the distance between the cells of S that contain i and j is even);

4. none of the numbers h that lie in the same column of S as i and j satisfy $i < h < j$.

For instance, the 9-tableau

<table><tr><td>4</td><td>5</td><td>8</td></tr><tr><td>7</td><td>6</td><td>9</td></tr><tr><td>1</td><td>3</td><td></td></tr><tr><td>2</td><td></td><td></td></tr></table>

has only one swappable pair, namely $(3,5)$. (The pairs $(1,4)$ and $(2,7)$ would fail the fourth condition.)

We note that a column-standard tableau S will never have any swappable pairs. (Indeed, if (i,j) is a swappable pair of S, then conditions 2 and 3 in the definition of "swappable pair" ensure that there is at least one cell between the cells containing i and j in S in the column that contains these two cells; but then the column-standardness of S implies that this cell contains some entry h satisfying $i < h < j$, and this contradicts condition 4 of the definition.)

It is easy to see that if (i,j) is a swappable pair of an n-tableau $S \in \operatorname{Tab}(\lambda)$, then the n-tableau $t_{i,j}S$ (that is, the tableau obtained from S by swapping the entries i and j) has the properties that

$$\begin{aligned} &S \text{ is domino-standard} \\ &\text{if and only if } t_{i,j}S \text{ is domino-standard} \end{aligned} \tag{17}$$

(since the condition "$r_S(i) \equiv r_S(j) \bmod 2$" ensures that each of the dominoes $D_1, D_2, \ldots, D_k$ contains at most one of i and j, and then the condition "none of the numbers h that lie in the same column of S as i and j satisfy $i < h < j$" ensures that swapping i with j does not disturb their order relations with their potential partners) and

$$\phi(t_{i,j}S) = \phi(S) \tag{18}$$

(since $t_{i,j}$ merely swaps two entries in the same column of S). Also, in this situation, the tableau $t_{i,j}S$ has the same swappable pairs as S.

We say that an n-tableau $S \in \operatorname{Tab}(\lambda)$ is *swappable* if it has a swappable pair; otherwise, we call it *unswappable*. We have

$$\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is domino-standard} \\ \text{and swappable;} \\ \phi(S) = Q}} (-1)^{Q \circ S^{-1}} = 0, \tag{19}$$

since we can find a sign-reversing involution on the indexing set of this sum (this involution takes any swappable domino-standard $S \in \operatorname{Tab}(\lambda)$ satisfying $\phi(S) = Q$, finds its lexicographically minimal swappable pair (i,j), and swaps the entries i and j in S, that is, sends S to $t_{i,j}S$; then, because of (18) and (19), the tableau $t_{i,j}S$ is again a swappable domino-standard n-tableau in $\operatorname{Tab}(\lambda)$ satisfying $\phi(t_{i,j}S) = Q$, so we have an involution on our hands[4]; and this involution is sign-reversing because

$$\begin{aligned}(-1)^{Q\circ\left(t_{i,j}S\right)^{-1}} &= (-1)^{Q\circ S^{-1}\circ t_{i,j}} \qquad \left(\text{since } \left(t_{i,j}S\right)^{-1} = S^{-1}\circ t_{i,j}\right)\\ &= (-1)^{Q\circ S^{-1}} \underbrace{(-1)^{t_{i,j}}}_{=-1} = (-1)^{Q\circ S^{-1}}\end{aligned}$$

for each $S \in \operatorname{Tab}(\lambda)$). Now, each $S \in \operatorname{Tab}(\lambda)$ is either swappable or unswappable (but never both); thus,

$$\begin{aligned}\sum_{\substack{S\in\operatorname{Tab}(\lambda)\\ \text{is domino-standard;}\\ \phi(S)=Q}} (-1)^{Q\circ S^{-1}} &= \underbrace{\sum_{\substack{S\in\operatorname{Tab}(\lambda)\\ \text{is domino-standard}\\ \text{and swappable;}\\ \phi(S)=Q}} (-1)^{Q\circ S^{-1}}}_{\substack{=0\\ (\text{by (19)})}} + \sum_{\substack{S\in\operatorname{Tab}(\lambda)\\ \text{is domino-standard}\\ \text{and unswappable;}\\ \phi(S)=Q}} (-1)^{Q\circ S^{-1}}\\ &= \sum_{\substack{S\in\operatorname{Tab}(\lambda)\\ \text{is domino-standard}\\ \text{and unswappable;}\\ \phi(S)=Q}} (-1)^{Q\circ S^{-1}}.\end{aligned}$$

Hence, in order to prove Claim 1, it suffices to show that

$$\sum_{\substack{S\in\operatorname{Tab}(\lambda)\\ \text{is domino-standard}\\ \text{and unswappable;}\\ \phi(S)=Q}} (-1)^{Q\circ S^{-1}} = p. \tag{20}$$

For this, we need to better understand the structure of unswappable domino-standard n-tableaux $S \in \operatorname{Tab}(\lambda)$.

To do so, we define yet another operation of n-tableaux of shape $Y(\lambda)$. Namely, a *biswap* shall mean an operation that can be applied to an n-tableau $S \in \operatorname{Tab}(\lambda)$. It proceeds by choosing two distinct dominoes D_a and D_b that lie in the same column, and swapping the values of S in cells c_a and c_b, and simultaneously swapping the values of S in cells d_a and d_b (so that the pair of values of S in the two cells of D_a gets swapped with the corresponding set in D_b). Visually speaking, this amounts to swapping the entire dominoes D_a and D_b (including all their entries). For instance, one biswap takes the 9-tableau $\begin{array}{|c|c|}\hline 3 & 7\\ \hline 6 & 5\\ \hline 2 & 9\\ \hline 4 & \\ \cline{1-1} 1 & \\ \cline{1-1} 8 & \\ \cline{1-1}\end{array}$ to $\begin{array}{|c|c|}\hline 1 & 7\\ \hline 8 & 5\\ \hline 2 & 9\\ \hline 4 & \\ \cline{1-1} 3 & \\ \cline{1-1} 6 & \\ \cline{1-1}\end{array}$ (where the yellow and the green

[4]Here we are using the fact that the tableau $t_{i,j}S$ has the same swappable pairs as S.

rectangles are the two dominoes D_a and D_b that we have swapped). Note that entries in cells outside of $D_1, D_2, \ldots, D_k$ do not change at all under biswaps.

Note a few properties of biswaps:

- Any biswap can be undone by another biswap. Thus, there is an equivalence relation on $\operatorname{Tab}(\lambda)$ in which two n-tableaux $S_1, S_2 \in \operatorname{Tab}(\lambda)$ are equivalent if and only if one of them can be transformed into the other by a sequence of biswaps.
- Each equivalence class of this equivalence relation has size p. (*Proof:* This is saying that any given n-tableau $S \in \operatorname{Tab}(\lambda)$ can be transformed into exactly p distinct n-tableaux using sequences of biswaps. But the definition of a biswap shows that a sequence of biswaps amounts to an arbitrary permutation of the dominoes (or, more precisely, their sets of entries) **within each column** of S (since any permutation of dominoes can be achieved by successively swapping pairs of dominoes); since the number of dominoes inside the i-th column of S is $\lfloor \lambda_i^t / 2 \rfloor$, we thus conclude that the number of such permutations is $\prod_{i \geq 1} \lfloor \lambda_i^t / 2 \rfloor ! = p$. All these p permutations lead to different n-tableaux, since the entries of S are distinct. Hence, in total, we get p distinct tableaux by applying sequences of biswaps to S.)
- If we apply a biswap to a domino-standard tableau, then we obtain another domino-standard tableau. (Indeed, domino-standardness is preserved because a biswap "moves dominoes wholesale".) In other words, biswaps preserve domino-standardness.
- The set of swappable pairs of a tableau $S \in \operatorname{Tab}(\lambda)$ is preserved under biswaps (since a biswap keeps each entry in its original column, and can only move it by an even number of rows). Thus, if we apply a biswap to an unswappable tableau, then we obtain another unswappable tableau. In other words, biswaps preserve unswappability.
- When we apply a biswap to an n-tableau $S \in \operatorname{Tab}(\lambda)$, the sign $(-1)^{Q \circ S^{-1}}$ is unchanged. (This is because a biswap amounts to **two** swaps of two entries, so that the permutation $Q \circ S^{-1} \in S_n$ is multiplied by **two** transpositions and therefore preserves its sign.)

Now, let $\operatorname{BSTab}(Q)$ be the set of all tableaux $S \in \operatorname{Tab}(\lambda)$ that can be obtained from Q by a sequence of biswaps. This is an equivalence class of the equivalence relation mentioned in the first bullet point above, and thus has size p (by the second bullet point). Moreover, it contains the tableau Q, which is domino-standard (since it is column-standard) and unswappable (since a column-standard tableau will never have any swappable pairs). Hence, **all** tableaux $S \in \operatorname{BSTab}(Q)$ are domino-standard and unswappable (since biswaps preserve domino-standardness and unswappability), and of course satisfy $\phi(S) = Q$ (since S is obtained from Q

by a sequence of biswaps, hence is column-equivalent to Q, and thus transforms back into Q when we apply the column-sorting map ϕ). Thus,

$$\operatorname{BSTab}(Q) \subseteq \{S \in \operatorname{Tab}(\lambda) \text{ is domino-standard and unswappable } \mid \phi(S) = Q\}. \qquad (21)$$

We shall now prove the converse inclusion. For this purpose, let $S \in \operatorname{Tab}(\lambda)$ be domino-standard and unswappable such that $\phi(S) = Q$. We shall show that $S \in \operatorname{BSTab}(Q)$.

Indeed, consider the elements of S in the **even rows** (i.e., in the 2-nd, 4-th, 6-th, etc. rows). In each column of S, we can permute the entries in the even rows arbitrarily using a sequence of biswaps (at the cost of also permuting some entries in odd rows). (For instance, we can swap the entries $S(2r_1, j)$ and $S(2r_2, j)$ by applying the biswap that swaps the domino containing the former with the domino containing the latter. As a side-effect, this biswap will also swap the entries $S(2r_1 - 1, j)$ and $S(2r_2 - 1, j)$, but this does not trouble us, since these entries do not lie in even rows. By a sequence of such swaps, we can achieve any permutation of the entries in the even rows in the j-th column of S.)

Thus, in particular, we can apply a sequence of biswaps to S that results in a tableau S' whose entries in the even rows increase top-to-bottom down each column (i.e., that satisfies $S'(2,j) < S'(4,j) < S'(6,j) < \cdots < S'\left(2\left\lfloor \lambda_j^t/2\right\rfloor, j\right)$ for each $j \geq 1$). Consider this tableau S'. Then, S' is obtained from S by a sequence of biswaps. Since S is domino-standard and unswappable, the same holds for S' (because biswaps preserve domino-standardness and unswappability). Fix $j \geq 1$. Then,

$$S'(2,j) < S'(4,j) < S'(6,j) < \cdots < S'\left(2\left\lfloor \lambda_j^t/2\right\rfloor, j\right)$$

(by the construction of S') and

$$S'(i-1,j) < S'(i,j) \qquad \text{for each even } i \in \left[\lambda_j^t\right]$$

(since S' is domino-standard), and furthermore, there exist no $a, b \in \left[\lambda_j^t\right]$ satisfying $a \equiv b \bmod 2$ and $S'(a,j) < S'(b,j)$ with the property that none of the numbers $h \in \left\{S'(1,j), S'(2,j), \ldots, S'\left(\lambda_j^t, j\right)\right\}$ satisfy $S'(a,j) < h < S'(b,j)$ (since S' is unswappable, but such a and b would make $(S'(a,j), S'(b,j))$ a swappable pair of S'). Therefore, Lemma 3.4.2 (applied to $m = \lambda_j^t$ and $t_i = S'(i,j)$) shows that

$$S'(1,j) < S'(2,j) < \cdots < S'\left(\lambda_j^t, j\right).$$

We have proved this for each $j \geq 1$, so we have proved that S' is column-standard. Thus, $\phi(S') = S'$ (since ϕ fixes every column-standard tableau). But S' is obtained from S by a sequence of biswaps, hence is column-equivalent to S; thus, $\phi(S') = \phi(S) = Q$. Therefore, $Q = \phi(S') = S'$. Since S' is obtained from S by a sequence of biswaps, we thus conclude that Q is obtained from S by a sequence of biswaps.

Therefore, in turn, S is obtained from Q by a sequence of biswaps. Hence, $S \in \operatorname{BSTab}(Q)$.

Forget that we fixed S. We thus have shown that $S \in \operatorname{BSTab}(Q)$ whenever $S \in \operatorname{Tab}(\lambda)$ is domino-standard and unswappable such that $\phi(S) = Q$. In other words,

$$\{S \in \operatorname{Tab}(\lambda) \text{ is domino-standard and unswappable } \mid \phi(S) = Q\} \subseteq \operatorname{BSTab}(Q).$$

Combining this with (21), we obtain

$$\{S \in \operatorname{Tab}(\lambda) \text{ is domino-standard and unswappable } \mid \phi(S) = Q\} = \operatorname{BSTab}(Q).$$

Hence,

$$\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is domino-standard} \\ \text{and unswappable;} \\ \phi(S) = Q}} (-1)^{Q \circ S^{-1}} = \sum_{S \in \operatorname{BSTab}(Q)} (-1)^{Q \circ S^{-1}}.$$

Furthermore, if $S \in \operatorname{BSTab}(Q)$, then $(-1)^{Q \circ S^{-1}} = 1$ (because when we apply a biswap to an n-tableau $S \in \operatorname{Tab}(\lambda)$, the sign $(-1)^{Q \circ S^{-1}}$ is unchanged; but $S \in \operatorname{BSTab}(Q)$ shows that S can be obtained from Q by a sequence of biswaps; hence, $(-1)^{Q \circ S^{-1}} = (-1)^{Q \circ Q^{-1}} = (-1)^{\operatorname{id}} = 1$). Hence,

$$\sum_{S \in \operatorname{BSTab}(Q)} \underbrace{(-1)^{Q \circ S^{-1}}}_{=1} = \sum_{S \in \operatorname{BSTab}(Q)} 1 = |\operatorname{BSTab}(Q)| = p$$

(since $\operatorname{BSTab}(Q)$ has size p). Therefore,

$$\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is domino-standard} \\ \text{and unswappable;} \\ \phi(S) = Q}} (-1)^{Q \circ S^{-1}} = \sum_{S \in \operatorname{BSTab}(Q)} (-1)^{Q \circ S^{-1}} = p.$$

This proves (20), and thus concludes the proof of Claim 1. □

Now,

$$\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is domino-standard}}} \mathbf{e}_S = \sum_{\substack{Q \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \ \sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is domino-standard;} \\ \phi(S) = Q}} \mathbf{e}_S \tag{22}$$

(here, we have split up the sum according to the value of $\phi(S)$, noting that $\phi(S)$ is always column-standard). However, if $Q, S \in \operatorname{Tab}(\lambda)$ are two n-tableaux satisfying $\phi(S) = Q$, then Q is column-equivalent to S (since ϕ is the column-sorting map, which only moves the entries within their columns), so that $Q \circ S^{-1} \in \mathcal{C}(S)$ and thus $\mathbf{e}_Q = (-1)^{Q \circ S^{-1}} \mathbf{e}_S$ (by [Grinbe24, Lemma 5.4.6 **(a)**], since $Q = (Q \circ S^{-1}) S$ and

$Q \circ S^{-1} \in \mathcal{C}(S)$), so that $\mathbf{e}_S = (-1)^{Q \circ S^{-1}} \mathbf{e}_Q$ (since the number $(-1)^{Q \circ S^{-1}} \in \{1, -1\}$ is its own inverse). Hence, (22) can be rewritten as

$$\begin{aligned}
&\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is domino-standard}}} \mathbf{e}_S \\
&= \sum_{\substack{Q \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \underbrace{\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is domino-standard;} \\ \phi(S) = Q}} (-1)^{Q \circ S^{-1}}}_{\substack{= p \\ \text{(by Claim 1)}}} \mathbf{e}_Q \\
&= \sum_{\substack{Q \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} p \mathbf{e}_Q = \sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} p \mathbf{e}_S = p \cdot \sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \mathbf{e}_S.
\end{aligned}$$

This proves Lemma 3.4.4. □

Lemma 3.4.5. In $\mathcal{S}^\lambda$, we have

$$\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \mathbf{e}_S \neq 0.$$

Proof. We recall that the Specht module $\mathcal{S}^\lambda$ is a submodule of the Young module $\mathcal{M}^\lambda$, which has a basis consisting of all n-tabloids $\overline{T}$ of shape λ (see [Grinbe24, Definition 5.3.15]). The set $\{n\text{-tabloids of shape } \lambda\}$ is equipped with a total order called the *Young last letter order* (see [Grinbe24, Proposition 5.7.7]); explicitly, two n-tabloids $\overline{P}$ and $\overline{Q}$ satisfy $\overline{P} < \overline{Q}$ if and only if the **largest** number $i \in [n]$ that lies in different rows in $\overline{P}$ and $\overline{Q}$ appears further north in $\overline{P}$ than in $\overline{Q}$. Note that the largest n-tabloid $\overline{S}$ of shape λ with respect to this order is the n-tabloid $\overline{S_{\max}}$, where $S_{\max} \in \operatorname{Tab}(\lambda)$ is the standard tableau that contains the numbers $1, 2, \ldots, \lambda_1$ in its first row, the numbers $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$ in its second row, and so on (i.e., the cells of $S_{\max}$ are filled with the numbers $1, 2, \ldots, n$ in lexicographic order).

It is well-known ([Grinbe24, Lemma 5.7.9 **(b)**]) that each column-standard n-tableau $S \in \operatorname{Tab}(\lambda)$ satisfies

$$\mathbf{e}_S = \overline{S} + \left(\text{a linear combination of } n\text{-tabloids } \overline{P} \text{ with } \overline{P} < \overline{S}\right).$$

Hence,

$$\begin{aligned}
&\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \mathbf{e}_S \\
&= \sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \left(\overline{S} + \left(\text{a linear combination of } n\text{-tabloids } \overline{P} \text{ with } \overline{P} < \overline{S}\right)\right) \\
&= \kappa \overline{S_{\max}} + \left(\text{a linear combination of } n\text{-tabloids } \overline{P} \text{ with } \overline{P} < \overline{S_{\max}}\right) \qquad (23)
\end{aligned}$$

(since $\overline{S_{\max}}$ is the largest n-tabloid $\overline{S}$ of shape λ), where κ is the number of column-standard n-tableaux $S \in \operatorname{Tab}(\lambda)$ satisfying $\overline{S} = \overline{S_{\max}}$. Now, it is clear that κ is a positive integer, and in fact κ can be easily computed explicitly: **Any** n-tableau $S \in \operatorname{Tab}(\lambda)$ that satisfies $\overline{S} = \overline{S_{\max}}$ (that is, is row-equivalent to $S_{\max}$) is column-standard (since **any** entry in the i-th row of $S_{\max}$ is larger than **any** entry in the j-th row of $S_{\max}$ when $i < j$, and thus a horizontal permutation cannot break the column-standardness of $S_{\max}$). Thus, κ is simply the number of all n-tableaux $S \in \operatorname{Tab}(\lambda)$ satisfying $\overline{S} = \overline{S_{\max}}$. In other words, κ is the number of all n-tableaux S that are row-equivalent to $S_{\max}$. Therefore,

$$\kappa = |\mathcal{R}(S_{\max})| = \lambda_1!\lambda_2!\lambda_3!\cdots$$

(by [Grinbe24, Proposition 5.5.7 **(a)**]). Either way, we conclude that $\kappa \neq 0$. Thus, the right hand side of the equality (23) has the form "nonzero multiple of $\overline{S_{\max}}$ plus a linear combination of smaller n-tabloids", and therefore is nonzero. Hence, so is the left hand side. This proves Lemma 3.4.5. □

Proof of Proposition 3.4.1. Let $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k$ be the $2k$ distinct integers

$$T(c_1), \ T(c_2), \ \ldots, \ T(c_k), \ T(d_1), \ T(d_2), \ \ldots, \ T(d_k)$$

(these are distinct, since all entries of T are distinct). Then,

$$\begin{aligned} G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}\mathbf{e}_T &= G^*_{T(c_1),\ T(c_2),\ \ldots,\ T(c_k);\ T(d_1),\ T(d_2),\ \ldots,\ T(d_k)}\mathbf{e}_T \\ &= \sum_{\substack{S\in\operatorname{Tab}(\lambda) \\ \text{is domino-standard}}} \mathbf{e}_S \qquad \text{(by Lemma 3.4.3)} \\ &= p \cdot \sum_{\substack{S\in\operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \mathbf{e}_S \qquad \text{(by Lemma 3.4.4)}. \end{aligned}$$

Since $p = \prod_{i\geq 1} \lfloor \lambda_i^t/2 \rfloor! \neq 0$ and $\sum_{\substack{S\in\operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \mathbf{e}_S \neq 0$ (by Lemma 3.4.5), the right hand side of this equality is $\neq 0$. Thus, so is the left hand side. In other words, $G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}\mathbf{e}_T \neq 0$. This proves Proposition 3.4.1. □

3.4.4. Consequences of nonvanishing

In order to draw conclusions from Proposition 3.4.1, we need the following fact about Specht modules:

Lemma 3.4.6. Let V be a left ideal of $\mathcal{A}$. Let λ be a partition of n. Then, $\operatorname{Hom}_{\mathcal{A}}(\mathcal{S}^\lambda, V) \cong V^*\mathcal{S}^\lambda$ as $\mathbf{k}$-vector spaces.

Proof. We have $\mathcal{S}^\lambda \cong \mathcal{A}\mathbf{E}_T$, where $\mathbf{E}_T = \nabla^-_{\operatorname{Col} T}\nabla_{\operatorname{Row} T}$ is the Young symmetrizer corresponding to any n-tableau T of shape λ (see [Grinbe24, §5.5] for all these notations). We know that $\mathbf{E}_T$ is quasi-idempotent, i.e., we have $\mathbf{E}_T^2 = h^\lambda \mathbf{E}_T$ for a nonzero scalar h^λ. Thus, setting $f := \dfrac{\mathbf{E}_T}{h^\lambda}$, we have $f^2 = f$ and $\mathcal{S}^\lambda \cong \mathcal{A}f$.

From $\mathcal{S}^\lambda \cong \mathcal{A}f$, we obtain the $\mathbf{k}$-vector space isomorphism

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, V\right) &\cong \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}f, V\right) \\
&\cong fV \qquad \left(\text{by Lemma 2.4.3, since } f^2 = f\right) \\
&\cong (fV)^* \qquad \left(\begin{array}{c}\text{since the antipode } \mathcal{A} \to \mathcal{A},\ x \mapsto x^* \\ \text{is a } \mathbf{k}\text{-vector space isomorphism}\end{array}\right) \\
&= V^* f^* \qquad \left(\text{since the antipode is a } \mathbf{k}\text{-algebra anti-morphism}\right).
\end{aligned}$$

But $f = \dfrac{\mathbf{E}_T}{h^\lambda}$ yields $f^* = \dfrac{\mathbf{E}_T^*}{h^\lambda} = \dfrac{\mathbf{F}_T}{h^\lambda}$ using the notation $\mathbf{F}_T$ from [Grinbe24, Proposition 5.11.19]. Thus, $\mathcal{A}f^* = \mathcal{A}\mathbf{F}_T$. But $\mathcal{A}\mathbf{F}_T \cong \mathcal{S}^\lambda$ by [Grinbe24, Proposition 5.11.19 **(c)**]. Hence, $\mathcal{A}f^* = \mathcal{A}\mathbf{F}_T \cong \mathcal{S}^\lambda$.

Moreover, V^* is a right ideal of $\mathcal{A}$ (since V is a left ideal), and therefore $V^* = V^*\mathcal{A}$. Hence,

$$\underbrace{V^*}_{=V^*\mathcal{A}} f^* = V^* \underbrace{\mathcal{A}f^*}_{\cong \mathcal{S}^\lambda} \cong V^* \mathcal{S}^\lambda.$$

Therefore, our above isomorphism becomes

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, V\right) \cong V^* f^* \cong V^* \mathcal{S}^\lambda.$$

This proves Lemma 3.4.6. □

Corollary 3.4.7. Let λ be a partition of n. Then:

(a) We have $\mathcal{G}^* \mathcal{S}^\lambda \neq 0$.

(b) We have $\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{G}\right) \neq 0$.

Proof. **(a)** This follows from Proposition 3.4.1, since $G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k} \in \mathcal{G}$ and $\mathbf{e}_T \in \mathcal{S}^\lambda$.

(b) Lemma 3.4.6 yields $\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{G}\right) \cong \mathcal{G}^* \mathcal{S}^\lambda \neq 0$ by part **(a)**. This completes the proof of part **(b)**. □

3.4.5. Proof of the Gelfand model

Proof of Theorem 3.2.1. A *matching* of $[n]$ shall mean a set of disjoint 2-element subsets of $[n]$ (that is, an m-matching of $[n]$ for some $m \in \mathbb{Z}$).

Proposition 3.4.7 **(b)** shows that each Specht module $\mathcal{S}^\lambda$ for $\lambda \vdash n$ is contained in $\mathcal{G}$ (that is, can be embedded in $\mathcal{G}$). Since these Specht modules are irreducible and mutually non-isomorphic, this shows that their direct sum $\bigoplus_{\lambda \vdash n} \mathcal{S}^\lambda$ is contained in $\mathcal{G}$ as well (since Proposition 2.10.1 says that a sum of non-isomorphic irreducible representations is always a direct sum).

Now we shall show that the dimensions of $\bigoplus_{\lambda \vdash n} \mathcal{S}^\lambda$ and $\mathcal{G}$ are the same. Indeed, recall the filtration $(0 = \mathcal{G}_{-1} \subseteq \mathcal{G}_0 \subseteq \cdots \subseteq \mathcal{G}_n = \mathcal{G})$ from Subsection 3.3. Thus,

$$\begin{aligned}
\dim \mathcal{G} &= \sum_{m=0}^{n} \underbrace{\dim \left(\mathcal{G}_m / \mathcal{G}_{m-1}\right)}_{\substack{\leq (\text{\# of } m\text{-matchings of } [n]) \\ (\text{by (10)})}} \\
&\leq \sum_{m=0}^{n} (\text{\# of } m\text{-matchings of } [n]) \\
&= (\text{\# of matchings of } [n]) \\
&= (\text{\# of involutions of } [n])
\end{aligned} \tag{24}$$

(since the matchings of $[n]$ are in bijection with the involutions of $[n]$ [5]). But a well-known enumerative result (see, e.g., [Leeuwe95, Proposition 1.3.2] or [Grinbe24, Corollary 5.21.11]) says that

$$\begin{aligned}
&(\text{\# of involutions of } [n]) \\
&= \sum_{\lambda \vdash n} (\text{\# of standard tableaux of shape } \lambda) .
\end{aligned} \tag{25}$$

Furthermore, it is well-known that

$$\dim \mathcal{S}^\lambda = (\text{\# of standard tableaux of shape } \lambda) \tag{26}$$

for each $\lambda \vdash n$. Hence, (24) becomes

$$\begin{aligned}
\dim \mathcal{G} &\leq (\text{\# of involutions of } [n]) \\
&= \sum_{\lambda \vdash n} \underbrace{(\text{\# of standard tableaux of shape } \lambda)}_{\substack{= \dim \mathcal{S}^\lambda \\ (\text{by (26)})}} \qquad (\text{by (25)}) \\
&= \sum_{\lambda \vdash n} \dim \mathcal{S}^\lambda = \dim \bigoplus_{\lambda \vdash n} \mathcal{S}^\lambda .
\end{aligned}$$

Since $\bigoplus_{\lambda \vdash n} \mathcal{S}^\lambda$ is contained in $\mathcal{G}$, we thus conclude that the direct sum $\bigoplus_{\lambda \vdash n} \mathcal{S}^\lambda$ is $\mathcal{G}$. In other words, $\mathcal{G}$ is a Gelfand model, qed. □

[5] The bijection is well-known and simple: If $M = \{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_m, j_m\}\}$ is a matching of $[n]$, then $t_{i_1,j_1} t_{i_2,j_2} \cdots t_{i_m,j_m} \in S_n$ is an involution of $[n]$. Conversely, if σ is an involution on $[n]$, then the corresponding matching is $\{\{i, \sigma(i)\} \mid i \in [n] \text{ with } \sigma(i) \neq i\}$.

3.5. Further corollaries

Having proved Theorem 3.2.1, let us now reap some rewards from our above arguments.

Corollary 3.5.1. Let λ be a partition of n. Then:

(a) We have $\dim\left(\mathcal{G}^*\mathcal{S}^\lambda\right) = 1$.

(b) We have $\dim\left(\operatorname{Hom}_\mathcal{A}\left(\mathcal{S}^\lambda, \mathcal{G}\right)\right) = 1$.

(c) We have
$$\mathcal{G}^*\mathcal{S}^\lambda = \mathbf{k} \cdot \sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \mathbf{e}_S.$$

Proof. **(b)** Theorem 3.2.1 yields that $\mathcal{G}$ is multiplicity-free as a left $\mathcal{A}$-module. Meanwhile, $\mathcal{S}^\lambda$ is an irreducible representation of S_n. Thus, Lemma 2.5.1 (applied to $J = \mathcal{G}$ and $K = \mathcal{S}^\lambda$) yields $\dim\left(\operatorname{Hom}_\mathcal{A}\left(\mathcal{S}^\lambda, \mathcal{G}\right)\right) \leq 1$. But Corollary 3.4.7 **(b)** yields $\operatorname{Hom}_\mathcal{A}\left(\mathcal{S}^\lambda, \mathcal{G}\right) \neq 0$, thus $\dim\left(\operatorname{Hom}_\mathcal{A}\left(\mathcal{S}^\lambda, \mathcal{G}\right)\right) \geq 1$. Combining these two inequalities, we obtain $\dim\left(\operatorname{Hom}_\mathcal{A}\left(\mathcal{S}^\lambda, \mathcal{G}\right)\right) = 1$. This proves part **(b)**.

(a) Lemma 3.4.6 yields $\operatorname{Hom}_\mathcal{A}\left(\mathcal{S}^\lambda, \mathcal{G}\right) \cong \mathcal{G}^*\mathcal{S}^\lambda$. Hence, $\mathcal{G}^*\mathcal{S}^\lambda \cong \operatorname{Hom}_\mathcal{A}\left(\mathcal{S}^\lambda, \mathcal{G}\right)$, so that $\dim\left(\mathcal{G}^*\mathcal{S}^\lambda\right) = \dim\left(\operatorname{Hom}_\mathcal{A}\left(\mathcal{S}^\lambda, \mathcal{G}\right)\right) = 1$ by part **(b)**. This proves part **(a)**.

(c) Part **(a)** shows that $\dim\left(\mathcal{G}^*\mathcal{S}^\lambda\right) = 1$. Hence, in order to prove part **(c)**, it suffices to show that the vector $\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \mathbf{e}_S$ is nonzero and belongs to $\mathcal{G}^*\mathcal{S}^\lambda$. The former follows from Lemma 3.4.5. The latter is because $\mathcal{G}^*\mathcal{S}^\lambda$ contains

$$\begin{aligned}
& G^*_{T(c_1),\ T(c_2),\ \ldots,\ T(c_k);\ T(d_1),\ T(d_2),\ \ldots,\ T(d_k)} \mathbf{e}_T \\
&= \sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is domino-standard}}} \mathbf{e}_S \qquad \text{(by Lemma 3.4.3)} \\
&= p \cdot \sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \mathbf{e}_S \qquad \text{(by Lemma 3.4.4)}
\end{aligned}$$

and thus also $\sum_{\substack{S \in \operatorname{Tab}(\lambda) \\ \text{is column-standard}}} \mathbf{e}_S$ (since p is invertible). $\square$

3.6. The decomposition of $\mathcal{G}_m$ into Specht modules

Our next goal is to extend Theorem 3.2.1 from the entire representation $\mathcal{G}$ to its submodules $\mathcal{G}_m$, by identifying which Specht modules $\mathcal{S}^\lambda$ it contains. This requires some precursory work.

For each partition λ, let us set

$$k_\lambda := \sum_{i \geq 1} \left\lfloor \lambda_i^t / 2 \right\rfloor . \qquad (27)$$

If $m \in \mathbb{Z}$, and if S is a set, then an *m-matching* of S shall mean a set of m disjoint 2-element subsets of S. For example, $\{\{1,5\}, \{2,3\}\}$ is a 2-matching of $[7]$. An m-matching of $[n]$ will simply be called an "m-matching", without mention of $[n]$. We recall a well-known formula for counting m-matchings:

Proposition 3.6.1. Let $m \in \mathbb{N}$. Let N be an n-element set. Then,

$$(\text{\# of } m\text{-matchings of } N) = \binom{n}{2m} \frac{(2m)!}{2^m m!}.$$

Proof. We shall use the following terminology: If M is an m-matching of N, then:

- The elements of N that are contained in the edges of M are called the *participants* of M.
- For any participant p of M, there is a unique participant q of M such that $\{p, q\} \in M$; this q will be called the *M-partner* of p.

Note that any m-matching M of N necessarily has exactly $2m$ participants (since it has m edges, all of which are disjoint, and each of which contains 2 elements).

Now, pick an arbitrary total order on the set N. Each m-matching M of N can be constructed as follows:

1. Choose the $2m$ participants of M. This amounts to choosing a $2m$-element subset of N, and thus can be done in $\binom{n}{2m}$ ways.
2. Let p be the smallest participant of M. Choose the M-partner q of p. This can be done in $2m-1$ ways (since q must be one of the $2m$ participants but also differ from p). The edge $\{p, q\}$ will belong to M. We shall refer to the participants p and q as *matched* (since they have already been assigned their M-partners), and to the remaining $2m-2$ participants as *unmatched*.
3. Let p' be the smallest unmatched participant of M. Choose the M-partner q' of p'. This can be done in $2m-3$ ways (since q' must be one of the $2m-2$ unmatched participants but also differ from p'). The edge $\{p', q'\}$ will belong to M. The participants p' and q' now change their status from "unmatched" to "matched", so that $2m-4$ participants remain unmatched.
4. Let p'' be the smallest unmatched participant of M. Choose the M-partner q'' of p''. This can be done in $2m-5$ ways (since q'' must be one of the $2m-4$ unmatched participants but also differ from p''). The edge $\{p'', q''\}$ will belong to M. The participants p'' and q'' now change their status from "unmatched" to "matched", so that $2m-6$ participants remain unmatched.

5. And so on, until we are left with no more unmatched participants, and we have discovered all m edges in M.

The total number of ways to perform this construction is

$$\binom{n}{2m} \cdot \underbrace{(2m-1)(2m-3)(2m-5)\cdots 1}_{=\frac{(2m)!}{(2m)(2m-2)(2m-4)\cdots 2}=\frac{(2m)!}{2^m m!}} = \binom{n}{2m} \frac{(2m)!}{2^m m!}.$$

Hence, the # of m-matchings of N is $\binom{n}{2m} \frac{(2m)!}{2^m m!}$. This proves Proposition 3.6.1. $\square$

Let us now show that the inequality in Lemma 3.3.1 is an equality:

Corollary 3.6.2. Let $m \in \mathbb{Z}$. Then,

$$\dim(\mathcal{G}_m / \mathcal{G}_{m-1}) = (\text{\# of } m\text{-matchings of } [n]).$$

Proof. Forget that we fixed m. We must prove that

$$\dim(\mathcal{G}_m / \mathcal{G}_{m-1}) = (\text{\# of } m\text{-matchings of } [n]) \tag{28}$$

for all $m \in \mathbb{Z}$. However, in our above proof of Theorem 3.2.1, we added together the inequalities

$$\dim(\mathcal{G}_m / \mathcal{G}_{m-1}) \leq (\text{\# of } m\text{-matchings of } [n]) \tag{29}$$

for all $m \in \{0, 1, \ldots, n\}$, and obtained the inequality $\dim \mathcal{G} \leq \dim \bigoplus_{\lambda \vdash n} \mathcal{S}^\lambda$, which is actually an equality (since we showed shortly thereafter that the direct sum $\bigoplus_{\lambda \vdash n} \mathcal{S}^\lambda$ is $\mathcal{G}$). Hence, all the inequalities (29) must be equalities. In other words, (28) holds for all $m \in \{0, 1, \ldots, n\}$. Since (28) also holds for all $m \in \mathbb{Z} \setminus \{0, 1, \ldots, n\}$ (since for all such m, both sides of (28) are 0), we thus conclude that (28) holds for all $m \in \mathbb{Z}$. This proves Corollary 3.6.2. $\square$

Next, we need a combinatorial lemma:

Lemma 3.6.3. Let $m \in \mathbb{Z}$. Then,

$$(\text{\# of } m\text{-matchings of } [n]) = \sum_{\substack{\lambda \vdash n; \\ k_\lambda = m}} (\text{\# of standard tableaux of shape } \lambda).$$

First proof of Lemma 3.6.3 (sketched). Each m-matching $M = \{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_m, j_m\}\}$ of $[n]$ gives rise to an involution $t_{i_1,j_1} t_{i_2,j_2} \cdots t_{i_m,j_m} \in S_n$ that has exactly $n - 2m$ fixed points (viz., all elements of $[n]$ other than the $2m$ numbers $i_1, j_1, i_2, j_2, \ldots, i_m, j_m$). Thus, we obtain a bijection from the set $\{m\text{-matchings of } [n]\}$ to the set $\{\text{involutions of } [n] \text{ with exactly } n-2m \text{ fixed points}\}$. Hence,

$$\begin{aligned} & (\text{\# of } m\text{-matchings of } [n]) \\ & = (\text{\# of involutions of } [n] \text{ with exactly } n - 2m \text{ fixed points}) . \end{aligned} \tag{30}$$

But it is well-known that there is a bijection between the involutions of $[n]$ and the standard tableaux of all shapes $\lambda \vdash n$, given by the RSK algorithm (see, e.g., [Fulton97, last paragraph of §4.1] or [Stanle23, proof of Corollary 7.13.9]). Furthermore, it is known (see [Fulton97, §4.2, Exercise 4] or [Stanle23, Exercise 7.28 (a)]) that when this bijection sends an involution w to a standard tableau T of some shape λ, we have

$$\begin{aligned} (\text{\# of fixed points of } w) &= (\text{\# of odd-length columns of } Y(\lambda)) \\ &= n - 2k_\lambda . \end{aligned}$$

Hence, this bijection restricts to a bijection between the involutions of $[n]$ with exactly $n - 2m$ fixed points and the standard tableaux of all shapes $\lambda \vdash n$ satisfying $n - 2k_\lambda = n - 2m$, that is, $k_\lambda = m$. Thus, by the bijection principle,

$$\begin{aligned} & (\text{\# of involutions of } [n] \text{ with exactly } n - 2m \text{ fixed points}) \\ & = \sum_{\substack{\lambda \vdash n; \\ k_\lambda = m}} (\text{\# of standard tableaux of shape } \lambda) . \end{aligned}$$

In view of this, we can rewrite (30) as

$$(\text{\# of } m\text{-matchings of } [n]) = \sum_{\substack{\lambda \vdash n; \\ k_\lambda = m}} (\text{\# of standard tableaux of shape } \lambda) .$$

This proves Lemma 3.6.3. □

Second proof of Lemma 3.6.3 (sketched). The following proof avoids the use of the RSK algorithm. Instead, we proceed similarly to the proof of (25) in [Leeuwe95, §1.3]. (See also [Grinbe2X, §3.2].)

Given two partitions λ and μ, we write $\lambda \lessdot \mu$ if and only if $Y(\lambda) \subseteq Y(\mu)$ and $|Y(\mu) \setminus Y(\lambda)| = 1$ (in other words: if and only if μ covers λ in Young's lattice). Note that $\lambda \lessdot \mu$ entails $|\lambda| = |\mu| - 1$.

For any $n \in \mathbb{N}$, we define the polynomial

$$\sigma(n) := \sum_{\lambda \vdash n} f^\lambda t^{k_\lambda} \in \mathbb{Z}[t] ,$$

where f^{λ} is the number of standard tableaux of shape λ. Our main goal will be proving the recurrence

$$\sigma(n) = \sigma(n-1) + (n-1)\, t\, \sigma(n-2) \tag{31}$$

for all $n \geq 2$.

To this end, we begin by showing the following statement (which generalizes [Leeuwe95, (1)]):

Sublemma A: For any partition λ, we have

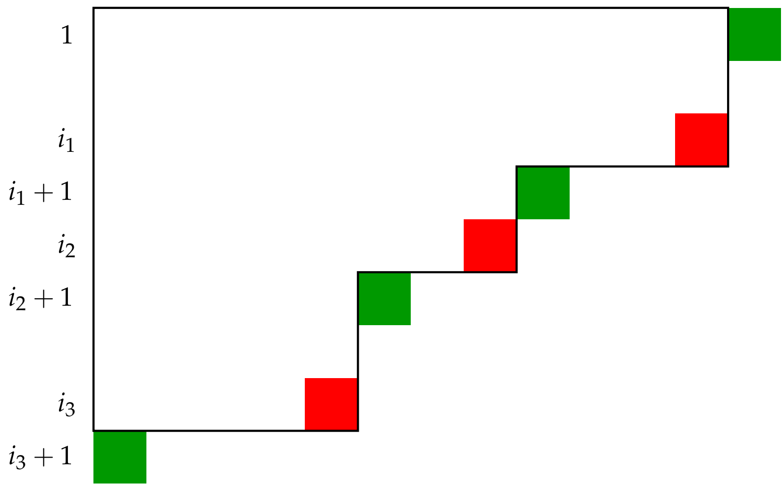
$$\sum_{\substack{\mu \text{ is a partition;} \\ \lambda \lessdot \mu}} t^{k_\mu} = t^{k_\lambda} + t \sum_{\substack{\mu \text{ is a partition;} \\ \mu \lessdot \lambda}} t^{k_\mu}.$$

Proof of Sublemma A. Regard partitions as infinite sequences of integers (with infinitely many 0's in their tails), and regard such sequences as elements of the $\mathbb{Z}$-module $\mathbb{Z}^{\infty}$. In particular, write λ as $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ (so that $\lambda_i = 0$ for all sufficiently large i). Let $\{i_1 < i_2 < \cdots < i_m\}$ be the set of all $i \geq 1$ satisfying $\lambda_i > \lambda_{i+1}$. For each $p \geq 1$, let ϵ_p be the vector $(0, 0, \ldots, 0, 1, 0, 0, 0, \ldots) \in \mathbb{Z}^{\infty}$ with the 1 in its p-th position. Then:[6]

- The partitions μ satisfying $\lambda \lessdot \mu$ are precisely

$$\lambda + \epsilon_1, \quad \lambda + \epsilon_{i_1+1}, \quad \lambda + \epsilon_{i_2+1}, \quad \ldots, \quad \lambda + \epsilon_{i_m+1},$$

[6] Here is an example, showing a partition λ as well as all partitions μ satisfying $\lambda \lessdot \mu$ (these are obtained by adding one of the green cells to the Young diagram of $Y(\lambda)$) as well as as all partitions μ satisfying $\mu \lessdot \lambda$ (these are obtained by removing one of the red cells from the Young diagram of $Y(\lambda)$):



and their respective numbers k_μ are

$$k_{\lambda+\epsilon_1} = k_\lambda \qquad \text{and}$$
$$k_{\lambda+\epsilon_{i_p+1}} = \begin{cases} k_\lambda, & \text{if } i_p \text{ is even;} \\ k_\lambda + 1, & \text{if } i_p \text{ is odd.} \end{cases} \tag{32}$$

Hence,

$$\sum_{\substack{\mu \text{ is a partition;} \\ \lambda \lessdot \mu}} t^{k_\mu} = t^{k_\lambda} + \sum_{p=1}^{m} \begin{cases} t^{k_\lambda}, & \text{if } i_p \text{ is even;} \\ t^{k_\lambda+1}, & \text{if } i_p \text{ is odd.} \end{cases} \tag{33}$$

- The partitions μ satisfying $\mu \lessdot \lambda$ are precisely

$$\lambda - \epsilon_{i_1}, \ \lambda - \epsilon_{i_2}, \ \ldots, \ \lambda - \epsilon_{i_m},$$

and their respective numbers k_μ are

$$k_{\lambda-\epsilon_{i_p}} = \begin{cases} k_\lambda - 1, & \text{if } i_p \text{ is even;} \\ k_\lambda, & \text{if } i_p \text{ is odd.} \end{cases} \tag{34}$$

Hence,

$$\sum_{\substack{\mu \text{ is a partition;} \\ \mu \lessdot \lambda}} t^{k_\mu} = \sum_{p=1}^{m} \begin{cases} t^{k_\lambda-1}, & \text{if } i_p \text{ is even;} \\ t^{k_\lambda}, & \text{if } i_p \text{ is odd.} \end{cases} \tag{35}$$

Now, our goal is to prove that

$$\sum_{\substack{\mu \text{ is a partition;} \\ \lambda \lessdot \mu}} t^{k_\mu} = t^{k_\lambda} + t \sum_{\substack{\mu \text{ is a partition;} \\ \mu \lessdot \lambda}} t^{k_\mu}.$$

Using (33) and (35), we can rewrite this as

$$t^{k_\lambda} + \sum_{p=1}^{m} \begin{cases} t^{k_\lambda}, & \text{if } i_p \text{ is even;} \\ t^{k_\lambda+1}, & \text{if } i_p \text{ is odd} \end{cases} = t^{k_\lambda} + t \sum_{p=1}^{m} \begin{cases} t^{k_\lambda-1}, & \text{if } i_p \text{ is even;} \\ t^{k_\lambda}, & \text{if } i_p \text{ is odd.} \end{cases}$$

But this is obvious. Hence, Sublemma A is proved. □

Sublemma B: Given two partitions $\lambda \neq \nu$, we have

$$\begin{aligned} &(\# \text{ of partitions } \mu \text{ such that } \lambda \lessdot \mu \text{ and } \nu \lessdot \mu) \\ &= (\# \text{ of partitions } \mu \text{ such that } \mu \lessdot \lambda \text{ and } \mu \lessdot \nu). \end{aligned}$$

Proof of Sublemma B. This is [Leeuwe95, (2)]. □

Hereon for the rest of this proof, all summation indices are understood to be partitions. Thus, for example, the summation sign "$\sum\limits_{\mu;\ \mu \lessdot \lambda}$" means a sum over all partitions μ satisfying $\mu \lessdot \lambda$.

Sublemma C: For any nonempty partition λ, we have

$$f^{\lambda} = \sum_{\mu;\ \mu \lessdot \lambda} f^{\mu}.$$

Proof of Sublemma C. This is [Leeuwe95, (3)], and is pretty obvious: Any standard tableau of shape λ must have the number n in exactly one of its "corner cells" (i.e., those cells of $Y(\lambda)$ whose removal from $Y(\lambda)$ would yield the diagram $Y(\mu)$ of a partition μ), and upon removing this entry, produces a standard tableau of shape μ for a unique partition μ satisfying $\mu \lessdot \lambda$. This gives a bijection from the set of all standard tableaux of shape λ to the set of all standard tableaux of shape μ with $\mu \lessdot \lambda$. Sublemma C follows by the bijection principle. □

Sublemma D: For any partition λ, we have

$$\sum_{\mu;\ \lambda \lessdot \mu} f^{\mu} = (|\lambda| + 1)\, f^{\lambda}.$$

Proof of Sublemma D. See [Leeuwe95, Lemma 1.3.1] or [Grinbe2X, Lemma D]. □

Now, let $n \geq 2$. Then,

$$\begin{aligned}
\sigma(n) &= \sum_{\lambda \vdash n} f^{\lambda} t^{k_\lambda} = \sum_{\lambda \vdash n} \left(\sum_{\mu;\ \mu \lessdot \lambda} f^{\mu} \right) t^{k_\lambda} \qquad \text{(by Sublemma C)} \\
&= \sum_{\mu} f^{\mu} \sum_{\substack{\lambda \vdash n; \\ \mu \lessdot \lambda}} t^{k_\lambda} \\
&= \sum_{\mu \vdash n-1} f^{\mu} \sum_{\lambda;\ \mu \lessdot \lambda} t^{k_\lambda} \qquad \left(\begin{array}{c} \text{because under the condition } \mu \lessdot \lambda, \\ \text{we have } |\mu| = |\lambda| - 1 \text{, and thus the} \\ \text{condition } \lambda \vdash n \text{ is equivalent to } \mu \vdash n-1 \end{array} \right) \\
&= \sum_{\lambda \vdash n-1} f^{\lambda} \sum_{\mu;\ \lambda \lessdot \mu} t^{k_\mu} \qquad \left(\begin{array}{c} \text{here, we have renamed } \mu \text{ and } \lambda \\ \text{as } \lambda \text{ and } \mu \end{array} \right) \\
&= \sum_{\lambda \vdash n-1} f^{\lambda} \left(t^{k_\lambda} + t \sum_{\mu;\ \mu \lessdot \lambda} t^{k_\mu} \right) \qquad \text{(by Sublemma A)} \\
&= \underbrace{\sum_{\lambda \vdash n-1} f^{\lambda} t^{k_\lambda}}_{= \sigma(n-1)} + \underbrace{\sum_{\lambda \vdash n-1} f^{\lambda} t \sum_{\mu;\ \mu \lessdot \lambda} t^{k_\mu}}_{\substack{= \sum\limits_{\mu \vdash n-1} f^{\mu} t \sum\limits_{\lambda;\ \lambda \lessdot \mu} t^{k_\lambda} \\ \text{(here, we renamed } \lambda \text{ and } \mu \text{ as } \mu \text{ and } \lambda)}} \\
&= \sigma(n-1) + \sum_{\mu \vdash n-1} f^{\mu} t \sum_{\lambda;\ \lambda \lessdot \mu} t^{k_\lambda}.
\end{aligned}$$

In view of

$$\begin{aligned}
&\sum_{\mu\vdash n-1} f^{\mu} t \sum_{\lambda;\ \lambda\lessdot\mu} t^{k_\lambda} \\
&= \sum_{\mu} f^{\mu} t \sum_{\substack{\lambda\vdash n-2;\\ \lambda\lessdot\mu}} t^{k_\lambda} \qquad \left(\begin{array}{c}\text{because under the condition } \lambda\lessdot\mu,\\ \text{we have } |\lambda| = |\mu|-1, \text{ and thus the}\\ \text{condition } \mu\vdash n-1 \text{ is equivalent to } \lambda\vdash n-2\end{array}\right) \\
&= \sum_{\lambda\vdash n-2} t^{k_\lambda} t \underbrace{\sum_{\mu;\ \lambda\lessdot\mu} f^{\mu}}_{\substack{=(|\lambda|+1)f^{\lambda}\\ \text{(by Sublemma D)}}} = \sum_{\lambda\vdash n-2} t^{k_\lambda} t \underbrace{(|\lambda|+1)}_{\substack{=n-1\\ \text{(since } \lambda\vdash n-2)}} f^{\lambda} \\
&= (n-1)\, t \sum_{\lambda\vdash n-2} t^{k_\lambda} f^{\lambda} = (n-1)\, t \underbrace{\sum_{\lambda\vdash n-2} f^{\lambda} t^{k_\lambda}}_{=\sigma(n-2)} = (n-1)\, t\, \sigma(n-2),
\end{aligned}$$

we can rewrite this as

$$\sigma(n) = \sigma(n-1) + (n-1)\, t\, \sigma(n-2).$$

This proves (31).

Let $[t^m] f$ denote the coefficient of t^m in an arbitrary polynomial $f \in \mathbb{Z}[t]$. (This is 0 when $m < 0$ or $m > \deg f$.) The recurrence (31) can be rewritten as the recurrence

$$[t^m](\sigma(n)) = [t^m](\sigma(n-1)) + (n-1) \cdot \left[t^{m-1}\right](\sigma(n-2))$$

for the coefficients of $\sigma(n)$, where $m \in \mathbb{Z}$ is arbitrary and where $n \geq 2$. Using this recurrence (and the fact that $\sigma(0) = \sigma(1) = 1$), it is now straightforward to see that

$$[t^m](\sigma(n)) = \binom{n}{2m} \frac{(2m)!}{2^m m!} \qquad \text{for each } m \in \mathbb{N}.$$

But the left hand side $[t^m](\sigma(n))$ of this equality is $\sum_{\substack{\lambda\vdash n;\\ k_\lambda = m}} f^{\lambda}$ by the definition of $\sigma(n)$. Hence, for each $m \in \mathbb{N}$, we have

$$\sum_{\substack{\lambda\vdash n;\\ k_\lambda = m}} f^{\lambda} = [t^m](\sigma(n)) = \binom{n}{2m} \frac{(2m)!}{2^m m!} = (\#\text{ of } m\text{-matchings of } [n])$$

(by Proposition 3.6.1). This equality also holds for negative integers m (since both sides are 0). Thus, it holds for all $m \in \mathbb{Z}$. Lemma 3.6.3 is proved again. □

We can now describe the submodules $\mathcal{G}_m$ of $\mathcal{G}$ and their quotients as representations of S_n:

Theorem 3.6.4. Let $m \in \mathbb{Z}$. Then,

$$\mathcal{G}_m \cong \bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m}} \mathcal{S}^\lambda \qquad \text{and} \qquad \mathcal{G}_m / \mathcal{G}_{m-1} \cong \bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda = m}} \mathcal{S}^\lambda.$$

Proof. Let λ be a partition of n satisfying $k_\lambda \leq m$. Then, we can choose any n-tableau T of shape λ, and then Proposition 3.4.1 (applied to $k = k_\lambda$) shows that there exist some $2k_\lambda$ distinct elements $i_1, i_2, \ldots, i_{k_\lambda}, j_1, j_2, \ldots, j_{k_\lambda} \in [n]$ such that

$$G^*_{i_1,i_2,\ldots,i_{k_\lambda};\, j_1,j_2,\ldots,j_{k_\lambda}} \mathbf{e}_T \neq 0.$$

This shows that $\mathcal{G}^*_m \mathcal{S}^\lambda \neq 0$ (because $G^*_{i_1,i_2,\ldots,i_{k_\lambda};\, j_1,j_2,\ldots,j_{k_\lambda}} \in \mathcal{G}^*_m$ (since $k_\lambda \leq m$) and $\mathbf{e}_T \in \mathcal{S}^\lambda$). However, Lemma 3.4.6 yields $\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{G}_m\right) \cong \mathcal{G}^*_m \mathcal{S}^\lambda \neq 0$. That is, the Specht module $\mathcal{S}^\lambda$ is contained in $\mathcal{G}_m$ (that is, can be embedded in $\mathcal{G}_m$).

Forget that we fixed λ. We thus have shown that for any partition λ of n satisfying $k_\lambda \leq m$, the Specht module $\mathcal{S}^\lambda$ is contained in $\mathcal{G}_m$. Since these Specht modules are irreducible and mutually non-isomorphic, this shows that their direct sum $\bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m}} \mathcal{S}^\lambda$ is contained in $\mathcal{G}_m$ as well (since Proposition 2.10.1 says that a sum of non-isomorphic irreducible representations is always a direct sum).

Now we shall show that the dimensions of $\bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m}} \mathcal{S}^\lambda$ and $\mathcal{G}_m$ are the same. Indeed, recall the filtration $(0 = \mathcal{G}_{-1} \subseteq \mathcal{G}_0 \subseteq \cdots \subseteq \mathcal{G}_n = \mathcal{G})$ from Subsection 3.3. Thus, $(0 = \mathcal{G}_{-1} \subseteq \mathcal{G}_0 \subseteq \cdots \subseteq \mathcal{G}_m = \mathcal{G}_m)$ is a filtration of $\mathcal{G}_m$. Hence,

$$\begin{aligned}
\dim(\mathcal{G}_m) &= \sum_{i=0}^{m} \underbrace{\dim(\mathcal{G}_i / \mathcal{G}_{i-1})}_{\substack{\leq (\text{\# of } i\text{-matchings of } [n]) \\ (\text{by Lemma 3.3.1})}} \\
&\leq \sum_{i=0}^{m} \underbrace{(\text{\# of } i\text{-matchings of } [n])}_{\substack{= \sum_{\substack{\lambda \vdash n; \\ k_\lambda = i}} (\text{\# of standard tableaux of shape } \lambda) \\ (\text{by Lemma 3.6.3, applied to } i \text{ instead of } m)}} \\
&= \underbrace{\sum_{i=0}^{m} \sum_{\substack{\lambda \vdash n; \\ k_\lambda = i}}}_{= \sum_{\substack{\lambda \vdash n; \\ k_\lambda \leq m}}} \underbrace{(\text{\# of standard tableaux of shape } \lambda)}_{\substack{= \dim \mathcal{S}^\lambda \\ (\text{by (26)})}} \\
&= \sum_{\substack{\lambda \vdash n; \\ k_\lambda \leq m}} \dim \mathcal{S}^\lambda = \dim \bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m}} \mathcal{S}^\lambda.
\end{aligned}$$

Since $\bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m}} \mathcal{S}^\lambda$ is contained in $\mathcal{G}_m$, we thus conclude that the direct sum $\bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m}} \mathcal{S}^\lambda$ is $\mathcal{G}_m$. This proves

$$\mathcal{G}_m \cong \bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m}} \mathcal{S}^\lambda. \tag{36}$$

The same reasoning (applied to $m-1$ instead of m) shows that

$$\mathcal{G}_{m-1} \cong \bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m-1}} \mathcal{S}^\lambda. \tag{37}$$

Now it remains to prove that $\mathcal{G}_m / \mathcal{G}_{m-1} \cong \bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda = m}} \mathcal{S}^\lambda$. For this purpose, we recall the following fact (an easy consequence of the Krull–Remak–Schmidt theorem [EGHetc11, Theorem 3.8.1], or – because $\mathcal{A}$ is semisimple – of the Jordan–Hölder theorem [EGHetc11, Theorem 3.7.1]):

> *Cancellativity of $\mathcal{A}$-modules:* Let U, V, W be three finite-dimensional left $\mathcal{A}$-modules. If $U \oplus W \cong V \oplus W$, then $U \cong V$.

The semisimplicity of $\mathcal{A}$ yields

$$\begin{aligned} \mathcal{G}_m &\cong \mathcal{G}_{m-1} \oplus (\mathcal{G}_m / \mathcal{G}_{m-1}) \cong (\mathcal{G}_m / \mathcal{G}_{m-1}) \oplus \mathcal{G}_{m-1} \\ &\cong (\mathcal{G}_m / \mathcal{G}_{m-1}) \oplus \left(\bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m-1}} \mathcal{S}^\lambda \right) \qquad \text{(by (37))}, \end{aligned}$$

so that

$$\begin{aligned} (\mathcal{G}_m / \mathcal{G}_{m-1}) \oplus \left(\bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m-1}} \mathcal{S}^\lambda \right) &\cong \mathcal{G}_m \cong \bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m}} \mathcal{S}^\lambda \qquad \text{(by (36))} \\ &\cong \left(\bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda = m}} \mathcal{S}^\lambda \right) \oplus \left(\bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m-1}} \mathcal{S}^\lambda \right). \end{aligned}$$

Applying the cancellativity of $\mathcal{A}$-modules to $U = \mathcal{G}_m / \mathcal{G}_{m-1}$ and $V = \bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda = m}} \mathcal{S}^\lambda$ and $W = \bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda \leq m-1}} \mathcal{S}^\lambda$, we thus obtain

$$\mathcal{G}_m / \mathcal{G}_{m-1} \cong \bigoplus_{\substack{\lambda \vdash n; \\ k_\lambda = m}} \mathcal{S}^\lambda.$$

This completes the proof of Theorem 3.6.4. □

Corollary 3.6.5. Let λ be a partition of n. Let $m \in \mathbb{Z}$ be such that $m < k_\lambda$. Then, $\mathcal{G}_m^* \mathcal{S}^\lambda = 0$.

Proof. Lemma 3.4.6 yields $\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{G}_m\right) \cong \mathcal{G}_m^* \mathcal{S}^\lambda$ as vector spaces.

If μ is a partition of n satisfying $k_\mu \leq m$, then $\mu \neq \lambda$ (since $k_\mu \leq m < k_\lambda$ entails $k_\mu \neq k_\lambda$ and thus $\mu \neq \lambda$) and therefore $\lambda \neq \mu$, so that

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{S}^\mu\right) = 0 \tag{38}$$

by Schur's lemma (since the irreducible S_n-representations $\mathcal{S}^\lambda$ and $\mathcal{S}^\mu$ are not isomorphic (since $\lambda \neq \mu$)).

But Theorem 3.6.4 (with the index λ renamed as μ) yields $\mathcal{G}_m \cong \bigoplus_{\substack{\mu \vdash n; \\ k_\mu \leq m}} \mathcal{S}^\mu$. Hence,

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{G}_m\right) \cong \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \bigoplus_{\substack{\mu \vdash n; \\ k_\mu \leq m}} \mathcal{S}^\mu\right) \cong \bigoplus_{\substack{\mu \vdash n; \\ k_\mu \leq m}} \underbrace{\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{S}^\mu\right)}_{\substack{=0 \\ \text{(by (38))}}} = 0.$$

Comparing this with $\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{G}_m\right) \cong \mathcal{G}_m^* \mathcal{S}^\lambda$, we obtain $\mathcal{G}_m^* \mathcal{S}^\lambda = 0$, and so Corollary 3.6.5 is proved. □

3.7. Relation to the involution Gelfand model

The Gelfand model $\mathcal{G}$ of S_n is closely related to the famous involution Gelfand model V_n from [AdPoRo08, §1.1] (see also [KodVer04]).

Indeed, let $m \in \mathbb{Z}$. Let $F_{n,m}$ be the free $\mathbf{k}$-module with basis consisting of all the $2m$-tuples $(i_1, i_2, \ldots, i_m;\ j_1, j_2, \ldots, j_m)$ of distinct elements of $[n]$ (viewed as formal symbols). This $F_{n,m}$ becomes an S_n-representation, where S_n acts entrywise on the tuples (i.e., a permutation is applied to each entry of the tuple). Now, let $V_{n,m}$ be the quotient vector space of $F_{n,m}$ modulo the relations

- $$(i_1, i_2, \ldots, i_m;\ j_1, j_2, \ldots, j_m) \equiv \left(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(m)};\ j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(m)}\right)$$

  for all $\sigma \in S_m$ and all $(i_1, i_2, \ldots, i_m;\ j_1, j_2, \ldots, j_m)$, and

- $$\begin{aligned} &\left(i_1, i_2, \ldots, i_{p-1}, j_p, i_{p+1}, \ldots, i_m;\ j_1, j_2, \ldots, j_{p-1}, i_p, j_{p+1}, \ldots, j_m\right) \\ &\equiv -\left(i_1, i_2, \ldots, i_m;\ j_1, j_2, \ldots, j_m\right) \end{aligned}$$

  for all $(i_1, i_2, \ldots, i_m;\ j_1, j_2, \ldots, j_m)$ and all $p \in [m]$.

This quotient space $V_{n,m}$ is still an S_n-representation (since the relations are preserved under the S_n-action). It is easy to see that it has a basis consisting of the elements

$$(i_1, i_2, \ldots, i_m;\ j_1, j_2, \ldots, j_m)$$

indexed by all m-matchings $\{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_m, j_m\}\}$ of $[n]$, as long as we make sure to pick one representative $(i_1, i_2, \ldots, i_m;\ j_1, j_2, \ldots, j_m)$ for each m-matching (different representatives will lead to different signs of $(i_1, i_2, \ldots, i_m;\ j_1, j_2, \ldots, j_m)$).

As we saw in the proof of Lemma 3.3.1, the quotient vector space $\mathcal{G}_m / \mathcal{G}_{m-1}$ is spanned by the family

$$\left(\overline{G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m}}\right)_{i_1,i_2,\ldots,i_m,j_1,j_2,\ldots,j_m \text{ are } 2m \text{ distinct elements of } [n]},$$

and this family is subject to the obvious relation

$$\overline{G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m}} = \overline{G_{i_{\sigma(1)},i_{\sigma(2)},\ldots,i_{\sigma(m)};\ j_{\sigma(1)},j_{\sigma(2)},\ldots,j_{\sigma(m)}}}$$

for all $\sigma \in S_m$ (indeed, this holds even in $\mathcal{G}_m$, without the overlines) and the less obvious relation (12) for all $p \in [m]$. Thus, there is an S_n-equivariant linear map

$$\psi_m : V_{n,m} \to \mathcal{G}_m / \mathcal{G}_{m-1},$$
$$(i_1, i_2, \ldots, i_m;\ j_1, j_2, \ldots, j_m) \mapsto \overline{G_{i_1,i_2,\ldots,i_m;\ j_1,j_2,\ldots,j_m}}.$$

This map ψ_m is surjective, since $\mathcal{G}_m / \mathcal{G}_{m-1}$ is spanned by the family (11). Since its domain $V_{n,m}$ and its codomain $\mathcal{G}_m / \mathcal{G}_{m-1}$ have the same dimension (by Corollary 3.6.2), this entails that this map ψ_m is also injective, and thus is an isomorphism of S_n-representations. Hence,

$$\mathcal{G}_m / \mathcal{G}_{m-1} \cong V_{n,m} \qquad \text{as } S_n\text{-representations.}$$

But the S_n-representation $V_{n,m}$ is just an isomorphic copy of the m-th degree component of the famous involution Gelfand model V_n from [AdPoRo08, §1.1] (see also [KodVer04]): Indeed, we can replace each $2m$-tuple $(i_1, i_2, \ldots, i_m;\ j_1, j_2, \ldots, j_m) \in V_{n,m}$ by the involution $t_{i_1,j_1} t_{i_2,j_2} \cdots t_{i_m j_m}$ multiplied by $(-1)^{\#\text{ of } s \text{ satisfying } i_s > j_s}$ to get an isomorphism from the former to the latter. Hence, $\mathcal{G}_m / \mathcal{G}_{m-1}$ is isomorphic to the m-th degree component of V_n as well.

Now, forget that we fixed m. We thus have shown that for each $m \in \mathbb{Z}$, the S_n-representation $\mathcal{G}_m / \mathcal{G}_{m-1}$ is isomorphic to the m-th degree component of the involution Gelfand model V_n. Hence, the associated graded object $\bigoplus_{m=0}^{n} (\mathcal{G}_m / \mathcal{G}_{m-1})$ of our filtered S_n-representation $\mathcal{G}$ is isomorphic to the Gelfand model V_n. But since we are in characteristic 0, Maschke's theorem ensures that all exact sequences of S_n-representations split, and therefore $\mathcal{G} \cong \bigoplus_{m=0}^{n} (\mathcal{G}_m / \mathcal{G}_{m-1})$ as representations of S_n. Hence, $\mathcal{G}$ is also isomorphic to the Gelfand model V_n. This gives a new proof of the fact that V_n is a Gelfand model.

3.8. Questions

Question 3.8.1. What is $\mathcal{G}^*\mathcal{G}$? Proposition 2.9.1 **(c)** shows that this is a nonunital subalgebra of $\mathcal{A}$ isomorphic to $\mathbf{k}^{p(n)}$, where $p(n) = |\{\lambda \vdash n\}|$. But how exactly does it lie inside $\mathcal{A}$? Is there a simple basis?

Question 3.8.2. How much of the above still holds in the Hecke algebra?

4. The dyadic shuffles

We shall now reap some concrete harvest of our theory.

As we recall, $\mathcal{A} = \mathbf{k}[S_n]$ is the group algebra of the symmetric group $S_n = \{\text{permutations of } [n]\}$ over the characteristic-0 field $\mathbf{k}$. The antipode map of $\mathcal{A}$ is the $\mathbf{k}$-linear map $a \mapsto a^*$ that sends each $w \in S_n$ to w^{-1}. This is a $\mathbf{k}$-algebra anti-automorphism of $\mathcal{A}$.

4.1. Definition via matchings

An *edge* will mean a 2-element subset of $[n]$.

A permutation $w \in S_n$ is said to *increase* on a subset of $[n]$ if the restriction of w to this subset is an increasing function. In particular, a permutation $w \in S_n$ increases on an edge $\{i < j\}$ if and only if $w(i) < w(j)$.

For instance, the permutation $w \in S_4$ with one-line notation $[3124]$ increases on the edges $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ but not on the edges $\{1,2\}$, $\{1,3\}$.

A set of k disjoint edges will be called a *k-matching*. A permutation $w \in S_n$ is said to *increase* on a k-matching M if and only if w increases on each edge $P \in M$.

For instance, the permutation $w \in S_5$ with one-line notation $[24513]$ increases on the 2-matching $\{\{1,3\},\{4,5\}\}$, but not on the 2-matching $\{\{1,3\},\{2,5\}\}$, since it fails to increase on the edge $\{2,5\}$.

Given a permutation $w \in S_n$ and an integer $k \geq 0$, we define $\operatorname{incmat}_k(w)$ to be the number of all k-matchings on which w increases. This number is called $\operatorname{noninv}_{(2^k,1^{n-2k})}(w)$ in [ReSaWe11].

For instance, the permutation $[24513] \in S_5$ (written here in one-line notation) increases on exactly four 2-matchings, which are color-coded on its one-line notation here:

24513, 24513, 24513, 24513

(each color represents an edge, which consists of the positions at which this color appears). Thus, $\operatorname{incmat}_2([24513]) = 4$.

Definition 4.1.1. For any $k \in \mathbb{N}$, we define the *dyadic shuffle*

$$\mathcal{S}_{n,k} := \sum_{w \in S_n} \operatorname{incmat}_k(w)\, w \in \mathcal{A}.$$

This $\mathcal{S}_{n,k}$ is denoted by $\nu_{\left(2^k,1^{n-2k}\right)}$ in [ReSaWe11].

Example 4.1.2. **(a)** We have

$$\mathcal{S}_{n,0} = \sum_{w \in S_n} \underbrace{\operatorname{incmat}_0(w)}_{\substack{=1 \\ \text{(since there is only one 0-matching,} \\ \text{namely } \{\}, \text{ and each } w \text{ increases on it)}}} w = \sum_{w \in S_n} w.$$

(b) To compute $\mathcal{S}_{n,1}$, we note that the 1-matchings are just singletons consisting of a single edge $\{i < j\}$, and a given permutation $w \in S_n$ increases on such a matching if and only if $w(i) < w(j)$. Thus, for any permutation $w \in S_n$, the number $\operatorname{incmat}_1(w)$ counts the pairs $(i < j)$ of elements of $[n]$ satisfying $w(i) < w(j)$. These pairs are known as the *noninversions* of w. For instance, for $n = 3$, we have

$$\begin{aligned} \mathcal{S}_{3,1} &= \sum_{w \in S_3} \operatorname{incmat}_1(w)\, w \\ &= 3\,[123] + 2\,[132] + 2\,[213] + [231] + [312], \end{aligned}$$

where $[i_1 i_2 \cdots i_n]$ denotes the permutation with one-line notation $(i_1, i_2, \ldots, i_n)$.

(c) If $2k > n$, then there exist no k-matchings, since there are no $2k$ distinct elements in $[n]$. Hence, in this case, we have $\operatorname{incmat}_k(w) = 0$ for all $w \in S_n$, and thus $\mathcal{S}_{n,k} = 0$.

Among the main results of [ReSaWe11] are the following two theorems (both parts of [ReSaWe11, Theorem 1.6]):

Theorem 4.1.3. The dyadic shuffles $\mathcal{S}_{n,0}, \mathcal{S}_{n,1}, \mathcal{S}_{n,2}, \ldots$ pairwise commute. That is, $\left[\mathcal{S}_{n,i}, \mathcal{S}_{n,j}\right] = 0$ for all $i, j \in \mathbb{N}$.

Theorem 4.1.4. For each $k \in \mathbb{N}$, there exists a multiset Z_k of integers such that $\prod_{\lambda \in Z_k} \left(\mathcal{S}_{n,k} - \lambda\right) = 0$. (In other words, the minimal polynomial of $\mathcal{S}_{n,k}$ over $\mathbf{k}$ factors into linear factors defined over $\mathbb{Z}$.)

We shall recover these results as consequences of our theory in this section. First, however, we shall give another definition of the dyadic shuffles.

4.2. Definition via descents

The *descent set* $\operatorname{Des} w$ of a permutation $w \in S_n$ is defined to be the set of all $i \in [n-1]$ such that $w(i) > w(i+1)$. Its elements i are called the *descents* of w.

A *composition* of n means a tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers satisfying $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$.

For any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n, define a subset $D(\alpha)$ of $[n-1]$ by

$$D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\},$$

and define the element

$$\mathcal{X}_\alpha := \sum_{\substack{w \in S_n; \\ \operatorname{Des} w \subseteq D(\alpha)}} w^{-1} \in \mathcal{A}. \tag{39}$$

In other words, $\mathcal{X}_\alpha$ is the sum of the minimum-length right coset representatives in $S_\alpha \setminus S_n$, where S_α is the Young subgroup of S_n corresponding to α (that is, the group of permutations that permute the smallest α_1 elements of $[n]$ among themselves, the next-smallest α_2 elements of $[n]$ among themselves, and so on).

These $\mathcal{X}_\alpha$ have significant overlap with the $G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$ defined in (8):

Proposition 4.2.1. Let α be any composition of n that consists of k many 2's and $n-2k$ many 1's in any order. Then,

$$\mathcal{X}_\alpha = G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$$

for an appropriate choice of $2k$ distinct elements $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in [n]$.

Proof. An example first: If α is the composition $(1,2,2,1,2)$, then $X_\alpha = G_{2,4,7;\ 3,5,8}$, because the condition $\operatorname{Des} w \subseteq D(1,2,2,1,2)$ on a permutation $w \in S_8$ is equivalent to the condition

$$(w(2) < w(3) \text{ and } w(4) < w(5) \text{ and } w(7) < w(8)).$$

Let us now handle the general case. The composition α has $k + (n-2k) = n-k$ entries, which include k many 2's and $n-2k$ many 1's. Any of the k many 2's in α leads to two consecutive elements $\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1}$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_i$ of $D(\alpha) \cup \{0, n\}$ having a "gap" of $\alpha_i = 2$ between them, so that the intermediate integer $\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} + 1$ belongs to the complementary set $[n-1] \setminus D(\alpha)$. This accounts for all elements of $[n-1] \setminus D(\alpha)$. Thus, we can write the set $[n-1] \setminus D(\alpha)$ in the form

$$[n-1] \setminus D(\alpha) = \{i_1 < i_2 < \cdots < i_k\}$$

for some k numbers $i_1 < i_2 < \cdots < i_k$ in $[n-1]$ that are "socially distanced" (i.e., each i_s differs from the previous i_{s-1} by at least 2). Consider these $i_1, i_2, \ldots, i_k$. Hence, the $2k$ numbers $i_1, i_1+1, i_2, i_2+1, \ldots, i_k, i_k+1$ are distinct (since each i_s differs from the previous i_{s-1} by at least 2). Hence, $G_{i_1,i_2,\ldots,i_k;\ i_1+1,i_2+1,\ldots,i_k+1}$ is well-defined.

Now, a permutation $w \in S_n$ satisfies $\operatorname{Des} w \subseteq D(\alpha)$ if and only if **none** of $i_1, i_2, \ldots, i_k$ is a descent of w (since $D(\alpha) = [n-1] \setminus \{i_1, i_2, \ldots, i_k\}$), that is, if and only if it satisfies $w(i_s) < w(i_s+1)$ for each $s \in [k]$. Thus, the definition (39) of $\mathcal{X}_\alpha$ can be rewritten as

$$\mathcal{X}_\alpha = \sum_{\substack{w \in S_n; \\ w(i_s) < w(i_s+1) \text{ for all } s \in [k]}} w^{-1} = G_{i_1,i_2,\ldots,i_k;\ i_1+1,i_2+1,\ldots,i_k+1}$$

(by (8)). Hence, $\mathcal{X}_\alpha = G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$ for an appropriate choice of $2k$ distinct elements $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in [n]$ (namely, for $j_s = i_s + 1$). This proves Proposition 4.2.1. □

Remark 4.2.2. Let $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in [n]$ be $2k$ distinct elements. If $i_s < j_s$ for all $s \in [k]$, then the permutations w^{-1} for $w \in S_n$ satisfying $w(i_s) < w(j_s)$ for all $s \in [k]$ are the minimum-length right coset representatives in $\left(S_{\{i_1,j_1\}} \times S_{\{i_2,j_2\}} \times \cdots \times S_{\{i_k,j_k\}}\right) \setminus S_n$, where $S_{\{i_1,j_1\}} \times S_{\{i_2,j_2\}} \times \cdots \times S_{\{i_k,j_k\}}$ is the order-2^k subgroup of S_n generated by the commuting transpositions $t_{i_1,j_1}, t_{i_2,j_2}, \ldots, t_{i_k,j_k}$. Thus, $G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$ is the sum of these minimum-length right coset representatives in this case. But this does not hold if some $i_s > j_s$.

Now we claim:

Theorem 4.2.3. Let $k \in \mathbb{N}$ be such that $2k \leq n$. Then,

$$\mathcal{S}_{n,k} = \frac{1}{k!\,(n-2k)!} \mathcal{X}^*_{\left(2^k,1^{n-2k}\right)} \mathcal{X}_{\left(2^k,1^{n-2k}\right)}. \tag{40}$$

Here, $\left(2^k, 1^{n-2k}\right)$ denotes the composition of n that begins with k many 2's and continues with $n-2k$ many 1's.

Proof. We have

$$\begin{aligned} D\left(\left(2^k, 1^{n-2k}\right)\right) &= \{2,4,6,\ldots,2k\} \cup \underbrace{\{2k+1, 2k+2, 2k+3, \ldots, n-1\}}_{\text{all integers from } 2k+1 \text{ to } n-1} \\ &= [n-1] \setminus \{1,3,5,\ldots,2k-1\}. \end{aligned}$$

Thus, a permutation $w \in S_n$ satisfies $\operatorname{Des} w \subseteq D\left(\left(2^k, 1^{n-2k}\right)\right)$ if and only if it satisfies $\operatorname{Des} w \subseteq [n-1] \setminus \{1,3,5,\ldots,2k-1\}$, that is, if and only if **none** of $1,3,5,\ldots,2k-1$ is a descent of w, that is, if and only if it satisfies $w(2s-1) < w(2s)$ for all $s \in [k]$. Hence, the definition (39) of $\mathcal{X}_{\left(2^k,1^{n-2k}\right)}$ can be rewritten as

$$\mathcal{X}_{\left(2^k,1^{n-2k}\right)} = \sum_{\substack{w \in S_n; \\ w(2s-1) < w(2s) \text{ for all } s \in [k]}} w^{-1}. \tag{41}$$

Applying the antipode map $a \mapsto a^*$ to this equality, we obtain

$$\mathcal{X}^*_{\left(2^k,1^{n-2k}\right)} = \sum_{\substack{x \in S_n; \\ x(2s-1) < x(2s) \text{ for all } s \in [k]}} x \tag{42}$$

(since the antipode map sends w^{-1} to w for each $w \in S_n$). Multiplying the equalities

(42) and (41) together, we find

$$\begin{aligned}
\mathcal{X}^*_{\left(2^k,1^{n-2k}\right)}\mathcal{X}_{\left(2^k,1^{n-2k}\right)} &= \sum_{\substack{x\in S_n;\\ x(2s-1)<x(2s) \text{ for all } s\in[k]}} x \sum_{\substack{w\in S_n;\\ w(2s-1)<w(2s) \text{ for all } s\in[k]}} w^{-1} \\
&= \sum_{\substack{x,w\in S_n;\\ x(2s-1)<x(2s) \text{ for all } s\in[k];\\ w(2s-1)<w(2s) \text{ for all } s\in[k]}} xw^{-1} \\
&= \sum_{\substack{u,w\in S_n;\\ (uw)(2s-1)<(uw)(2s) \text{ for all } s\in[k];\\ w(2s-1)<w(2s) \text{ for all } s\in[k]}} u
\end{aligned}$$

(here, we have substituted uw for x in the sum). In other words,

$$\mathcal{X}^*_{\left(2^k,1^{n-2k}\right)}\mathcal{X}_{\left(2^k,1^{n-2k}\right)} = \sum_{u\in S_n} \rho\left(u\right) u, \tag{43}$$

where $\rho\left(u\right) \in \mathbb{Z}$ is the number of all permutations $w \in S_n$ that satisfy

$$\left(uw\right)\left(2s-1\right) < \left(uw\right)\left(2s\right) \text{ and } w\left(2s-1\right) < w\left(2s\right) \text{ for all } s \in [k].$$

Clearly, it suffices to show that

$$\rho\left(u\right) = k!\left(n-2k\right)!\operatorname{incmat}_k\left(u\right) \qquad \text{for each } u \in S_n$$

(because then, dividing (43) by $k!\left(n-2k\right)!$ will result in (40)).

This we can show bijectively: Fix $u \in S_n$. Let K be the set of all permutations $w \in S_n$ that satisfy

$$\left(uw\right)\left(2s-1\right) < \left(uw\right)\left(2s\right) \text{ and } w\left(2s-1\right) < w\left(2s\right) \text{ for all } s \in [k].$$

Thus, $\rho\left(u\right) = |K|$. For each permutation $w \in K$, the k-matching

$$M_w := \left\{\left\{w\left(1\right), w\left(2\right)\right\},\ \left\{w\left(3\right), w\left(4\right)\right\},\ \ldots,\ \left\{w\left(2k-1\right), w\left(2k\right)\right\}\right\}$$

has the property that u increases on M_w (by the definition of K). Conversely, any k-matching on which u increases can be written as M_w for $k!\left(n-2k\right)!$ many distinct permutations $w \in K$ (indeed, we must ensure that the k edges of the k-matching are the k edges $\left\{w\left(1\right), w\left(2\right)\right\},\ \left\{w\left(3\right), w\left(4\right)\right\},\ \ldots,\ \left\{w\left(2k-1\right), w\left(2k\right)\right\}$ in one of $k!$ possible orders[7], and then the values $w\left(2k+1\right),\ w\left(2k+2\right),\ \ldots,\ w\left(n\right)$ must be the remaining $n-2k$ elements of $[n]$ in one of $\left(n-2k\right)!$ possible orders). Hence, there is a $k!\left(n-2k\right)!$-to-1 correspondence between the permutations $w \in K$ and the k-matchings on which u increases. Therefore, $|K|$ equals $k!\left(n-2k\right)!$ times the number of the latter matchings, which of course is $\operatorname{incmat}_k\left(u\right)$. In other words, $\rho\left(u\right) = k!\left(n-2k\right)!\operatorname{incmat}_k\left(u\right)$, since $\rho\left(u\right) = |K|$. This completes the proof of $\rho\left(u\right) = k!\left(n-2k\right)!\operatorname{incmat}_k\left(u\right)$. This, in turn, completes the proof of Theorem 4.2.3. □

[7]In order to satisfy the condition $w\left(2s-1\right) < w\left(2s\right)$ for all $s \in [k]$, we must make sure to let $w\left(2s-1\right)$ be the smaller and $w\left(2s\right)$ the larger of the two elements of the corresponding edge of the k-matching.

Proposition 4.2.4. Let $k \in \mathbb{N}$. Then, $\mathcal{S}_{n,k}^* = \mathcal{S}_{n,k}$.

Proof. This follows from (40), since $(a^*a)^* = a^*a$ for every $a \in \mathcal{A}$. Alternatively, this can be derived from the definition of $\mathcal{S}_{n,k}$, since every permutation $w \in S_n$ satisfies $\operatorname{incmat}_k(w) = \operatorname{incmat}_k(w^{-1})$ (this follows from a direct bijection: w increases on an edge $\{i,j\}$ if and only if w^{-1} increases on $\{w(i), w(j)\}$). $\square$

Corollary 4.2.5. Recall the left ideal $\mathcal{G}$ defined in Theorem 3.2.1. Then, each $k \in \mathbb{N}$ satisfies

$$\mathcal{X}_{(2^k,1^{n-2k})} \in \mathcal{G} \qquad \text{if } 2k \leq n, \tag{44}$$

and

$$\mathcal{S}_{n,k} \in \mathcal{G}^*\mathcal{G}. \tag{45}$$

Proof. WLOG assume that $2k \leq n$ (since otherwise, (44) is vacuously true, whereas (45) is obvious because Example 4.1.2 **(c)** shows that $\mathcal{S}_{n,k} = 0$).

Thus, Proposition 4.2.1 (applied to $\alpha = (2^k, 1^{n-2k})$) shows that

$$\mathcal{X}_{(2^k,1^{n-2k})} = G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$$

for an appropriate choice of $2k$ distinct elements $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in [n]$ (indeed, it is easy to see that $\mathcal{X}_{(2^k,1^{n-2k})} = G_{1,3,\ldots,2k-1;\ 2,4,\ldots,2k}$). Hence, $\mathcal{X}_{(2^k,1^{n-2k})} \in \mathcal{G}$ by the definition of $\mathcal{G}$. This proves (44). Furthermore, (40) becomes

$$\mathcal{S}_{n,k} = \frac{1}{k!\,(n-2k)!} \underbrace{\mathcal{X}^*_{(2^k,1^{n-2k})}}_{\substack{\in \mathcal{G}^* \\ (\text{since } \mathcal{X}_{(2^k,1^{n-2k})} \in \mathcal{G})}} \underbrace{\mathcal{X}_{(2^k,1^{n-2k})}}_{\in \mathcal{G}} \in \mathcal{G}^*\mathcal{G}.$$

This proves (45). $\square$

We can generalize Theorem 4.2.3 further:

Theorem 4.2.6. Let $k \in \mathbb{N}$ be such that $2k \leq n$. Then:

(a) Let α be any composition of n that consists of k many 2's and $n-2k$ many 1's in any order. Then,

$$\mathcal{S}_{n,k} = \frac{1}{k!\,(n-2k)!} \mathcal{X}_\alpha^* \mathcal{X}_\alpha. \tag{46}$$

(b) Let $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k$ be any $2k$ distinct elements of $[n]$. Then,

$$\mathcal{S}_{n,k} = \frac{1}{k!\,(n-2k)!} G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k} G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$$

(where $G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$ is as defined in (8)).

Proof. **(b)** The $2k$ numbers $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k$ are distinct. In other words, the $2k$ numbers $i_1, j_1, i_2, j_2, \ldots, i_k, j_k$ are distinct. Thus, there is a permutation $u \in S_n$ that sends the numbers $1, 2, 3, 4, \ldots, 2k-1, 2k$ to the numbers $i_1, j_1, i_2, j_2, \ldots, i_k, j_k$, respectively. Pick such a u. Thus, for each $s \in [k]$, we have

$$i_s = u(2s-1) \qquad \text{and} \qquad j_s = u(2s). \tag{47}$$

Now, we shall show that

$$G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k} = u\mathcal{X}_{\left(2^k,1^{n-2k}\right)}.$$

Indeed, (8) shows that

$$\begin{aligned}
G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k} &= \sum_{\substack{w\in S_n;\\ w(i_s)<w(j_s) \text{ for all } s\in[k]}} w^{-1} \\
&= \sum_{\substack{w\in S_n;\\ w(u(2s-1))<w(u(2s)) \text{ for all } s\in[k]}} w^{-1} \qquad (\text{by } (47)) \\
&= \underbrace{\sum_{\substack{w\in S_n;\\ \left(wu^{-1}\right)(u(2s-1))<\left(wu^{-1}\right)(u(2s)) \text{ for all } s\in[k]}}}_{\substack{=\sum\limits_{\substack{w\in S_n;\\ w(2s-1)<w(2s) \text{ for all } s\in[k]}}\\ \left(\text{since } \left(wu^{-1}\right)(u(2s-1))=w(2s-1)\right.\\ \left.\text{and } \left(wu^{-1}\right)(u(2s))=w(2s)\right)}} \underbrace{\left(wu^{-1}\right)^{-1}}_{=uw^{-1}} \\
&\qquad \left(\text{here, we have substituted } wu^{-1} \text{ for } w \text{ in the sum}\right) \\
&= \sum_{\substack{w\in S_n;\\ w(2s-1)<w(2s) \text{ for all } s\in[k]}} uw^{-1} = u \underbrace{\sum_{\substack{w\in S_n;\\ w(2s-1)<w(2s) \text{ for all } s\in[k]}} w^{-1}}_{\substack{=\mathcal{X}_{\left(2^k,1^{n-2k}\right)}\\ (\text{by } (41))}} \\
&= u\mathcal{X}_{\left(2^k,1^{n-2k}\right)}.
\end{aligned}$$

Applying the antipode map $a \mapsto a^*$ to this equality (recalling that this map is an anti-algebra morphism), we obtain

$$G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k} = \mathcal{X}^*_{\left(2^k,1^{n-2k}\right)} \underbrace{u^*}_{\substack{=u^{-1}\\ (\text{since } u\in S_n)}} = \mathcal{X}^*_{\left(2^k,1^{n-2k}\right)} u^{-1}.$$

Hence,

$$\frac{1}{k!\,(n-2k)!}\underbrace{G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}}_{=\mathcal{X}^*_{(2^k,1^{n-2k})}u^{-1}}\underbrace{G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}}_{=u\mathcal{X}_{(2^k,1^{n-2k})}}$$
$$=\frac{1}{k!\,(n-2k)!}\mathcal{X}^*_{(2^k,1^{n-2k})}\underbrace{u^{-1}u}_{=1}\mathcal{X}_{(2^k,1^{n-2k})}=\frac{1}{k!\,(n-2k)!}\mathcal{X}^*_{(2^k,1^{n-2k})}\mathcal{X}_{(2^k,1^{n-2k})}$$
$$=\mathcal{S}_{n,k} \qquad (\text{by (40)}).$$

This proves Theorem 4.2.6 **(b)**.

(a) Proposition 4.2.1 yields that

$$\mathcal{X}_\alpha = G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$$

for an appropriate choice of $2k$ distinct elements $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in [n]$. Consider these $2k$ elements. Then, using $\mathcal{X}_\alpha = G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$, we obtain

$$\frac{1}{k!\,(n-2k)!}\mathcal{X}^*_\alpha\mathcal{X}_\alpha=\frac{1}{k!\,(n-2k)!}G^*_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}G_{i_1,i_2,\ldots,i_k;\ j_1,j_2,\ldots,j_k}$$
$$=\mathcal{S}_{n,k} \qquad (\text{by Theorem 4.2.6 **(b)**}).$$

This proves Theorem 4.2.6 **(a)**. □

4.3. Commutativity and integrality of eigenvalues

We can now prove Theorem 4.1.3 and Theorem 4.1.4:

Proof of Theorem 4.1.3. Theorem 3.2.1 shows that the left ideal $\mathcal{G}$ of $\mathcal{A}$ is a Gelfand model of S_n. In particular, it is thus multiplicity-free. Hence, Theorem 2.4.1 (applied to $J = \mathcal{G}$) yields $[\mathcal{G}^*\mathcal{G},\ \mathcal{G}^*\mathcal{G}] = 0$. In other words, any two elements of $\mathcal{G}^*\mathcal{G}$ commute. Since (45) shows that all of $\mathcal{S}_{n,0}, \mathcal{S}_{n,1}, \mathcal{S}_{n,2}, \ldots$ are elements of $\mathcal{G}^*\mathcal{G}$, we thus conclude that $\mathcal{S}_{n,0}, \mathcal{S}_{n,1}, \mathcal{S}_{n,2}, \ldots$ pairwise commute. This proves Theorem 4.1.3. □

Various variants of Theorem 4.1.3 can be proved in the same way: for instance, we conclude that if $\alpha, \beta, \gamma, \delta$ are four compositions of n that consist only of 1's and 2's, then $\mathcal{X}^*_\alpha\mathcal{X}_\beta$ and $\mathcal{X}^*_\gamma\mathcal{X}_\delta$ commute. We can also use Theorem 2.2.2 to conclude that $\mathcal{X}_\alpha\left[\mathcal{X}_\beta, \mathcal{X}_\gamma\right] = 0$ and $\mathcal{S}_{n,k}\left[\mathcal{X}_\alpha, \mathcal{X}_\beta\right] = 0$ and so on.

Proof of Theorem 4.1.4. Let $k \in \mathbb{N}$. Theorem 3.2.1 shows that the left ideal $\mathcal{G}$ of $\mathcal{A}$ is a Gelfand model of S_n. In particular, it is thus multiplicity-free. But (45) yields $\mathcal{S}_{n,k} \in \mathcal{G}^*\mathcal{G} \subseteq \mathcal{G}$ (since $\mathcal{G}$ is a left ideal of $\mathcal{A}$). Therefore, Theorem 2.7.1 (applied to $J = \mathcal{G}$ and $a = \mathcal{S}_{n,k}$) shows that there exists a multiset Z_a of elements of $\mathbf{k}$ such that $\prod\limits_{\lambda \in Z_a}(\mathcal{S}_{n,k} - \lambda) = 0$.

But we want a multiset of integers, not a multiset of elements of $\mathbf{k}$. For this, we use a standard trick: Apply the argument above to $\mathbf{k} = \mathbb{Q}$; then, Z_a is a multiset of elements of $\mathbb{Q}$. In other words, the left action of $\mathcal{S}_{n,k}$ on $\mathbb{Q}[S_n]$ is a linear endomorphism whose minimal polynomial factors into linear factors over $\mathbb{Q}$. That is, all eigenvalues of this endomorphism belong to $\mathbb{Q}$. But this endomorphism is defined over $\mathbb{Z}$, so it is represented by an integer matrix, and thus its eigenvalues (being the roots of the characteristic polynomial of this matrix) must be integral over $\mathbb{Z}$. Thus, these eigenvalues belong to $\mathbb{Q}$ but are integral over $\mathbb{Z}$. The only such numbers are integers. Hence, these eigenvalues are integers. Denoting their multiset (with their multiplicities) by Z_k, we then conclude that Z_k is a multiset of integers satisfying $\prod_{\lambda \in Z_k} (\mathcal{S}_{n,k} - \lambda) = 0$. This holds over $\mathbb{Q}$, thus also over $\mathbb{Z}$, and thus (by base change) over all $\mathbf{k}$. This proves Theorem 4.1.4. □

4.4. The bottom-to-random connection

Let us set

$$\mathcal{B}_n := \mathcal{X}_{(n-1,1)} \qquad \text{for any } n \geq 1$$

(and $\mathcal{B}_0 := 0$). This element $\mathcal{B}_n$ is known as the *bottom-to-random shuffle*, and is part of a family of shuffles studied in [BCGS25] and references therein.

Remark 4.4.1. For each $j \in [n]$, let $z_j \in S_n$ be the cycle $(n, n-1, \ldots, j)$ (which is id when $j = n$). Then,

$$\mathcal{B}_n = \mathcal{X}_{(n-1,1)} = z_1 + z_2 + \cdots + z_n. \tag{48}$$

Proof. We have $D((n-1,1)) = \{n-1\}$. Hence, the permutations $w \in S_n$ satisfying $\operatorname{Des} w \subseteq D((n-1,1))$ are the permutations $w \in S_n$ whose only descent (if they have any) is $n-1$. But these permutations are exactly the cycles $(j, j+1, \ldots, n)$ for $j \in [n]$, and their inverses are the cycles $(n, n-1, \ldots, j) = z_j$ for $j \in [n]$. Hence, $\mathcal{X}_{(n-1,1)} = z_1 + z_2 + \cdots + z_n$ follows from the definition of $\mathcal{X}_{(n-1,1)}$. □

Let us set $\mathcal{S}_{n,-1} := 0$. We now claim the following:

Theorem 4.4.2. Let $k \in \mathbb{N}$. Then,

$$\binom{n-2(k-1)}{2} \mathcal{S}_{n,k-1} = \mathcal{S}_{n,k}(\mathcal{B}_n - (n-2k)) \tag{49}$$

$$= (\mathcal{B}_n^* - (n-2k)) \mathcal{S}_{n,k}. \tag{50}$$

Proof. The antipode map $a \mapsto a^*$ fixes the elements $\mathcal{S}_{n,k}$ and $\mathcal{S}_{n,k-1}$ (by Proposition 4.2.4), but is an anti-algebra isomorphism. Hence, applying it to the equality (49) yields the equality (50), and vice versa (since $a^{**} = a$ for each $a \in \mathcal{A}$). Therefore, it will suffice to prove (50).

For any $i \in \mathbb{Z}$, let $\mathcal{M}_i([n])$ denote the set of all i-matchings of $[n]$. We shall use Iverson bracket notation for truth values (that is, if $\mathcal{A}$ is a statement, then $[\mathcal{A}]$ shall be the integer 1 if $\mathcal{A}$ is true and the integer 0 otherwise); thus, each $w \in S_n$ satisfies

$$\operatorname{incmat}_k(w) = \sum_{P \in \mathcal{M}_k([n])} [w \text{ increases on } P] \tag{51}$$

(by the definition of $\operatorname{incmat}_k(w)$). Hence, the definition of $\mathcal{S}_{n,k}$ becomes

$$\begin{aligned} \mathcal{S}_{n,k} &= \sum_{w \in S_n} \operatorname{incmat}_k(w)\, w \\ &= \sum_{w \in S_n} \sum_{P \in \mathcal{M}_k([n])} [w \text{ increases on } P]\, w \qquad (\text{by (51)}) \\ &= \sum_{P \in \mathcal{M}_k([n])} \sum_{w \in S_n} [w \text{ increases on } P]\, w. \end{aligned} \tag{52}$$

Also, applying the antipode map $a \mapsto a^*$ to (48), we obtain

$$\mathcal{B}_n^* = z_1^{-1} + z_2^{-1} + \cdots + z_n^{-1} = \sum_{j \in [n]} z_j^{-1}.$$

These two equalities yield

$$\begin{aligned} &(\mathcal{B}_n^* - (n-2k))\, \mathcal{S}_{n,k} \\ &= \left(\sum_{j \in [n]} z_j^{-1} - (n-2k) \right) \sum_{P \in \mathcal{M}_k([n])} \sum_{w \in S_n} [w \text{ increases on } P]\, w \\ &= \sum_{P \in \mathcal{M}_k([n])} \sum_{w \in S_n} [w \text{ increases on } P] \left(\sum_{j \in [n]} z_j^{-1} - (n-2k) \right) w \\ &= \sum_{P \in \mathcal{M}_k([n])} \sum_{j \in [n]} \sum_{w \in S_n} [w \text{ increases on } P]\, z_j^{-1} w \\ &\qquad - (n-2k) \sum_{P \in \mathcal{M}_k([n])} \sum_{w \in S_n} [w \text{ increases on } P]\, w \\ &= \sum_{P \in \mathcal{M}_k([n])} \sum_{j \in [n]} \sum_{w \in S_n} [z_j w \text{ increases on } P]\, w \\ &\qquad - (n-2k) \sum_{P \in \mathcal{M}_k([n])} \sum_{w \in S_n} [w \text{ increases on } P]\, w \end{aligned} \tag{53}$$

(here, we have substituted $z_j w$ for w in the third sum).

On the other hand, applying (52) to $k-1$ instead of k, we find

$$\mathcal{S}_{n,k-1} = \sum_{P \in \mathcal{M}_{k-1}([n])} \sum_{w \in S_n} [w \text{ increases on } P]\, w.$$

[8] Multiplying this equality by $\dbinom{n-2\left(k-1\right)}{2}$, we obtain

$$\begin{aligned}
&\binom{n-2\left(k-1\right)}{2}\mathcal{S}_{n,k-1}\\
&=\sum_{P\in\mathcal{M}_{k-1}([n])}\binom{n-2\left(k-1\right)}{2}\sum_{w\in S_n}[w \text{ increases on } P]\, w\\
&=\sum_{Q\in\mathcal{M}_{k-1}([n])}\binom{n-2\left(k-1\right)}{2}\sum_{w\in S_n}[w \text{ increases on } Q]\, w. \qquad (54)
\end{aligned}$$

However, the map

$$\begin{aligned}
\{(P,e)\ \mid\ P\in\mathcal{M}_k([n]) \text{ and } e\in P\}\to\mathcal{M}_{k-1}([n]),\\
(P,e)\mapsto P\setminus\{e\}
\end{aligned}$$

(in common language: remove an edge from a k-matching to obtain a $(k-1)$-matching) is a $\dbinom{n-2\left(k-1\right)}{2}$-to-1 map (i.e., each $Q\in\mathcal{M}_{k-1}([n])$ has exactly $\dbinom{n-2\left(k-1\right)}{2}$ many preimages under this map)[9]. Hence, for any $w\in S_n$, we have

$$\begin{aligned}
&\sum_{Q\in\mathcal{M}_{k-1}([n])}\binom{n-2\left(k-1\right)}{2}\sum_{w\in S_n}[w \text{ increases on } Q]\, w\\
&=\sum_{P\in\mathcal{M}_k([n])}\ \sum_{e\in P}\ \sum_{w\in S_n}[w \text{ increases on } P\setminus\{e\}]\, w.
\end{aligned}$$

Thus, (54) rewrites as

$$\begin{aligned}
&\binom{n-2\left(k-1\right)}{2}\mathcal{S}_{n,k-1}\\
&=\sum_{P\in\mathcal{M}_k([n])}\ \sum_{e\in P}\ \sum_{w\in S_n}[w \text{ increases on } P\setminus\{e\}]\, w. \qquad (55)
\end{aligned}$$

[8] This equality holds even when $k=0$, since both of its sides are 0 in this case (because $\mathcal{S}_{n,-1}=0$ and because $\mathcal{M}_{-1}([n])=\varnothing$).

[9] To prove this, we need to check that for each $Q\in\mathcal{M}_{k-1}([n])$, there are exactly $\dbinom{n-2(k-1)}{2}$ many pairs (P,e) satisfying $P\in\mathcal{M}_k([n])$ and $e\in P$ and $P\setminus\{e\}=Q$. But this is clear: Any such pair is obtained by picking an edge e that is disjoint from all the $k-1$ edges of Q (this can be done in $\dbinom{n-2(k-1)}{2}$ many ways, since there are $n-2\left(k-1\right)$ elements to choose the elements of e from), and setting $P=Q\cup\{e\}$ (this can be done in only one way).

In view of (53) and (55), we can rewrite our goal (52) as

$$\begin{aligned}
&\sum_{P\in\mathcal{M}_k([n])}\ \sum_{j\in[n]}\ \sum_{w\in S_n} \left[z_j w \text{ increases on } P\right] w \\
&\qquad - (n-2k) \sum_{P\in\mathcal{M}_k([n])}\ \sum_{w\in S_n} [w \text{ increases on } P]\, w \\
&= \sum_{P\in\mathcal{M}_k([n])}\ \sum_{e\in P}\ \sum_{w\in S_n} [w \text{ increases on } P\setminus\{e\}]\, w. \qquad (56)
\end{aligned}$$

We shall prove this "addend by addend"; i.e., we shall show that every $P \in \mathcal{M}_k([n])$ and every $w \in S_n$ satisfy

$$\begin{aligned}
&\sum_{j\in[n]} \left[z_j w \text{ increases on } P\right] - (n-2k)\,[w \text{ increases on } P] \\
&= \sum_{e\in P} [w \text{ increases on } P\setminus\{e\}]\,. \qquad (57)
\end{aligned}$$

Once this is shown, multiplying this equality by w and summing it over all $P \in \mathcal{M}_k([n])$ and $w \in S_n$ will yield the desired (56).

Thus, let us now show (57). Let $P \in \mathcal{M}_k([n])$ and $w \in S_n$. Then, it is easy to see that

$$[w \text{ increases on } P] = \prod_{f\in P} [w \text{ increases on } f]$$

and likewise

$$[w \text{ increases on } P\setminus\{e\}] = \prod_{f\in P\setminus\{e\}} [w \text{ increases on } f] \qquad \text{for every } e\in P,$$

and furthermore, each $j \in [n]$ satisfies

$$\begin{aligned}
&\left[z_j w \text{ increases on } P\right] \\
&= \prod_{f\in P} \left[z_j w \text{ increases on } f\right] \\
&= \prod_{f\in P} \begin{cases} [w \text{ increases on } f]\,, & \text{if } j\notin w(f)\,; \\ 1, & \text{if } j = w(\max f)\,; \\ 0, & \text{if } j = w(\min f) \end{cases} \qquad (58)
\end{aligned}$$

(the last equality sign requires a moment of thought[10]). Note that the product in (58) either agrees entirely with the product $\prod_{f\in P} [w \text{ increases on } f]$ or differs from

[10]*Proof:* We just need to show that

$$\left[z_j w \text{ increases on } f\right] = \begin{cases} [w \text{ increases on } f]\,, & \text{if } j\notin w(f)\,; \\ 1, & \text{if } j = w(\max f)\,; \\ 0, & \text{if } j = w(\min f) \end{cases}$$

the latter product in only one factor, since $j = w(\max f)$ or $j = w(\min f)$ can happen only for one edge $f \in P$.

Now, we have three possible cases:

- *Case 1:* The permutation w increases on fewer than $k-1$ edges of P. In this case, the equality (57) is true since all the truth values in it are 0 (this includes $[z_j w \text{ increases on } P]$, since the product in (58) agrees with the product $\prod_{f \in P} [w \text{ increases on } f]$ in all but at most one factors, but at least two factors of the latter product are 0).

- *Case 2:* The permutation w increases on exactly $k-1$ edges of P. In this case, let g be the unique edge of P on which w does not increase. Then, (58) shows that $[z_j w \text{ increases on } P]$ equals 1 for $j = w(\max g)$ and equals 0 for all other j's. Hence, the equality (57) rewrites as

$$1 - (n-2k) \cdot 0 = 1,$$

---

for each $j \in [n]$ and $f \in P$. So let $j \in [n]$ and $f \in P$. Write the edge f as $\{p < q\}$. Hence, the permutation $w \in S_n$ increases on f if and only if $w(p) < w(q)$. Likewise, $z_j w$ increases on f if and only if $(z_j w)(p) < (z_j w)(q)$, that is, $z_j(w(p)) < z_j(w(q))$. Now, we are in one of the following three cases:

- *Case 1:* We have $j \notin w(f)$. In this case, we have $j \notin w(f) = \{w(p), w(q)\}$ (since $f = \{p, q\}$), so that $w(p), w(q) \in [n] \setminus \{j\}$. Hence, the inequality $z_j(w(p)) < z_j(w(q))$ is equivalent to $w(p) < w(q)$ (because since the permutation z_j is increasing on the set $[n] \setminus \{j\}$, which contains both $w(p)$ and $w(q)$). In other words, the permutation $z_j w$ increases on f if and only if the permutation w does. In other words,

$$[z_j w \text{ increases on } f] = [w \text{ increases on } f].$$

- *Case 2:* We have $j = w(\max f)$. But $f = \{p < q\}$ entails $q = \max f$, so that $w(q) = w(\max f) = j$ and therefore $z_j(w(q)) = z_j(j) = n$. But $p \neq q$ and thus $z_j(w(p)) \neq z_j(w(q)) = n$. Hence, $z_j(w(p)) < n = z_j(w(q))$ holds automatically in this case. In other words, $z_j w$ increases on f automatically. Hence,

$$[z_j w \text{ increases on } f] = 1.$$

- *Case 3:* We have $j = w(\min f)$. In view of $f = \{p < q\}$, this rewrites as $j = w(p)$. Hence, $z_j(w(p)) = z_j(j) = n$ is always larger than $z_j(w(q))$ (since $p \neq q$). In other words, $z_j w$ does not increase on f. Hence,

$$[z_j w \text{ increases on } f] = 0.$$

Combining the results of these three cases, we obtain

$$[z_j w \text{ increases on } f] = \begin{cases} [w \text{ increases on } f], & \text{if } j \notin w(f); \\ 1, & \text{if } j = w(\max f); \\ 0, & \text{if } j = w(\min f), \end{cases}$$

qed.

which is true.

- *Case 3:* The permutation w increases on all k edges of P. In this case, (58) shows that $\left[z_j w \text{ increases on } P\right]$ equals 1 for $(n-2k)+k$ values of j (namely, for all $j \in [n]$ that belong to no edge in P, as well as for the $w(\max f)$ for all $f \in P$). Hence, the equality (57) rewrites as

$$((n-2k)+k)-(n-2k)\cdot 1 = k,$$

which is true again.

So (57) has been proved, and thus (50) as well. As we said above, this completes the proof of Theorem 4.4.2. □

Corollary 4.4.3. Let $k \in \mathbb{N}$. Then, $\mathcal{S}_{n,k}\mathcal{B}_n = \mathcal{B}_n^* \mathcal{S}_{n,k}$.

Proof. Theorem 4.4.2 yields $\mathcal{S}_{n,k}\left(\mathcal{B}_n - (n-2k)\right) = \left(\mathcal{B}_n^* - (n-2k)\right)\mathcal{S}_{n,k}$. Adding $(n-2k)\,\mathcal{S}_{n,k}$ to both sides of this equality, we obtain $\mathcal{S}_{n,k}\mathcal{B}_n = \mathcal{B}_n^*\mathcal{S}_{n,k}$. This proves Corollary 4.4.3. □

The following corollary was conjectured by Nadia Lafrenière:

Corollary 4.4.4. Let $k \in \mathbb{N}$. Let V be a left $\mathcal{A}$-module. Let $v \in V$ be such that $\mathcal{S}_{n,k-1}v = 0$ and $\mathcal{S}_{n,k}v = \mu v$ for some nonzero scalar $\mu \in \mathbf{k}$. Then, $\mathcal{B}_n^* v = (n-2k)\,v$.

Proof. Theorem 4.4.2 yields

$$\left(\mathcal{B}_n^* - (n-2k)\right)\mathcal{S}_{n,k} = \binom{n-2(k-1)}{2}\mathcal{S}_{n,k-1}.$$

Hence,

$$\left(\mathcal{B}_n^* - (n-2k)\right)\mathcal{S}_{n,k}v = \binom{n-2(k-1)}{2}\underbrace{\mathcal{S}_{n,k-1}v}_{=0} = 0.$$

Since $\mathcal{S}_{n,k}v = \mu v$, we can rewrite this as $\left(\mathcal{B}_n^* - (n-2k)\right)\mu v = 0$. Dividing this equality by μ (since μ is nonzero), we obtain $\left(\mathcal{B}_n^* - (n-2k)\right)v = 0$, so that $\mathcal{B}_n^* v = (n-2k)\,v$. □

4.5. Connection with w_0

Let $w_0 \in S_n$ be the permutation that sends $1, 2, \ldots, n$ to $n, n-1, \ldots, 1$. Clearly, w_0 is an involution, i.e., we have $w_0^{-1} = w_0$. Moreover, the map w_0 is strictly decreasing. We claim the following:

Proposition 4.5.1. Let $k \in \mathbb{N}$. Then, $w_0 \mathcal{S}_{n,k} = \mathcal{S}_{n,k} w_0$.

Proof. This is equivalent to $w_0 \mathcal{S}_{n,k} w_0^{-1} = \mathcal{S}_{n,k}$. This, in turn, is equivalent to $\operatorname{incmat}_k \left(w_0^{-1} u w_0\right) = \operatorname{incmat}_k (u)$ for all $u \in S_n$ (by the definition of $\mathcal{S}_{n,k}$). So let us prove the latter equality.

Fix $u \in S_n$. The symmetric group S_n acts on the set $[n]$; hence it also acts on the set of edges (a permutation $w \in S_n$ sends an edge $\{i, j\}$ to $w(\{i, j\}) := \{w(i), w(j)\}$) and also acts on the set of k-matchings (a permutation $w \in S_n$ sends a k-matching M to the k-matching $w(M) := \{w(e) \mid e \in M\}$).

We claim the following:

> *Claim 1:* Let e be an edge. Then, u increases on e if and only if $w_0^{-1} u w_0$ increases on $w_0(e)$.

Proof of Claim 1. Write the edge e as $e = \{i < j\}$. Then, from $i < j$, we obtain $w_0(i) > w_0(j)$ (since w_0 is strictly decreasing). Now, from $e = \{i < j\}$, we obtain

$$w_0(e) = \{w_0(i),\ w_0(j)\} = \{w_0(j) < w_0(i)\}$$

(since $w_0(i) > w_0(j)$). Hence, we have the following chain of equivalences:

$$\begin{aligned}
&\Big(\underbrace{w_0^{-1}}_{=w_0} u \underbrace{w_0}_{=w_0^{-1}} \text{ increases on } \underbrace{w_0(e)}_{=\{w_0(j) < w_0(i)\}}\Big) \\
\iff & \left(w_0 u w_0^{-1} \text{ increases on } \{w_0(j) < w_0(i)\}\right) \\
\iff & \Big(\underbrace{\left(w_0 u w_0^{-1}\right)(w_0(j))}_{=w_0(u(j))} < \underbrace{\left(w_0 u w_0^{-1}\right)(w_0(i))}_{=w_0(u(i))}\Big) \\
\iff & \left(w_0(u(j)) < w_0(u(i))\right) \\
\iff & \left(u(j) > u(i)\right) \qquad \text{(since } w_0 \text{ is strictly decreasing)} \\
\iff & \left(u(i) < u(j)\right) \\
\iff & \left(u \text{ increases on } e\right) \qquad \left(\text{since } e = \{i < j\}\right).
\end{aligned}$$

In other words, $(u \text{ increases on } e) \iff \left(w_0^{-1} u w_0 \text{ increases on } w_0(e)\right)$. This proves Claim 1. □

> *Claim 2:* Let M be a k-matching. Then, u increases on M if and only if $w_0^{-1} u w_0$ increases on $w_0(M)$.

Proof of Claim 2. We have the following chain of equivalences:

$$\begin{aligned}
&(u \text{ increases on } M) \\
\iff &(u \text{ increases on } e \text{ for each } e \in M) \\
\iff &\left(w_0^{-1}uw_0 \text{ increases on } w_0(e) \text{ for each } e \in M\right) \\
&\qquad (\text{by Claim 1}) \\
\iff &\left(w_0^{-1}uw_0 \text{ increases on each } f \in w_0(M)\right) \\
&\qquad \left(\begin{array}{c}\text{since the edges } f \in w_0(M) \\ \text{are precisely the } w_0(e) \text{ for } e \in M\end{array}\right) \\
\iff &\left(w_0^{-1}uw_0 \text{ increases on } w_0(M)\right).
\end{aligned}$$

This proves Claim 2. □

Now, the definition of $\operatorname{incmat}_k\left(w_0^{-1}uw_0\right)$ yields

$$\begin{aligned}
&\operatorname{incmat}_k\left(w_0^{-1}uw_0\right) \\
&= \left(\# \text{ of all } k\text{-matchings } M \text{ such that } w_0^{-1}uw_0 \text{ increases on } M\right) \\
&= \left(\# \text{ of all } k\text{-matchings } M \text{ such that } w_0^{-1}uw_0 \text{ increases on } w_0(M)\right) \\
&\qquad \left(\begin{array}{c}\text{here, we substituted } w_0(M) \text{ for } M\text{, since} \\ \text{the action of } w_0 \text{ on } \{k\text{-matchings}\} \text{ is bijective}\end{array}\right) \\
&= (\# \text{ of all } k\text{-matchings } M \text{ such that } u \text{ increases on } M) \\
&\qquad (\text{by Claim 2}) \\
&= \operatorname{incmat}_k(u) \qquad (\text{by the definition of } \operatorname{incmat}_k(u)).
\end{aligned}$$

This completes our proof. □

We can actually improve Proposition 4.5.1:

Proposition 4.5.2. Let $k \in \mathbb{N}$. Then,

$$w_0\mathcal{S}_{n,k} = \mathcal{S}_{n,k}w_0 = \sum_{i=0}^{k} (-1)^i \binom{n-2i}{2k-2i} \frac{(2k-2i)!}{2^{k-i}(k-i)!}\mathcal{S}_{n,i}.$$

Proof. We begin with some simple observations:

If M is a k-matching, then any subset I of M is an i-matching, where $i = |I|$. Moreover:

> *Claim 1:* Let $i \in \{0, 1, \ldots, k\}$. Let I be an i-matching. Then, the number of all k-matchings M satisfying $I \subseteq M$ is $\dbinom{n-2i}{2k-2i} \dfrac{(2k-2i)!}{2^{k-i}(k-i)!}$.

Proof of Claim 1. The i edges of I are disjoint (since I is a matching), and each of them has size 2. Let N_I be their union. Thus, N_I is a $2i$-element subset of $[n]$. Hence, $[n] \setminus N_I$ is an $(n-2i)$-element set.

In order to construct a k-matching M satisfying $I \subseteq M$, we must only choose its remaining $k-i$ edges (after the i edges inherited from I). These $k-i$ edges must be $k-i$ disjoint 2-element subsets of $[n] \setminus N_I$ (since they must be disjoint from the edges in I). In other words, they must form a $(k-i)$-matching of $[n] \setminus N_I$. This requirement is both necessary and sufficient. Hence,

$$\begin{aligned}
&(\#\text{ of } k\text{-matchings } M \text{ satisfying } I \subseteq M) \\
&= (\#\text{ of } (k-i)\text{-matchings of } [n] \setminus N_I) \\
&= \binom{n-2i}{2(k-i)} \frac{(2(k-i))!}{2^{k-i}(k-i)!} \qquad \left(\begin{array}{c}\text{by Proposition 3.6.1,}\\ \text{applied to } n-2i,\ k-i \text{ and } [n] \setminus N_I \\ \text{instead of } n,\ k \text{ and } N\end{array}\right) \\
&= \binom{n-2i}{2k-2i} \frac{(2k-2i)!}{2^{k-i}(k-i)!}.
\end{aligned}$$

This proves Claim 1. □

> *Claim 2:* Let $w \in S_n$ be a permutation. Let e be an edge. Then, the permutation w increases on e if and only if the permutation $w_0 w$ does not increase on e.

Proof of Claim 2. Write the edge e as $e = \{i < j\}$. Then, $w_0(e) = w_0(\{i < j\}) = \{w_0(i) > w_0(j)\}$ (since w_0 is strictly decreasing, so that $i < j$ entails $w_0(i) > w_0(j)$). Moreover, from $i < j$, we obtain $i \neq j$, so that $w(i) \neq w(j)$ (since w is a permutation and thus injective). Now, we have the following chain of equivalences:

$$\begin{aligned}
&(w \text{ increases on } e) \\
\Longleftrightarrow\ & (w(i) < w(j)) \qquad (\text{since } e = \{i < j\}) \\
\Longleftrightarrow\ & (w(i) \leq w(j)) \qquad (\text{since } w(i) \neq w(j)) \\
\Longleftrightarrow\ & (w_0(w(i)) \geq w_0(w(j))) \qquad (\text{since } w_0 \text{ is strictly decreasing}) \\
\Longleftrightarrow\ & ((w_0 w)(i) \geq (w_0 w)(j)) \qquad \left(\begin{array}{c}\text{since } w_0(w(i)) = (w_0 w)(i) \\ \text{and } w_0(w(j)) = (w_0 w)(j)\end{array}\right) \\
\Longleftrightarrow\ & (\text{we don't have } (w_0 w)(i) < (w_0 w)(j)) \\
\Longleftrightarrow\ & (w_0 w \text{ does not increase on } e) \qquad (\text{since } e = \{i < j\}).
\end{aligned}$$

This proves Claim 2. □

However, for each $w \in S_n$, we have

$$\operatorname{incmat}_k(w) = \sum_{\substack{M \text{ is a } k\text{-matching;} \\ w \text{ increases on } M}} 1 \qquad (59)$$

(since a sum of 1s equals the number of addends). Hence, the definition of $\mathcal{S}_{n,k}$ becomes

$$\mathcal{S}_{n,k} = \sum_{w \in S_n} \operatorname{incmat}_k(w)\, w = \sum_{w \in S_n} \sum_{\substack{M \text{ is a } k\text{-matching;} \\ w \text{ increases on } M}} 1w \qquad \text{(by (59))}$$
$$= \sum_{M \text{ is a } k\text{-matching}} \sum_{\substack{w \in S_n \text{ increases} \\ \text{on } M}} w. \tag{60}$$

By the same argument, for any $i \in \mathbb{N}$, we have

$$\mathcal{S}_{n,i} = \sum_{I \text{ is an } i\text{-matching}} \sum_{\substack{w \in S_n \text{ increases} \\ \text{on } I}} w. \tag{61}$$

However, multiplying the equality (60) by w_0 from the left, we find

$$w_0 \mathcal{S}_{n,k} = w_0 \sum_{M \text{ is a } k\text{-matching}} \sum_{\substack{w \in S_n \text{ increases} \\ \text{on } M}} w$$
$$= \sum_{M \text{ is a } k\text{-matching}} \sum_{\substack{w \in S_n \text{ increases} \\ \text{on } M}} w_0 w. \tag{62}$$

However, for any k-matching M and any $w \in S_n$, we have the chain of equivalences

$$\begin{aligned}
& (w \text{ increases on } M) \\
\Longleftrightarrow\ & ((w \text{ increases on } e) \text{ for each edge } e \in M) \\
\Longleftrightarrow\ & ((w_0 w \text{ does not increase on } e) \text{ for each edge } e \in M) \qquad \text{(by Claim 2)} \\
\Longleftrightarrow\ & (w_0 w \text{ increases on no edge } e \in M)\,.
\end{aligned}$$

Thus, for each k-matching M, we have

$$\begin{aligned}
&\sum_{\substack{w\in S_n \text{ increases}\\ \text{on } M}} w_0 w \\
&= \sum_{\substack{w\in S_n;\\ w_0 w \text{ increases on no edge } e\in M}} w_0 w \\
&= \sum_{\substack{w\in S_n \text{ increases}\\ \text{on no edge } e\in M}} w \qquad \left(\begin{array}{c}\text{here, we have substituted } w \text{ for } w_0 w\\ \text{in the sum}\end{array}\right) \\
&= \sum_{I\subseteq M} (-1)^{|I|} \sum_{\substack{w\in S_n \text{ increases}\\ \text{on each edge } e\in I}} w \qquad \left(\begin{array}{c}\text{by the Principle of Inclusion and Exclusion,}\\ \text{specifically [Grinbe21, Theorem 6.2.9]}\\ \text{with } M \text{ instead of } [n]\end{array}\right) \\
&= \sum_{I\subseteq M} (-1)^{|I|} \sum_{\substack{w\in S_n \text{ increases}\\ \text{on } I}} w \qquad \left(\begin{array}{c}\text{since ``} w \text{ increases on each edge } e\in I\text{''}\\ \text{is equivalent to ``} w \text{ increases on } I\text{''}\end{array}\right) \\
&= \sum_{i=0}^{k} \sum_{\substack{I\subseteq M;\\ |I|=i}} (-1)^{|I|} \sum_{\substack{w\in S_n \text{ increases}\\ \text{on } I}} w
\end{aligned}$$

(here, we have split up the first sum according to the value $|I|$, since $|I| \in \{0, 1, \ldots, k\}$

for each $I \subseteq M$). Plugging this into (62), we obtain

$$w_0 \mathcal{S}_{n,k} = \sum_{M \text{ is a } k\text{-matching}} \sum_{i=0}^{k} \sum_{\substack{I \subseteq M; \\ |I|=i}} (-1)^{|I|} \sum_{\substack{w \in S_n \text{ increases} \\ \text{on } I}} w$$

$$= \sum_{i=0}^{k} \underbrace{\sum_{M \text{ is a } k\text{-matching}} \sum_{\substack{I \subseteq M; \\ |I|=i}}}_{\substack{= \sum\limits_{I \text{ is an } i\text{-matching}} \sum\limits_{\substack{M \text{ is a } k\text{-matching;} \\ I \subseteq M}} \\ \text{(since an } i\text{-element subset of a } k\text{-matching} \\ \text{is always an } i\text{-matching)}}} \underbrace{(-1)^{|I|}}_{\substack{=(-1)^i \\ (\text{since } |I|=i)}} \sum_{\substack{w \in S_n \text{ increases} \\ \text{on } I}} w$$

$$= \sum_{i=0}^{k} \sum_{I \text{ is an } i\text{-matching}} \underbrace{\sum_{\substack{M \text{ is a } k\text{-matching;} \\ I \subseteq M}} (-1)^i}_{\substack{= \binom{n-2i}{2k-2i} \frac{(2k-2i)!}{2^{k-i}(k-i)!} \cdot (-1)^i \\ \text{(since Claim 1 says that this sum has} \\ \binom{n-2i}{2k-2i} \frac{(2k-2i)!}{2^{k-i}(k-i)!} \text{ addends,} \\ \text{all of which equal } (-1)^i)}} \sum_{\substack{w \in S_n \text{ increases} \\ \text{on } I}} w$$

$$= \sum_{i=0}^{k} \binom{n-2i}{2k-2i} \frac{(2k-2i)!}{2^{k-i}(k-i)!} \cdot (-1)^i \underbrace{\sum_{I \text{ is an } i\text{-matching}} \sum_{\substack{w \in S_n \text{ increases} \\ \text{on } I}} w}_{\substack{= \mathcal{S}_{n,i} \\ \text{(by (61))}}}$$

$$= \sum_{i=0}^{k} (-1)^i \binom{n-2i}{2k-2i} \frac{(2k-2i)!}{2^{k-i}(k-i)!} \mathcal{S}_{n,i}.$$

Combining this with the equality $w_0 \mathcal{S}_{n,k} = \mathcal{S}_{n,k} w_0$ from Proposition 4.5.1, we obtain the claim of Proposition 4.5.2. □

References

[AdPoRo08] Ron M. Adin, Alexander Postnikov, Yuval Roichman, *Combinatorial Gelfand models*, Journal of Algebra **320** (2008), Issue 3, pp. 1311–1325.

[AFBCCL24] Ilani Axelrod-Freed, Sarah Brauner, Judy Hsin-Hui Chiang, Patricia Commins, Veronica Lang, *Spectrum of random-to-random shuffling in the Hecke algebra*, arXiv:2407.08644v2.

[AguAra01] J. L. Aguado and J. O. Araujo, *A Gel'fand model for the symmetric group*, Communications in Algebra **29** (4) (2001), pp. 1841–1851.

[BCGS25] Sarah Brauner, Patricia Commins, Darij Grinberg, Franco Saliola, *The q-deformed random-to-random family in the Hecke algebra*, arXiv:2503.17580v3.

[BleLau92] D. Blessenohl and H. Laue, *Algebraic combinatorics related to the free Lie algebra*, Séminaire Lotharingien de Combinatoire, B29e (1992), 24 pp.

[DiaFul23] Persi Diaconis, Jason Fulman, *The Mathematics of Shuffling Cards*, AMS 2023.

[DieSal18] Antonius B. Dieker, Franco V. Saliola, *Spectral analysis of random-to-random Markov chains*, Advances in Mathematics **323** (2018), pp. 427–485.

[EGHetc11] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, Elena Yudovina, *Introduction to Representation Theory*, with historical interludes by Slava Gerovitch, Student Mathematical Library **59**, AMS 2011, updated version 2018.

[Fulton97] William Fulton, *Young Tableaux, With Applications to Representation Theory and Geometry*, London Mathematical Society Student Texts **35**, Cambridge University Press 1997.
See `https://mathoverflow.net/questions/456463` for errata.

[GerSch91] M. Gerstenhaber, S. D. Schack, *A Hodge-type decomposition for commutative algebra cohomology*, Journal of Pure and Applied Algebra **48** (1987), Issues 1–2, pp. 229–247.

[Grinbe21] Darij Grinberg, *An Introduction to Algebraic Combinatorics (Math 531, Winter 2024 lecture notes)*, 8 February 2025.
`http://www.cip.ifi.lmu.de/~grinberg/t/21s/lecs.pdf`

[Grinbe24] Darij Grinberg, *An introduction to the symmetric group algebra [Math 701, Spring 2024 lecture notes]*, 27 July 2025, arXiv:2507.20706v1.
`https://www.cip.ifi.lmu.de/~grinberg/t/24s/sga.pdf`

[Grinbe25] Darij Grinberg, *The representation theory of somewhere-to-below shuffles*, arXiv:2508.00752v2.

[Grinbe2X] Darij Grinberg, *The hook length formula [talk slides]*, 2025-01-09.
`https://www.cip.ifi.lmu.de/~grinberg/t/24s/hooktalk.pdf`

[KarPur23] Steven N. Karp, Kevin Purbhoo, *Universal Plücker coordinates for the Wronski map and positivity in real Schubert calculus*, arXiv:2309.04645v1.

[KodVer04] Vijay Kodiyalam, D.-N. Verma, *A natural representation model for symmetric groups*, arXiv:math/0402216v1.

[Lafren19] Nadia Lafrenière, *Valeurs propres des opérateurs de mélanges symétrisés*, thesis at UQAM, 2019, 1912.07718v1.

[Leeuwe95] Marc van Leeuwen, *The Robinson-Schensted and Schützenberger algorithms, an elementary approach*, The Electronic Journal of Combinatorics **3** (1996), Issue 2 (1996), #R15.
A preprint is found at `http://wwwmathlabo.univ-poitiers.fr/~maavl/pdf/foata-fest.pdf` .

[ProSta98] James Propp, Richard Stanley, *Domino tilings with barriers*, J. Combin. Theory Ser. A **87** (1999) no. 2, pp. 347–356.
A preprint is found at `https://arxiv.org/abs/math/9801067v3`

[ReSaWe11] Victor Reiner, Franco Saliola, Volkmar Welker, *Spectra of Symmetrized Shuffling Operators*, arXiv:1102.2460v2. Published in: Memoirs of the AMS **228** (2014).

[Reuten93] Christophe Reutenauer, *Free Lie Algebras*, Oxford University Press 1993.

[Sagan01] Bruce Sagan, *The Symmetric Group*, Graduate Texts in Mathematics **203**, 2nd edition 2001.
See `https://users.math.msu.edu/users/bsagan/Books/Sym/errata.pdf` for errata.

[Stanle23] Richard P. Stanley, *Enumerative Combinatorics, volume 2*, Second edition, Cambridge University Press 2023.
See `http://math.mit.edu/~rstan/ec/` for errata.

[Wildon18] Mark Wildon, *Representation theory of the symmetric group*, 5 April 2018.
`https://www.ma.rhul.ac.uk/~uvah099/Maths/Sym/SymGroup2014.pdf`

[Young77] Alfred Young, *The Collected Papers of Alfred Young 1873–1940*, University of Toronto Press 1977.