A quotient of the ring of symmetric functions generalizing quantum cohomology

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/drexel2019.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf
overview: http://www.cip.ifi.lmu.de/~grinberg/algebra/fpsac19.pdf

What is this about?

- From a modern point of view, **Schubert calculus** (a.k.a. classical enumerative geometry, or Hilbert's 15th problem) is about two cohomology rings:

$$\mathrm{H}^*\Bigg(\underbrace{\mathrm{Gr}\,(k,n)}_{\text{Grassmannian}}\Bigg) \quad \text{and} \quad \mathrm{H}^*\Bigg(\underbrace{\mathrm{Fl}\,(n)}_{\text{flag variety}}\Bigg)$$

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- Classical result: as rings,

$$\mathrm{H}^* (\mathrm{Gr}(k, n)) \cong (\text{symmetric polynomials in } x_1, x_2, \ldots, x_k \text{ over } \mathbb{Z}) \big/ (h_{n-k+1}, h_{n-k+2}, \ldots, h_n)_{\text{ideal}},$$

  where the h_i are complete homogeneous symmetric polynomials (to be defined soon).

- (Small) **Quantum cohomology** is a deformation of cohomology from the 1980–90s. For the Grassmannian, it is

$$\begin{aligned} &\mathrm{QH}^* (\mathrm{Gr}(k,n)) \\ &\cong (\text{symmetric polynomials in } x_1, x_2, \ldots, x_k \text{ over } \mathbb{Z}[q]) \\ &\qquad \diagup \left(h_{n-k+1}, h_{n-k+2}, \ldots, h_{n-1}, h_n + (-1)^k q \right)_{\text{ideal}} . \end{aligned}$$

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- Many properties of classical cohomology still hold here. In particular: $\mathrm{QH}^* (\operatorname{Gr}(k, n))$ has a $\mathbb{Z}[q]$-module basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ of (projected) Schur polynomials (to be defined soon), with λ ranging over all partitions with $\leq k$ parts and each part $\leq n-k$. The structure constants are the **Gromov–Witten invariants**. References:
  - Aaron Bertram, Ionut Ciocan-Fontanine, William Fulton, *Quantum multiplication of Schur polynomials*, 1999.
  - Alexander Postnikov, *Affine approach to quantum Schubert calculus*, 2005.

Where are we going?

- **Goal:** Deform $\mathrm{H}^*(\mathrm{Gr}(k, n))$ using k parameters instead of one, generalizing $\mathrm{QH}^*(\mathrm{Gr}(k, n))$.

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- The new ring has no geometric interpretation known so far, but various properties suggesting such an interpretation likely exists.
- I will now start from scratch and define standard notations around symmetric polynomials, then introduce the deformed cohomology ring algebraically.

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- Let $\mathcal{S}$ denote the ring of *symmetric* polynomials in $\mathcal{P}$.
  These are the polynomials $f \in \mathcal{P}$ satisfying

$$f(x_1, x_2, \ldots, x_k) = f\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}\right)$$

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- **Theorem (Artin ≤1944):** The $\mathcal{S}$-module $\mathcal{P}$ is free with basis

$$\left(x^\alpha\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i} \qquad \text{(or, informally: "}\left(x_1^{<1} x_2^{<2} \cdots x_k^{<k}\right)\text{")}.$$

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  Example: For $k = 3$, this basis is $\left(1, x_3, x_3^2, x_2, x_2x_3, x_2x_3^2\right)$.

- The ring $\mathcal{S}$ of symmetric polynomials in $\mathcal{P} = \mathbf{k}[x_1, x_2, \ldots, x_k]$ has several bases, usually indexed by certain sets of (integer) partitions.
  First, let us recall what partitions are:

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  $(2, 3, 2, 0, 0, 0, \ldots)$ and $(2, 1, 1, 1, \ldots)$ are not.

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- Thus there is a bijection

$$\{k\text{-partitions}\} \to \{\text{partitions with at most } k \text{ nonzero entries}\},$$
$$\lambda \mapsto (\lambda_1, \lambda_2, \ldots, \lambda_k, 0, 0, 0, \ldots).$$

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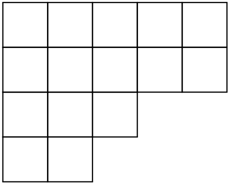
  $(2, 3, 2)$ is not.
- If $\lambda \in \mathbb{N}^k$ is a k-partition, then its *Young diagram $Y(\lambda)$* is defined as a table made out of k left-aligned rows, where the i-th row has λ_i boxes.
  Example: If $k = 6$ and $\lambda = (5, 5, 3, 2, 0, 0)$, then

$Y(\lambda) =$  .

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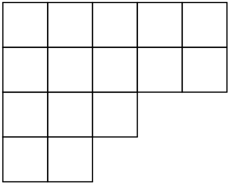
  $Y(\lambda) =$  .

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- The same convention applies to partitions.

Symmetric polynomials: the e-basis

- For each $m \in \mathbb{Z}$, we let e_m denote the m-th *elementary symmetric polynomial*:

$$e_m = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \{0,1\}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

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- Note that $e_m = 0$ when $m > k$.

- For each $m \in \mathbb{Z}$, we let h_m denote the m-th *complete homogeneous symmetric polynomial*:

$$h_m = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

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- **Theorem:** $(h_\lambda)_{\lambda \text{ is a } k\text{-partition}}$ is a basis of the **k**-module $\mathcal{S}$. (Another basis!)

- For each k-partition λ, we let s_λ be the *λ-th Schur polynomial*:

$$s_\lambda = \frac{\det\left(\left(x_i^{\lambda_j+k-j}\right)_{1\le i\le k,\ 1\le j\le k}\right)}{\det\left(\left(x_i^{k-j}\right)_{1\le i\le k,\ 1\le j\le k}\right)} \qquad \text{(alternant formula)}$$

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- **Theorem:** The equality above holds, and s_λ is a symmetric polynomial with nonnegative coefficients. Explicitly,

$$s_\lambda = \sum_{\substack{T \text{ is a semistandard } \lambda\text{-tableau} \\ \text{with entries } 1,2,\ldots,k}} \prod_{i=1}^{k} x_i^{(\text{number of } i\text{'s in } T)},$$

where a *semistandard λ-tableau with entries $1, 2, \ldots, k$* is a way of putting an integer $i \in \{1, 2, \ldots, k\}$ into each box of $Y(\lambda)$ such that the entries **weakly** increase along rows and **strictly** increase along columns.

Symmetric polynomials: the s-basis (Schur polynomials)

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- **Theorem:** $(s_\lambda)_{\lambda \text{ is a } k\text{-partition}}$ is a basis of the **k**-module $\mathcal{S}$.

- If λ and μ are two k-partitions, then the product $s_\lambda s_\mu$ can be again written as a **k**-linear combination of Schur polynomials (since these form a basis):

$$s_\lambda s_\mu = \sum_{\nu \text{ is a } k\text{-partition}} c^\nu_{\lambda,\mu} s_\nu,$$

where the $c^\nu_{\lambda,\mu}$ lie in **k**. These $c^\nu_{\lambda,\mu}$ are called the *Littlewood-Richardson coefficients*.

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  where the $c_{\lambda,\mu}^{\nu}$ lie in **k**. These $c_{\lambda,\mu}^{\nu}$ are called the *Littlewood-Richardson coefficients*.
- **Theorem:** These Littlewood-Richardson coefficients $c_{\lambda,\mu}^{\nu}$ are nonnegative integers (and count something).

- We have defined

$$s_\lambda = \det\left(\left(h_{\lambda_i - i + j}\right)_{1\le i\le k,\ 1\le j\le k}\right)$$

for k-partitions λ.
Apply the same definition to arbitrary $\lambda \in \mathbb{Z}^k$.

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Apply the same definition to arbitrary $\lambda \in \mathbb{Z}^k$.
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    - If β has two equal entries, then $s_\alpha = 0$.
    - Otherwise, let γ be the k-tuple obtained by sorting β in decreasing order, and let σ be the permutation of the indices that causes this sorting. Let λ be the k-partition $(\gamma_1 - (k-1), \gamma_2 - (k-2), \ldots, \gamma_k - (k-k))$. Then, $s_\alpha = (-1)^\sigma s_\lambda$.

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- Also, the alternant formula still holds if all $\lambda_i + (k-i)$ are ≥ 0.

A more general setting: $a_1, a_2, \ldots, a_k$ and J

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$$\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$$
$$\left(\text{informally: } ``\left(\overline{x_1^{<n-k+1} x_2^{<n-k+2} \cdots x_n^{<n}}\right)"\right)$$

where the overline $\overline{}$ means "projection" onto whatever quotient we need (here: from $\mathcal{P}$ onto $\mathcal{P} / J$).
(This basis has $n(n-1) \cdots (n-k+1)$ elements.)

A slightly less general setting: symmetric $a_1, a_2, \ldots, a_k$ and J

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- Let $\omega = \underbrace{(n-k, n-k, \ldots, n-k)}_{k \text{ entries}}$ and

$$P_{k,n} = \{\lambda \text{ is a } k\text{-partition} \mid \lambda_1 \leq n-k\} \\ = \{k\text{-partitions } \lambda \subseteq \omega\}.$$

- Here, for two k-partitions α and β, we say that $\alpha \subseteq \beta$ if and only if $\alpha_i \leq \beta_i$ for all i.
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  - **classical cohomology:** If $\mathbf{k} = \mathbb{Z}$ and $a_1 = a_2 = \cdots = a_k = 0$, then $\mathcal{S} / I$ becomes the cohomology ring $\mathrm{H}^* (\mathrm{Gr}(k, n))$; the basis $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$ corresponds to the Schubert classes.
  - **quantum cohomology:** If $\mathbf{k} = \mathbb{Z}[q]$ and $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = -(-1)^k q$, then $\mathcal{S} / I$ becomes the quantum cohomology ring $\mathrm{QH}^* (\mathrm{Gr}(k, n))$.

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- The above theorem lets us work in these rings (and more generally) without relying on geometry.

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- **Equivalent restatement:** Each $\nu \in P_{k,n}$ and $f \in \mathcal{S} / I$ satisfy $\operatorname{coeff}_\omega \left(\overline{s_\nu} f \right) = \operatorname{coeff}_{\nu^\vee} (f)$.

- **Theorem (G.):** The $\mathbf{k}$-module $\mathcal{S} / I$ is free with basis

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The h-basis

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- The transfer matrix between the two bases $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$ and $\left(\overline{h_\lambda}\right)_{\lambda \in P_{k,n}}$ is unitriangular wrt the "size-then-anti-dominance" order, but seems hard to describe.

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- **Proposition (G.):** Let m be a positive integer. Then,

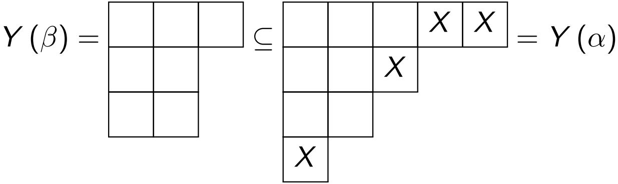
$$\overline{h_{n+m}} = \sum_{j=0}^{k-1} (-1)^j \, a_{k-j} \overline{s_{(m,1^j)}},$$

where $\left(m, 1^j\right) := (m, \underbrace{1, 1, \ldots, 1}_{j \text{ ones}}, 0, 0, 0, \ldots)$ (a hook-shaped k-partition).

The Pieri rule for symmetric polynomials

- If α and β are two k-partitions, then we say that *α / β is a horizontal strip* if and only if the Young diagram $Y(\alpha)$ is obtained from $Y(\beta)$ by adding some (possibly none) extra boxes with no two of these new boxes lying in the same column.

  Example: If $k = 4$ and $\alpha = (5, 3, 2, 1)$ and $\beta = (3, 2, 2, 0)$, then α / β is a horizontal strip, since



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- Equivalently, α / β is a horizontal strip if and only if

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \alpha_3 \geq \cdots \geq \alpha_k \geq \beta_k.$$

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- Furthermore, given $j \in \mathbb{N}$, we say that *$\alpha \diagup \beta$ is a horizontal j-strip* if $\alpha \diagup \beta$ is a horizontal strip and $|\alpha| - |\beta| = j$.

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- Furthermore, given $j \in \mathbb{N}$, we say that *α / β is a horizontal j-strip* if α / β is a horizontal strip and $|\alpha| - |\beta| = j$.
- **Theorem (Pieri).** Let λ be a k-partition. Let $j \in \mathbb{N}$. Then,

$$s_\lambda h_j = \sum_{\substack{\mu \text{ is a } k\text{-partition;} \\ \mu / \lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} s_\mu .$$

- **Theorem (G.):** Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^{k} (-1)^i \, a_i \sum_{\nu \subseteq \lambda} c^{\lambda}_{(n-k-j+1, 1^{i-1}), \nu} \overline{s_\nu}.$$

A Pieri rule for $\mathcal{S}/I$

- **Theorem (G.):** Let $\lambda \in P_{k,n}$. Let $j \in \{0, 1, \ldots, n-k\}$. Then,

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- This generalizes the h-Pieri rule from Bertram, Ciocan-Fontanine and Fulton, but note that $c^{\lambda}_{(n-k-j+1,1^{i-1}),\nu}$ may be > 1.

- Example: For $n = 7$ and $k = 3$, we have

$$\begin{aligned}\overline{s_{(4,3,2)}h_2} = \overline{s_{(4,4,3)}} &+ a_1\left(\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}}\right)\\ &- a_2\left(\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}}\right)\\ &+ a_3\left(\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}}\right).\end{aligned}$$

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- Multiplying by e_j appears harder: For $n = 5$ and $k = 3$, we have

$$\overline{s_{(2,2,1)} e_2} = a_1 \overline{s_{(2,2)}} - 2a_2 \overline{s_{(2,1)}} + a_3 \left(\overline{s_{(2)}} + \overline{s_{(1,1)}} \right) + a_1^2 \overline{s_{(1)}} - 2a_1 a_2 \overline{s_{()}}.$$

A "rim hook algorithm"

- For $\mathrm{QH}^*(\mathrm{Gr}(k,n))$, Bertram, Ciocan-Fontanine and Fulton give a "rim hook algorithm" that rewrites an arbitrary $\overline{s_\mu}$ as $(-1)^{\text{something}} q^{\text{something}} \overline{s_\lambda}$ with $\lambda \in P_{k,n}$.
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$$\overline{s_{(4,4,3)}} = a_2^2 \overline{s_{(1)}} - 2a_1 a_2 \overline{s_{(2)}} + a_1^2 \overline{s_{(3)}} + a_3 \overline{s_{(3,2)}} - a_2 \overline{s_{(3,3)}}.$$

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- **Theorem (G.):** Let μ be a k-partition with $\mu_1 > n - k$. Let

$$W = \left\{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k \;\middle|\; \lambda_1 = \mu_1 - n \right.$$
$$\left. \text{and } \lambda_i - \mu_i \in \{0,1\} \text{ for all } i \in \{2,3,\ldots,k\} \right\}.$$

  (Not all elements of W are k-partitions, but all belong to $\mathbb{Z}^k$, so we know how to define s_λ for them.)

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  Then,

$$\overline{s_\mu} = \sum_{j=1}^{k} (-1)^{k-j} a_j \sum_{\substack{\lambda \in W; \\ |\lambda| = |\mu| - (n-k+j)}} \overline{s_\lambda}.$$

Positivity?

- **Conjecture:** Let $b_i = (-1)^{n-k-1} a_i$ for each $i \in \{1, 2, \ldots, k\}$. Let $\lambda, \mu, \nu \in P_{k,n}$. Then, $(-1)^{|\lambda|+|\mu|-|\nu|} \operatorname{coeff}_\nu (\overline{s_\lambda s_\mu})$ is a polynomial in $b_1, b_2, \ldots, b_k$ with coefficients in $\mathbb{N}$.
- Verified for all $n \leq 7$ using SageMath.
- This would generalize positivity of Gromov–Witten invariants.

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- **Question:** Is there an analogous generalization of $\mathrm{QH}^*(\mathrm{Fl}(n))$? Is it connected to Fulton's "universal Schubert polynomials"?

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More questions

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- **Question:** What about quotients of the quasisymmetric polynomials?

S_k-module structure

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- What is the S_k-module structure on $\mathcal{P} / J$?
- **Almost-theorem (G., needs to be checked):** Assume that $\mathbf{k}$ is a $\mathbb{Q}$-algebra. Then, as S_k-modules,

$$\mathcal{P} / J \cong \left(\mathcal{P} / \mathcal{P}\mathcal{S}^+\right)^{\times \binom{n}{k}} \cong \left(\underbrace{\mathbf{k} S_k}_{\text{regular rep}} \right)^{\times \binom{n}{k}},$$

where $\mathcal{P}\mathcal{S}^+$ is the ideal of $\mathcal{P}$ generated by symmetric polynomials with constant term 0.

Deforming symmetric functions, 1

- Let us recall symmetric **functions** (not polynomials) now; we'll need them soon anyway.

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$$\mathcal{S} \cong \Lambda / (\mathbf{e}_{k+1}, \quad \mathbf{e}_{k+2}, \quad \mathbf{e}_{k+3}, \quad \ldots)_{\text{ideal}}, \qquad \text{thus}$$
$$\mathcal{S} / I \cong \Lambda / (\mathbf{h}_{n-k+1} - a_1, \quad \mathbf{h}_{n-k+2} - a_2, \quad \ldots, \quad \mathbf{h}_n - a_k, \quad \mathbf{e}_{k+1}, \quad \mathbf{e}_{k+2}, \quad \mathbf{e}_{k+3}, \quad \ldots)_{\text{ideal}}.$$

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- So why not replace the $\mathbf{e}_j$ by $\mathbf{e}_j - b_j$ too?

- **Theorem (G.):** Assume that $a_1, a_2, \ldots, a_k$ as well as $b_1, b_2, b_3, \ldots$ are elements of $\mathbf{k}$. Then,

$$\Lambda \diagup (\mathbf{h}_{n-k+1} - a_1, \ \ \mathbf{h}_{n-k+2} - a_2, \ \ \ldots, \ \ \mathbf{h}_n - a_k, \\ \mathbf{e}_{k+1} - b_1, \ \ \mathbf{e}_{k+2} - b_2, \ \ \mathbf{e}_{k+3} - b_3, \ \ \ldots)_{\text{ideal}}$$

  is a free $\mathbf{k}$-module with basis $\left(\overline{\mathbf{s}_\lambda}\right)_{\lambda \in P_{k,n}}$.

On the proofs, 1

- Proofs of all the above (except for the rim hook algorithm, which will be done soon, and the S_k-action) can be found in
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- **Main ideas:**
  - Use Gröbner bases to show that $\mathcal{P} / J$ is free with basis $\left(\overline{x^\alpha}\right)_{\alpha \in \mathbb{N}^k;\ \alpha_i < n-k+i \text{ for each } i}$.
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  - Using that + Jacobi–Trudi, show that $\mathcal{S} \diagup I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$.
  - As for the rest, compute in Λ... a lot.

On the proofs, 2: the Gröbner basis argument

- The Gröbner basis argument relies on the easy identity

$$h_p\left(x_{i..k}\right) = \sum_{t=0}^{i-1} \left(-1\right)^t e_t\left(x_{1..i-1}\right) h_{p-t}\left(x_{1..k}\right)$$

  for all $i \in \{1, 2, \ldots, k+1\}$ and $p \in \mathbb{N}$.
  Here, $x_{a..b}$ means $x_a, x_{a+1}, \ldots, x_b$.

- Use it to show that

$$\left(h_{n-k+i}\left(x_{i..k}\right) - \sum_{t=0}^{i-1} \left(-1\right)^t e_t\left(x_{1..i-1}\right) a_{i-t}\right)_{i\in\{1,2,\ldots,k\}}$$

  is a Gröbner basis of the ideal J wrt the degree-lexicographic term order, where the variables are ordered by $x_1 > x_2 > \cdots > x_k$.

- This Gröbner basis leads to a basis of $\mathcal{P} / J$, which is precisely our $\left(\overline{x^\alpha}\right)_{\alpha\in\mathbb{N}^k;\ \alpha_i<n-k+i \text{ for each } i}$.

- How to prove that $\mathcal{S} / I$ is free with basis $\left(\overline{s_\lambda}\right)_{\lambda \in P_{k,n}}$?

On the proofs, 3: the first basis of $\mathcal{S} / I$

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- Thus, $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n};\ \alpha \in \mathbb{N}^k;\ \alpha_i < i \text{ for each } i}$ is a basis of $\mathcal{P}/J$.
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- Maybe the most important one:
  Bernstein's identity: Let λ be a partition. Let $m \in \mathbb{Z}$ be such that $m \geq \lambda_1$. Then,

$$\sum_{i \in \mathbb{N}} (-1)^i \, \mathbf{h}_{m+i} \left(\mathbf{e}_i\right)^{\perp} \mathbf{s}_\lambda = \mathbf{s}_{(m, \lambda_1, \lambda_2, \lambda_3, \ldots)}.$$

  Here, $\mathbf{f}^{\perp}\mathbf{g}$ means "$\mathbf{g}$ skewed by $\mathbf{f}$" (so that $(\mathbf{s}_\mu)^{\perp} \mathbf{s}_\lambda = \mathbf{s}_{\lambda / \mu}$).

Thank you

- **Shari Moskow** for the invitation.
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